On a Projective Class of Finsler Metrics*

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1 Introduction

The Douglas (projective) curvature $D_j{}^i{}_{kl}$ and the Weyl (projective) curvature $W^i{}_k$ are two most important quantities in projective Finsler geometry. Finsler metrics with $D_j{}^i{}_{kl} = 0$ are called *Douglas metrics* and Finsler metrics with $W^i{}_k = 0$ are called *Weyl metrics*. It is well-known that a Finsler metric is a Weyl metric if and only if it is of scalar flag curvature, namely, the flag curvature $\mathbf{K}(P, y) = K(x, y)$ is independent of the section P containing y. Thus Weyl metrics are called *metrics of scalar (flag) curvature*, and being of scalar flag curvature is a projective property.

Equations $D_j{}^i{}_{kl} = 0$ and $W^i{}_k = 0$ are projectively invariant, namely, if a Finsler metric F satisfies one of the equations, then any Finsler metric projectively equivalent to F must satisfy the same equation. There is another projective invariant equation in Finsler geometry, that is, for some tensor T_{jkl},

$$D_j{}^i{}_{kl;m} y^m = T_{jkl} y^i, \tag{1}$$

where $D_j{}^i{}_{kl;m}$ denotes the horizontal covariant derivatives of $D_j{}^i{}_{kl}$ with respect to the Berwald connection of F. Equation (1) is equivalent to that for any linearly parallel veector fields $U = U(t), V = V(t)$ and $W = W(t)$ along a geodesic $c(t)$, there is a function $T = T(t)$ such that

$$\frac{d}{dt}\Big[D_{\dot{c}}(U, V, W)\Big] = T\dot{c}.$$

The geometric meaning of the above identity is that the rate of change of the Douglas curvature along a geodesic is tangent to the geodesic.

For a manifold M, let $\mathcal{GDW}(M)$ denote the class of all Finsler metrics satisfying (1) for some tensor T_{jkl} (T_{jkl} not fixed). In [2], Bacso-Papp show that $\mathcal{GDW}(M)$ is closed under projective changes. More precisely, if F is projectively equivalent to a Finsler metric in $\mathcal{GDW}(M)$, then $F \in \mathcal{GDW}(M)$.

A natural question is: how large is $\mathcal{GDW}(M)$ and what kind of interesting metrics does it have? It is obvious that all Douglas metrics belong to this class.

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On the other hand, all Weyl metrics (metrics of scalar flag curvature) also belong to this class. The later is really a surprising result, due to Sakaguchi [4]. In this sense, we shall call Finsler metrics in $\mathcal{GDW}(M)$ *GDW-metrics* (generalized Douglas-Weyl metrics).

In this paper, we are going to study and characterize GDW-metrics of Randers type on a manifold M. A Randers metric on a manifold M is a Finsler metric in the following form

$$F = \alpha + \beta,$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. One of the reasons why we would like to study Randers metrics for the above problem is because that Randers metrics are "computable".

Let

$$s_{ij} := \frac{1}{2}\Big(\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i}\Big).$$

Clearly, β is closed if and only if $s_{ij} = 0$. It is known that $F = \alpha+\beta$ is a Douglas metric if and only if β is closed (see [1]). On the other hand, Shen-Yildirim [6] recently find a sufficient and necessary condition for $F = \alpha + \beta$ to be of scalar flag curvature, that is,

$$\begin{aligned} \bar{R}^i{}_k &= \Big(\lambda - \frac{1}{n-1} t^m{}_m\Big)\Big\{\alpha^2 \delta^i_k - a_{jk} y^i y^j\Big\} \\ &\quad + \alpha^2 t^i{}_k + t_{00}\delta^i_k - t_{k0} y^i - t^i{}_0 y_k - 3 s^i{}_0 s_{k0}, \end{aligned} \tag{2}$$

$$s_{ij|k} = \frac{1}{n-1}\Big\{a_{ik} s^m{}_{j|m} - a_{jk} s^m{}_{i|m}\Big\}, \tag{3}$$

where $t_{ij} := s_{im} s^m_j$, $\bar{R}^i{}_k$ denotes the Riemann curvature tensor of α and $\lambda = \lambda(x)$ is a scalar function on M. Here $s_{ij|k}$ denote the coefficients of the covariant derivative of s_{ij} with respect to α. We use a_{ij} and a^{ij} to lower or lift the indices of a tensor.

Theorem 1.1 *Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M. F is a GDW metric if and only if (3) holds.*

Thus any Randers metrics of scalar curvature belongs to $\mathcal{GDW}(M)$. This verifies Sakaguchi's theorem for Randers metrics. The following Randers metric actually satisfies both (2) and (3).

