Funk Metrics and R-Flat Sprays*

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Abstract

Akbar-Zadeh's theorem says that every Finsler metric on a compact manifold with zero flag curvature $\mathbf{K} = 0$ must be trivial (i.e., locally Minkowskian). In this paper, we construct non-trivial R-flat sprays using a Funk metric. It is then an inverse problem in the calculus of variation to find Finsler metrics that induce a given R-flat spray. We find some explicit solutions to this inverse problem and obtain non-trivial projectively flat Finsler metrics with zero flag curvature $\mathbf{K} = 0$. These metrics are the special solutions to Hilbert's Fourth Problem and can be served as "spaceforms" in Finsler geometry.

1 Introduction

One of fundamental problems in Finsler geometry is to find and study non-trivial Finsler metrics of constant (flag) curvature. Several non-trivial Finsler metrics of constant curvature have been found. The simplest ones are the Funk metrics and the Hilbert metrics. The Funk metric $F = F(x, y)$ on a strongly convex domain Ω in R^n is defined by

$$x + \frac{y}{F} \in \partial\Omega, \qquad y \in T_x\Omega = \mathrm{R}^n. \tag{1}$$

The Funk metric is non-reversible, positively complete and projectively flat with $\mathbf{K} = -1/4$. The Hilbert metric on Ω is obtained from the Funk metric by symmetrization,

$$F_h(x, y) := \frac{1}{2}\Big(F(x, y) + F(x, -y)\Big), \qquad y \in T_x\Omega = \mathrm{R}^n. \tag{2}$$

It is reversible, complete and projectively flat with $\mathbf{K} = -1$. H. Akbar-Zadeh [AZ] shows that any Finsler metric on a compact manifold with $\mathbf{K} = -1$ must be Riemannian. Thus, there is no non-Riemannian example with $\mathbf{K} = -1$ on any compact manifold.

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According to the Bonnet-Meyers theorem in Finsler geometry [BCS], any positively complete Finsler manifold with constant curvature $\mathbf{K} = 1$ must be compact. Further, if the manifold is simply connected, then it must be diffeomorphic to the standard sphere. Five years ago, R. Bryant constructed a family of Finsler metrics on S^n of constant curvature $\mathbf{K} = 1$ (see [Br1][Br2][Br3]). The Bryant metrics are non-reversible and projectively flat (geodesics are great circles).

How about the case when $\mathbf{K} = 0$? We know that any Minkowski space has zero curvature $\mathbf{K} = 0$. H. Akbar-Zadeh [AZ] also shows that every Finsler metric on a compact manifold with $\mathbf{K} = 0$ must be trivial, i.e., locally Minkowskian. In [Sh2], we conjecture that there exist non-trivial positively complete, projectively flat Finsler metrics of constant curvature $\mathbf{K} = 0$. In this paper, we will prove the existence of projectively flat Finsler metrics of curvature $\mathbf{K} = 0$ by constructing a projectively flat and R-flat spray using the Funk metrics.

What are sprays? A spray on a manifold M is a global vector field $\mathbf{G}$ on TM which is expressed in a standard local coordinate system (x^i, y^i) by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

where $G^i(x,y)$ are local C^∞ functions on $TM \setminus \{0\}$ satisfying $G^i(x,\lambda y) = \lambda G^i(x,y)$, $\lambda > 0$. Every Finsler metric induces a spray (see (11)). The notion of Riemann curvature is defined for sprays [Bw][Dg][Ko][Sh1]. A spray is said to be *R-flat* if its Riemann curvature vanishes. A Finsler metric is of constant curvature $\mathbf{K} = 0$ if and only if its spray is R-flat. According to [GrMu], every isotropic spray is locally induced by a Finsler metric. Thus as long as we find a R-flat spray, we obtain a Finsler metric with $\mathbf{K} = 0$.

Theorem 1.1 *Let Ω be a strongly convex domain in R^n and F the Funk metric on Ω. Then the following spray is R-flat.*

$$\tilde{\mathbf{G}} := y^i \frac{\partial}{\partial x^i} - 2Fy^i \frac{\partial}{\partial y^i}. \tag{3}$$

From (3), we see that the geodesics of $\tilde{\mathbf{G}}$ are straight lines in Ω. Thus if a Finsler metric $\tilde{F}$ induces $\tilde{\mathbf{G}}$ on an open subset of Ω, then it is pointwise projectively flat with $\mathbf{K} = 0$.

Theorem 1.2 *Let $F = F(x,y)$ be the Funk metric on a strongly convex domain $\Omega \subset \mathrm{R}^n$. For an arbitrary point $a = (a^i) \in \Omega$, define a function $\tilde{F} = \tilde{F}(x,y)$ on $T\Omega = \Omega \times \mathrm{R}^n$ by*

$$\tilde{F} := F(x,y) + F_{x^i}(x,y)(x^i - a^i). \tag{4}$$

$\tilde{F}$ is a pointwise projectively flat Finsler metric with $\mathbf{K} = 0$.

If we take the Funk metric on $\mathbb{B}^n$ and $a = 0$, then the resulting Finsler metric $\tilde{F} = \tilde{F}(x,y)$ on $\mathbb{B}^n$ is given by

$$\tilde{F} = \frac{\left(\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)} + \langle x,y\rangle\right)^2}{(1-|x|^2)^2\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}, \tag{5}$$

where $y \in T_x\mathbb{B}^n = \mathrm{R}^n$. By Theorem 4.1, $\tilde{F}$ is a pointwise projectively flat Finsler metric on $\mathbb{B}^n$ with $\mathbf{K} = 0$.

Finsler metrics given in (4) are not all Finsler metrics that induce $\tilde{\mathbf{G}}$. We can express all analytic Finsler metrics that induces $\tilde{\mathbf{G}}$ in a power series. See Theorem 4.1 below.

2 Preliminaries

A Minkowski norm φ on a vector space V is a nonnegative function with the following properties

(a) φ is positively homogeneous of degree one, i.e.,

$$\varphi(\lambda \mathbf{y}) = \lambda \varphi(\mathbf{y}), \qquad \lambda > 0, \ \mathbf{y} \in \mathrm{V}. \tag{6}$$

(b) φ is C^∞ on $\mathrm{V} \setminus \{0\}$ and for any $\mathbf{y} \in \mathrm{V} \setminus \{0\}$,

$$g_{\mathbf{y}}(\mathbf{u}, \mathbf{v}) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[\varphi^2(\mathbf{y} + s\mathbf{u} + t\mathbf{v})\Big]_{|s=t=0}, \qquad \mathbf{u}, \mathbf{v} \in \mathrm{V}, \tag{7}$$

is a positive definite symmetric bilinear form.

A domain Ω in a vector space V is said to be *strongly convex* if there is a point $x_o \in \Omega$ and a Minkowski norm φ on V such that $\partial\Omega - \{x_o\} = \varphi^{-1}(1)$.

Let M be an n-manifold. A family of Minkowski norms $F = \{F_p\}_{p \in M}$ in tangent spaces T_pM is called a *Finsler metric* on M if it is C^∞ on $TM \setminus \{0\}$. Throughout this paper, Finsler metrics are always positive definite, unless otherwise stated.

Funk metrics: Let Ω be a strongly convex domain in R^n. By definition, there is a Minkowski norm φ on R^n and a point $x_o \in \Omega$ such that $\partial\Omega - \{x_o\} = \varphi^{-1}(1)$. Let F be the Funk metric on Ω. For any $y \in T_x\Omega = \mathrm{R}^n$, $F = F(x, y)$ is determined by

$$\varphi\Big(x - x_o + \frac{y}{F}\Big) = 1. \tag{8}$$

Differentiating (8) yields a system of PDEs,

$$F_{x^i} = F F_{y^i}, \qquad i = 1, \cdots, n. \tag{9}$$

Equation (9) is proved in [Ok]. The Funk metric F on the unit ball $\mathbb{B}^n \subset \mathrm{R}^n$ is given by

$$F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle}{1 - |x|^2}. \tag{10}$$

Every Finsler metric F on M induces a spray $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$ by

$$G^i(x, y) := \frac{1}{4} g^{il}(x, y)\Big\{2\frac{\partial g_{jl}}{\partial x^k}(x, y) - \frac{\partial g_{jk}}{\partial x^l}(x, y)\Big\} y^j y^k, \tag{11}$$

where $g_{ij}(x,y) := \frac{1}{2}[F^2]_{y^iy^j}(x,y)$.