Example 1.1 ([5]) Let $a \in \mathrm{R}^n$ be a constant vector. Define $F = \alpha + \beta$ on an open ball $\mathrm{B}^n(1/\sqrt{|a|})$ in R^n by

$$\begin{aligned} F : &= \frac{\sqrt{(1 - |a|^2|x|^4)|y|^2 + (|x|^2\langle a, y\rangle - 2\langle a, x\rangle\langle x, y\rangle)^2}}{1 - |a|^2|x|^4} \\ &\quad - \frac{|x|^2\langle a, y\rangle - 2\langle a, x\rangle\langle x, y\rangle}{1 - |a|^2|x|^4}. \end{aligned}$$

Then F is of scalar fuction. Thus it satisfies (2) and (3). See more examples in [3].

So far, we have not found a Randers metric satisfying (3), but not (2). We conjecture that such examples exist.

2 Randers Metrics

Let $F=\alpha+\beta$ be a Randers metric on a manifold M, where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ and $\beta=b_i(x)y^i$. Let $b_{i|j}$ denote the coefficients of the covariant derivative of β wirth respect to α. Let

$$r_{ij}:=\frac{1}{2}(b_{i|j}+b_{j|i}),\qquad s_{ij}:=\frac{1}{2}(b_{i|j}-b_{j|i}),$$

where $b_{i|j}$ denote the coefficients of the covariant derivative of β with respect to α. The spray coefficients of F are given by

$$G^i=\bar{G}^i+\frac{r_{00}-2s_0\alpha}{2F}y^i+\alpha s^i{}_0.$$

Let

$$\Pi^i:=G^i-\frac{1}{n+1}\frac{\partial G^m}{\partial y^m},\qquad \bar{\Pi}^i:=\bar{G}^i-\frac{1}{n+1}\frac{\partial\bar{G}^m}{\partial y^m}.$$

Observe that

$$\frac{\partial(\alpha s^m{}_0)}{\partial y^m}=\frac{y_m}{\alpha}s^m{}_0+\alpha s^m{}_m=0.$$

Thus we have

$$\Pi^i=\bar{\Pi}^i+\alpha s^i{}_0. \tag{4}$$

By definition, the Douglas curvature is given by

$$D_j{}^i{}_{kl}:=\frac{\partial^3\Pi^i}{\partial y^j\partial y^k\partial y^l}.$$

Since $\bar{\Pi}^i$ are always quadratic in y, we get

$$D_j{}^i{}_{kl}=\frac{\partial^3}{\partial y^j\partial y^k\partial y^l}\Big(\alpha s^i{}_0\Big)=\alpha_{jkl}s^i{}_0+\alpha_{jk}s^i{}_l+\alpha_{jl}s^i{}_k+\alpha_{kl}s^i{}_j, \tag{5}$$

where

$$\begin{aligned}\alpha_j &= \alpha^{-1}y_j\\ \alpha_{jk} &= \alpha^{-3}A_{jk}\\ \alpha_{jkl} &= -\alpha^{-5}\{A_{jk}y_l+A_{jl}y_k+A_{kl}y_j\},\end{aligned}$$

where $A_{ij}:=\alpha^2a_{ij}-y_iy_j$.

It is easy to show that F is a Douglas metric if and only if $s^i{}_0=0$. This fact is due to Bacso-Matsumoto [1].

Let $\tilde{G}=y^i\frac{\partial}{\partial x^i}-2\tilde{G}^i\frac{\partial}{\partial y^i}$ where

$$\tilde{G}^i:=\bar{G}^i+\alpha s^i{}_0. \tag{6}$$

Let "$\|$" and "$|$" denote the covariant differentiations with respect to $\tilde{G}$ and $\bar{G}$ respectively. Then

$$\begin{aligned} D_j{}^i{}_{kl\|m}y^m &= D_j{}^i{}_{kl|m}y^m - 2\alpha\frac{\partial}{\partial y^p}(D_j{}^i{}_{kl})s^p{}_0 \\ &+\alpha^{-1}[\alpha^2 s^i{}_p + y_p s^i{}_0]D_j{}^p{}_{kl} - \alpha^{-1}[\alpha^2 s^p{}_j + s^p{}_0 y_j]D_p{}^i{}_{kl} \\ &-\alpha^{-1}[\alpha^2 s^p{}_k + s^p{}_0 y_k]D_j{}^i{}_{pl} - \alpha^{-1}[\alpha^2 s^p{}_l + s^p{}_0 y_l]D_j{}^i{}_{kp}. \qquad (7) \end{aligned}$$

Since "$|$" is a differentiation with respect to α, $a_{ij|m} = 0$. Thus

$$\alpha_{|m} = 0, \quad \alpha_{j|m} = 0, \quad \alpha_{jk|m} = 0, \quad \alpha_{jkl|m} = 0.$$

We obtain

$$\begin{aligned} D_j{}^i{}_{kl|m}y^m &= \alpha_{jkl}s^i{}_{0|0} + \alpha_{jk}s^i{}_{l|0} + \alpha_{jl}s^i{}_{k|0} + \alpha_{kl}s^i{}_{j|0} \\ &= -\alpha^{-5}\Big\{A_{jk}y_l + A_{jl}y_k + A_{kl}y_j\Big\}s^i{}_{0|0} \\ &+\alpha^{-3}\Big\{A_{jk}s^i{}_{l|0} + A_{jl}s^i{}_{k|0} + A_{kl}s^i{}_{j|0}\Big\} \end{aligned}$$

Differentuating (5) yields

$$\frac{\partial}{\partial y^p}(D_j{}^i{}_{kl}) = \alpha_{jklp}s^i{}_0 + \alpha_{jkl}s^i{}_p + \alpha_{jkp}s^i{}_l + \alpha_{jlp}s^i{}_k + \alpha_{klp}s^i{}_j,$$

where

$$\begin{aligned} \alpha_{jklp} &= 3\alpha^{-5}y_p\{a_{jk}y_l + a_{jl}y_k + a_{kl}y_j\} \\ &-\alpha^{-3}\Big\{a_{jk}a_{lp} + a_{jl}a_{kp} + a_{kl}a_{jp}\Big\} \\ &+3\alpha^{-5}\Big\{y_k y_l a_{jp} + y_j y_l a_{kp} + y_j y_k a_{lp}\Big\} \\ &-15\alpha^{-7}y_j y_k y_l y_p, \end{aligned}$$

We get

$$\begin{aligned} \alpha_{jklp}s^p{}_0 &= -\alpha^{-5}\Big\{A_{jk}s_{l0} + A_{jl}s_{k0} + A_{kl}s_{j0}\Big\} \\ &\quad +2\alpha^{-5}\Big\{s_{j0}y_k y_l + s_{k0}y_i y_l + s_{l0}y_j y_k\Big\} \\ \alpha_{jkp}s^p{}_0 &= -\alpha^{-3}\Big\{y_k s_{j0} + y_j s_{k0}\Big\} \\ \alpha_{jlp}s^p{}_0 &= -\alpha^{-3}\Big\{y_l s_{j0} + y_j s_{l0}\Big\} \\ \alpha_{klp}s^p{}_0 &= -\alpha^{-3}\Big\{y_k s_{l0} + y_l s_{k0}\Big\}. \end{aligned}$$

Then we obtain

$$-2\alpha\frac{\partial}{\partial y^p}(D_j{}^i{}_{kl})s^p{}_0 = 2\alpha^{-4}\Big\{A_{jk}s_{l0} + A_{jl}s_{k0} + A_{kl}s_{j0}\Big\}s^i{}_0$$

$$
\begin{aligned}
&+2\alpha^{-4}\{A_{jk}y_l+A_{jl}y_k+A_{kl}y_j\}t^i{}_0\\
&-4\alpha^{-4}\Big\{y_ky_ls_{j0}+y_jy_ls_{k0}+y_jy_ks_{l0}\Big\}s^i{}_0\\
&+2\alpha^{-2}\Big\{y_ks_{j0}+y_js_{k0}\Big\}s^i{}_l\\
&+2\alpha^{-2}\Big\{y_ls_{j0}+y_js_{l0}\Big\}s^i{}_k\\
&+2\alpha^{-2}\Big\{y_ks_{l0}+y_ls_{k0}\Big\}s^i{}_j,
\end{aligned}
$$