For a vector $\mathbf{y} = y^i\frac{\partial}{\partial x^i}|_p \in T_pM$, set $\mathbf{R_y}(\mathbf{u}) := R^i{}_k u^k \frac{\partial}{\partial x^i}|_p$, where $\mathbf{u} = u^i\frac{\partial}{\partial x^i}|_p$ and $R^i{}_k = R^i{}_k(x,y)$ are given by

$$R^i{}_k := 2\frac{\partial G^i}{\partial x^k} - y^j\frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j\frac{\partial^2 G^i}{\partial y^j\partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{12}$$

Clearly,

$$\mathbf{R_y}(\mathbf{y}) = 0. \tag{13}$$

Assume that $\mathbf{G}$ is induced by a Finsler metric F, then $\mathbf{R_y}$ is self-adjoint with respect to $g_{\mathbf{y}}$, i.e.,

$$g_{\mathbf{y}}(\mathbf{R_y}(\mathbf{u}),\ \mathbf{v}) = g_{\mathbf{y}}(\mathbf{u},\ \mathbf{R_y}(\mathbf{v})). \tag{14}$$

For a tangent plane $P \subset T_pM$ and a vector $\mathbf{y} \in P \setminus \{0\}$, the *flag curvature* $\mathbf{K}(P,\mathbf{y})$ is defined by

$$\mathbf{K}(P,\mathbf{y}) := \frac{g_{\mathbf{y}}(\mathbf{R_y}(\mathbf{u}),\mathbf{u})}{g_{\mathbf{y}}(\mathbf{y},\mathbf{y})g_{\mathbf{y}}(\mathbf{u},\mathbf{u}) - g_{\mathbf{y}}(\mathbf{y},\mathbf{u})g_{\mathbf{y}}(\mathbf{y},\mathbf{u})},$$

where $\mathbf{u} \in P$ such that $P = \mathrm{span}\{\mathbf{y},\mathbf{u}\}$. By (13) and (14), we see that $\mathbf{K}(P,\mathbf{y})$ is independent of the choice of $\mathbf{u} \in P$. Clearly, the flag curvature is a constant, $\mathbf{K} = \lambda$ if and only if

$$\mathbf{R_y}(\mathbf{u}) = \lambda\Big\{ g_{\mathbf{y}}(\mathbf{y},\mathbf{y})\,\mathbf{u} - g_{\mathbf{y}}(\mathbf{y},\mathbf{u})\,\mathbf{y}\Big\}, \qquad \mathbf{y},\mathbf{u} \in T_pM.$$

In particular, $\mathbf{K} = 0$ if and only if $\mathbf{R} = 0$.

Proof of Theorem 1.1: Let $F = F(x,y)$ be the Funk metric on a strongly convex domain $\Omega \subset \mathrm{R}^n$ and

$$\tilde{\mathbf{G}} := y^i\frac{\partial}{\partial x^i} - 2Fy^i\frac{\partial}{\partial y^i}. \tag{15}$$

Let

$$G^i(x,y) := Fy^i.$$

Using (9), we obtain

$$\begin{aligned}
\frac{\partial G^i}{\partial x^k} &= FF_{y^k}y^i \\
\frac{\partial G^i}{\partial y^j} &= F_{y^j}y^i + F^i_j \\
y^j\frac{\partial^2 G^i}{\partial x^j\partial y^k} &= FF_{y^k}y^i + F^2\delta^i_k \\
G^j\frac{\partial^2 G^i}{\partial y^j\partial y^k} &= FF_{y^k}y^i + F^2\delta^i_k.
\end{aligned}$$

Plugging them into (12) yields

$$R^i{}_k = 0.$$

Thus $\tilde{\mathbf{G}}$ is R-flat. This proves Theorem 1.1. Q.E.D.

According to [GrMu], $\tilde{\mathbf{G}}$ can be induced by a Finsler metric. In the following sections, we are going to find the solutions to this inverse problem in a direct way.

3 Solving the inverse problem

We first convert the above inverse problem to solving a simple system of PDEs. First we need the following

Lemma 3.1 (Rapcsák [R]) *Let $\tilde{F} = \tilde{F}(x, y)$ be a Finsler metric on an open subset $\mathcal{U} \subset \mathrm{R}^n$. $\tilde{F}$ is pointwise projectively flat (i.e., geodesics are straight lines) if and only if $\tilde{F}$ satisfies*

$$\tilde{F}_{x^k y^l} y^k = \tilde{F}_{x^l}, \qquad l = 1, \cdots, n. \tag{16}$$

In this case, the spray coefficients G^i are in the form $G^i = Py^i$, where

$$P = \frac{\tilde{F}_{x^k} y^k}{2\tilde{F}}. \tag{17}$$

See [Sh1] for details.

Now we are going to prove our key lemma.

Lemma 3.2 *Let F be the Funk metric on a strongly convex domain Ω in* R^n *and*

$$\tilde{\mathbf{G}} = y^i \frac{\partial}{\partial x^i} - 2Fy^i \frac{\partial}{\partial y^i}.$$

Then a Finsler metric $\tilde{F}$ on Ω induces $\tilde{\mathbf{G}}$ if and only if $\tilde{F}$ satisfies

$$\tilde{F}_{x^k} = (F\tilde{F})_{y^k}, \qquad k = 1, \cdots, n. \tag{18}$$

Proof: Suppose that $\tilde{F}$ induces $\tilde{\mathbf{G}}$, that is, the geodesic coefficients of $\tilde{F}$ are given by $G^i = Fy^i$. Since $\tilde{\mathbf{G}}$ is pointwise projectively flat, so is $\tilde{F}$. By the Rapcsák lemma, $\tilde{F}$ satisfies (16) and the geodesic coefficients $G^i = Py^i$ of $\tilde{F}$ is given by (17). Thus

$$2F\tilde{F} = \tilde{F}_{x^k} y^k. \tag{19}$$

Differentiating (19) with respect to y^l and using (16), we obtain

$$2(F\tilde{F})_{y^l} = \tilde{F}_{x^l} + \tilde{F}_{x^k y^l} y^k = 2\tilde{F}_{x^l}.$$

That is, $\tilde{F}$ satisfies (18).

Conversely, assume that a Finsler metric $\tilde{F} = \tilde{F}(x, y)$ on Ω satisfies (18). Differentiating (18) with respect to y^l and then contracting it with y^k yield

$$\tilde{F}_{x^k y^l} y^k = (F\tilde{F})_{y^k y^l} y^k = (F\tilde{F})_{y^l} = \tilde{F}_{x^l}.$$

Thus $\tilde{F}$ satisfies (16) and $\tilde{F}$ is pointwise projectively flat with $G^i = Py^i$ given by (17). Contracting (18) with y^l yields

$$\tilde{F}_{x^l} y^l = (F\tilde{F})_{y^l} y^l = 2F\tilde{F}.$$

Thus

$$P = \frac{\tilde{F}_{x^l} y^l}{2\tilde{F}} = F.$$

Namely, the spray of $\tilde{F}$ is just $\tilde{\mathbf{G}}$. Q.E.D.

Proof of Theorem 1.2: Let $F = F(x, y)$ denote the Funk metric on a strongly convex domain $\Omega \subset \mathrm{R}^n$ and $a \in \Omega^n$. Let

$$\tilde{F} = F(x, y) + F_{x^i}(x, y)(x^i - a^i) = F(x, y) + F(x, y)F_{y^i}(x, y)(x^i - a^i).$$

To show that $\tilde{F}$ induces $\tilde{\mathbf{G}}$, it suffices to verify that $\tilde{F}$ satisfies (18).

By (9), we obtain

$$\begin{aligned} \tilde{F}_{x^k} &= F_{x^k} + F_{x^k} F_{y^i}(x^i - a^i) + F F_{x^k y^i}(x^i - a^i) + F F_{y^k} \\ &= F F_{y^k} + F F_{y^k} F_{y^i}(x^i - a^i) + F(F F_{y^k})_{y^i}(x^i - a^i) + F F_{y^k} \\ &= 2F F_{y^k} + 2F_{y^k} F_{y^i}(x^i - a^i) + F^2 F_{y^k y^i}(x^i - a^i) \end{aligned}$$

and

$$\begin{aligned} (F\tilde{F})_{y^k} &= (F^2 + F^2 F_{y^i}(x^i - a^i))_{y^k} \\ &= 2F F_{y^k} + 2F F_{y^k} F_{y^i}(x^i - a^i) + F^2 F_{y^k y^i}(x^i - a^i). \end{aligned}$$

Thus $\tilde{F}$ satisfies that $\tilde{F}_{x^k} = (F\tilde{F})_{y^k}$. This proves Theorem 1.2. Q.E.D.