By (5), we can also easily get

$$
\begin{aligned}
\alpha^{-1}[\alpha^2 s^i{}_p+y_ps^i{}_0]D_j{}^p{}_{kl} \;=\; & -\alpha^{-2}\{A_{jk}y_l+A_{jl}y_k+A_{kl}y_j\}t^i{}_0\\
&+\alpha^{-2}\Big\{A_{jk}t^i{}_l+A_{jl}t^i{}_k+A_{kl}t^i{}_j\Big\}\\
&-\alpha^{-4}\Big\{A_{jk}s_{l0}+A_{jl}s_{k0}s^i{}_0+A_{kl}s_{j0}\Big\}s^i{}_0.
\end{aligned}
$$

$$
\begin{aligned}
-\alpha^{-1}[\alpha^2 s^p{}_j+s^p{}_0y_j]D^i_{pkl} \;=\; & \alpha^{-2}\Big\{y_ks_{lj}+y_ls_{kj}\Big\}+2\alpha^{-4}y_ky_ls_{j0}s^i{}_0\\
&-\alpha^{-2}(\alpha^2s_{kj}-s_{0j}y_k)s^i{}_l\\
&-\alpha^{-2}(\alpha^2s_{lj}-s_{0j}y_l)s^i{}_k\\
&+\alpha^{-4}\Big\{y_jy_ks_{l0}+y_jy_ls_{k0}\Big\}s^i{}_0\\
&-\alpha^{-2}y_js_{k0}s^i{}_l-\alpha^{-2}y_js_{l0}s^i{}_k\\
&-\alpha^{-4}y_jA_{kl}t^i{}_0-\alpha^{-4}A_{kl}s_{j0}s^i{}_0-\alpha^{-2}A_{kl}t^i{}_j.
\end{aligned}
$$

$$
\begin{aligned}
-\alpha^{-1}[\alpha^2 s^p{}_k+s^p{}_0y_k]D^i_{plj} \;=\; & \alpha^{-2}\Big\{y_ls_{jk}+y_js_{lk}\Big\}+2\alpha^{-4}y_ly_js_{k0}s^i{}_0\\
&-\alpha^{-2}(\alpha^2s_{lk}-s_{0k}y_l)s^i{}_j\\
&-\alpha^{-2}(\alpha^2s_{jk}-s_{0k}y_j)s^i{}_l\\
&+\alpha^{-4}\Big\{y_ky_ls_{j0}+y_ky_js_{l0}\Big\}s^i{}_0\\
&-\alpha^{-2}y_ks_{l0}s^i{}_j-\alpha^{-2}y_ks_{j0}s^i{}_l\\
&-\alpha^{-4}y_kA_{jl}t^i{}_0-\alpha^{-4}A_{lj}s_{k0}s^i{}_0-\alpha^{-2}A_{jl}t^i{}_k
\end{aligned}
$$

$$
\begin{aligned}
-\alpha^{-1}[\alpha^2 s^p{}_l+s^p{}_0y_l]D^i_{pkj} \;=\; & \alpha^{-2}\Big\{y_ks_{jl}+y_js_{kl}\Big\}+2\alpha^{-4}y_ky_js_{l0}s^i{}_0\\
&-\alpha^{-2}(\alpha^2s_{kl}-s_{0l}y_k)s^i{}_j\\
&-\alpha^{-2}(\alpha^2s_{jl}-s_{0l}y_j)s^i{}_k\\
&+\alpha^{-4}\Big\{y_ly_ks_{j0}+y_ly_js_{k0}\Big\}s^i{}_0\\
&-\alpha^{-2}y_ls_{k0}s^i{}_j-\alpha^{-2}y_ls_{j0}s^i{}_k\\
&-\alpha^{-4}y_lA_{jk}t^i{}_0-\alpha^{-4}A_{kj}s_{l0}s^i{}_0-\alpha^{-2}A_{jk}t^i{}_l.
\end{aligned}
$$

Plugging the above identities into (7), we get

$$D_j{}^i{}_{kl\|m}y^m = \alpha^{-5}\Big\{A_{jk}H^i{}_l + A_{jl}H^i{}_k + A_{kl}H^i{}_j\Big\}, \tag{8}$$

where

$$H^i{}_j := \alpha^2 s^i{}_{j|0} - y_j s^i{}_{0|0}.$$

3 *Proof of Theorem 1.1*

First we are going to prove the following

Theorem 3.1 *Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M. F isa GDW-metric if and only if*

$$\alpha^2 s_{ij|0} = s_{i0|0}y_j - s_{j0|0}y_i. \tag{9}$$

Proof: $F = \alpha + \beta$ be a Randers metric on a manifold. Let G denote the spray of F and $\tilde{G}$ the spray defined in (6). Since $\tilde{G}$ and G are projectively equivalent, the following conditions are equivalent

(i) there is a tensor T_{jkl} such that

$$D^i_{jkl;m}y^m = D_{jkl}y^i,$$

(ii) there is a tensor D_{jkl} such that

$$D^i_{jkl\|m}y^m = D_{jkl}y^i, \tag{10}$$

where $D_j{}^i{}_{kl;m}$ and $D_j{}^i{}_{kl\|m}$ denote the covariant derivatives of $D_j{}^i{}_{kl}$ with respect to the Berwald connections of G and $\tilde{G}$, respectively. This equivalence is essentially proved in [2]. Thus the argument is omitted here.