Taking the Funk metric F on the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ in (10) and $a \in \mathrm{R}^n$ with $|a| < 1$, we obtain the following Finsler metric on $\mathbb{B}^n$.

$$\begin{aligned} \tilde{F} &= \frac{F^2}{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}} - \frac{\langle a, y\rangle F + \langle a, x\rangle F^2}{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}} \\ &= \frac{(1 - \langle a, x\rangle)F^2 - \langle a, y\rangle F}{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}. \end{aligned} \tag{20}$$

By Theorem 1.2, we know that $\tilde{F}$ is pointwise projectively flat with $\mathbf{K} = 0$.

4 Analytic Finsler Metrics with $\mathbf{K} = 0$

The system (18) is crucial in determining analytic Finsler metrics that induce $\tilde{\mathbf{G}}$.

A Finsler metric $\tilde{F} = \tilde{F}(x, y)$ on an open subset $\Omega \subset \mathbb{R}^n$ is said to be analytic at $x_o \in \Omega$ if it can be expressed as a Taylor series around x_o as follows,

$$\tilde{F} = \sum_{m=0}^{\infty} \sum_{i_1 \cdots i_m = 1}^{n} a_{i_1 \cdots i_m}(y)(x^{i_1} - x_o^{i_1}) \cdots (x^{i_m} - x_o^{i_m}),$$

where $a_{i_1 \cdots i_m}(y)$ are C^∞ functions on $\mathrm{R}^n \setminus \{0\}$ satisfying

$$a_{i_1 \cdots i_m}(\lambda y) = \lambda a_{i_1 \cdots i_m}(y), \qquad \lambda > 0.$$

Thus $a_0(y) = \tilde{F}(x_o, y)$ is a Minkowski norm on R^n.

Theorem 4.1 *Let $\varphi = \varphi(y)$ be a Minkowski norm and $\Omega = \{y \in \mathrm{R}^n \mid \varphi(y) < 1\}$. Let $\tilde{\mathbf{G}}$ denote the R-flat spray on Ω defined in (3). If $\tilde{F} = \tilde{F}(x, y)$ is a Finsler metric on a neighborhood of the origin $0 \in \Omega$ that induces $\tilde{\mathbf{G}}$, then $\tilde{F}$ is given by*

$$\tilde{F} := \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m}{dt^m} \Big[\varphi^m(y + tx)\psi(y + tx)\Big]|_{t=0}, \tag{21}$$

where $\psi(y) := \tilde{F}(0, y)$. Conversely, for any Minkowski norm $\psi = \psi(y)$ on R^n, the function $\tilde{F}$ defined in (21) induces $\tilde{\mathbf{G}}$, hence it is a pointwise projectively flat Finsler metric with $\mathbf{K} = 0$.

Let $\Omega \subset \mathrm{R}^n$ be a strongly convex defined by a Minkowski norm $\varphi = \varphi(y)$ on R^n ($\partial\Omega := \varphi^{-1}(1)$). Let $F = F(x, y)$ denote the Funk metric on Ω. From the definition of F,

$$F(0, y) = \varphi(y), \qquad y \in \mathrm{R}^n.$$

Suppose that there is a Finsler metric $\tilde{F} = \tilde{F}(x, y)$ on Ω satisfying (18),

$$\tilde{F}_{x^k} = (F\tilde{F})_{y^k}. \tag{22}$$

By (9) and (22), we obtain

$$\begin{aligned} \tilde{F}_{x^i x^j} &= (F\tilde{F})_{y^i x^j} \\ &= (F_{x^j}\tilde{F} + F\tilde{F}_{x^j})_{y^i} \\ &= (FF_{y^j}\tilde{F} + F(F\tilde{F})_{y^j})_{y^i} \\ &= (F^2\tilde{F})_{y^i y^j} \end{aligned}$$

By induction, we obtain

$$\tilde{F}_{x^{i_1} \cdots x^{i_m}} = (F^m \tilde{F})_{y^{i_1} \cdots y^{i_m}}. \tag{23}$$

Let

$$\psi(y) := \tilde{F}(0,y), \qquad y \in \mathrm{R}^n.$$

This gives

$$\tilde{F}_{x^{i_1}\cdots x^{i_m}}(0,y) = \Big[\varphi^m \psi\Big]_{y^{i_1}\cdots y^{i_m}}(y).$$

Thus, if $\tilde{F} = \tilde{F}(x,y)$ is analytic in x at $x = 0$ for a fixed $y \neq 0$, then it must be given by

$$\tilde{F} = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{i_1\cdots i_m} \Big[\varphi^m \psi\Big]_{y^{i_1}\cdots y^{i_m}}(y) x^{i_1}\cdots x^{i_m}. \tag{24}$$

We can also express the above power series in the following form

$$\tilde{F} := \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m}{dt^m}\Big[\varphi^m(y+tx)\psi(y+tx)\Big]|_{t=0}. \tag{25}$$

Let $\tilde{F} = \tilde{F}(x,y)$ be defined by the power series in (25). Assume that $\tilde{F}$ is convergent for x in a neighborhood of $0 \in \Omega$. We claim that $\tilde{F}$ induces the spray $\tilde{\mathbf{G}}$ in (3). By Lemma 3.2, it suffices to verify that the function $\tilde{F}$ satisfies (22). Differentiating (25) with respect to x^k,

$$\begin{aligned}
\tilde{F}_{x^k} &= \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{i_1\cdots i_m=1}^{n} \Big[\varphi^m \psi\Big]_{y^{i_1}\cdots y^{i_m}}(y) \sum_{j=1}^{m} x^{i_1}\cdots \delta^{i_j}_k \cdots x^{i_m} \\
&= \sum_{m=1}^{\infty} \frac{m}{m!} \sum_{i_1\cdots i_{m-1}=1}^{n} \Big[\varphi^m \psi\Big]_{y^k y^{i_1}\cdots y^{i_{m-1}}}(y) x^{i_1}\cdots x^{i_{m-1}} \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{i_1\cdots i_m=1}^{n} \Big[\varphi^{m+1} \psi\Big]_{y^k y^{i_1}\cdots y^{i_m}}(y) x^{i_1}\cdots x^{i_m}.
\end{aligned} \tag{26}$$

On the other hand, it follows from (9) that

$$F_{x^{i_1}\cdots x^{i_m}}(x,y) = \frac{1}{m+1}\Big[F^{m+1}\Big]_{y^{i_1}\cdots y^{i_m}}(x,y).$$

This gives

$$F_{x^{i_1}\cdots x^{i_m}}(0,y) = \frac{1}{m+1}\Big[\varphi^{m+1}\Big]_{y^{i_1}\cdots y^{i_m}}(0,y).$$

Thus the Funk metric $F = F(x,y)$ can be expressed by

$$F = \sum_{m=0}^{\infty} \frac{1}{(m+1)!} \sum_{i_1\cdots i_m=1}^{n} \Big[\varphi^{m+1}\Big]_{y^{i_1}\cdots y^{i_m}}(y) x^{i_1}\cdots x^{i_m}.$$

The power series multiplication gives

$$F\tilde{F} = \sum_{m=0}^{\infty} \frac{1}{(m+1)!} \sum_{i_1\cdots i_m=1}^{n} \Big[\varphi^{m+1}\psi\Big]_{y^{i_1}\cdots y^{i_m}}(y) x^{i_1}\cdots x^{i_m}.$$

Differentiating $F\tilde{F}$ with respect to y^k, we obtain

$$(F\tilde{F})_{y^k} = \sum_{m=0}^{\infty} \frac{1}{(m+1)!} \sum_{i_1 \cdots i_m = 1}^{n} \Big[\varphi^{m+1}\psi\Big]_{y^k y^{i_1} \cdots y^{i_m}} (y) x^{i_1} \cdots x^{i_m}. \tag{27}$$

From (26) and (27), we conclude that $\tilde{F}$ indeed satisfies (22). This proves Theorem 4.1. Q.E.D.

There are infinitely many choices for φ and ψ. Thus we obtain infinitely many projectively flat Finsler metrics with $\mathbf{K} = 0$. Taking

$$\varphi(y) := |y| =: \psi(y), \qquad y \in \mathrm{R}^n,$$

we obtain

$$\tilde{F} = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m}{dt^m} \Big[|y + tx|^{m+1}\Big]_{|t=0}. \tag{28}$$

This is just the Finsler metric given in (5).

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