Assume that F is a GDW-metric. Then (10) holds for some tensor D_{jkl}. By (8), we have

$$D_{jkl}y^i \quad = \quad \alpha^{-5}\Big\{A_{jk}H^i{}_l + A_{jl}H^i{}_k + A_{kl}H^i{}_j\Big\}. \tag{11}$$

Contracting (11) with y_i yields

$$D_{jkl} = -\alpha^{-5}\Big\{A_{jk}s_{l0|0} + A_{jl}s_{k0|0} + A_{kl}s_{j0|0}\Big\}. \tag{12}$$

Plugging (12) into (11), we get

$$A_{jk}\Big\{H^i{}_l + s_{l0|0}y^i\Big\} + A_{jl}\Big\{H^i{}_k + s_{k0|0}y^i\Big\} + A_{kl}\Big\{H^i{}_j + s_{j0|0}y^i\Big\} = 0. \tag{13}$$

Contracting (13) with a^{kl} we obtain

$$H^i{}_j + s_{j0|0}y^i = 0. \tag{14}$$

This is obviously equivalent to (9).

Conversely, if (9) holds, or equivalently, (14) holds, it follows from (8) that

$$D^{\ i}_{jkl\|m}y^m = D_{jkl}y^i,$$

where D_{jkl} are given by (12). Thus F is a GDW-metric. Q.E.D.

To Prove Theorem 1.1, one just needs to prove the equivalence between (3) and (9).

Lemma 3.2 *(3) is equivalent to (9).*

Proof: Supoose that (3) holds. Then

$$s_{ij|k} = \lambda\Big\{a_{ik}s^{m}_{\ j|m} - a_{jk}s^{m}_{\ i|m}\Big\}, \tag{15}$$

where $\lambda = 1/(n-1)$ (in fact, λ can be any scalar function). Contracting it with y^k yields

$$s_{ij|0} = \lambda\Big\{y_i s^{m}_{\ j|m} - y_j s^{m}_{\ i|m}\Big\}. \tag{16}$$

Contracting (16) with y^j yields

$$s_{i0|0} = \lambda\Big\{y_i s^{m}_{\ 0|m} - \alpha^2 s^{m}_{\ i|m}\Big\}. \tag{17}$$

Thus

$$s_{j0|0} = \lambda\Big\{y_j s^{m}_{\ 0|m} - \alpha^2 s^{m}_{\ j|m}\Big\}. \tag{18}$$

By (17)-(18),

$$\begin{aligned} s_{i0|0}y_j - s_{j0|0}y_i &= \lambda\alpha^2\Big\{s^{m}_{\ j|m}y_i - s^{m}_{\ i|m}y_j\Big\} \\ &= \lambda\alpha^2\Big\{a_{ik}s^{m}_{\ jm} - a_{jk}s^{m}_{\ m|i}\Big\}y^k \\ &= \alpha^2 s_{ij|0}. \end{aligned}$$

The last identity follows from (16). Then we obtain (9).

Conversely, assume that (9) holds. First differentiating (9) with respect to y^k, y^l and y^m yields

$$\begin{aligned} &2a_{kl}s_{ij|m} + 2a_{km}s_{ij|l} + 2a_{lm}s_{ij|k} \\ &\quad = s_{ik|l}a_{jm} + s_{ik|m}a_{jl} + s_{il|k}a_{jm} + s_{im|k}a_{jl} + s_{il|m}a_{jk} + s_{im|l}a_{jk} \\ &\quad -s_{jk|l}a_{im} - s_{jk|m}a_{il} - s_{jl|k}a_{im} - s_{jm|k}a_{il} - s_{jl|m}a_{ik} - s_{jm|l}a_{ik}. \end{aligned}$$

Contracting it with a^{lm}, we get

$$ns_{ij|k} = s_{ik|j} - s_{jk|i} + a_{ik}s^{m}_{\ j|m} - a_{jk}s^{m}_{\ im}. \tag{19}$$

It follows from (19) that

$$n s_{ik|j} = s_{ij|k} + s_{jk|i} + a_{ij} s^{m}_{\ km} - a_{jk} s^{m}_{\ im} \quad (20)$$
$$n s_{jk|i} = -s_{ij|k} + s_{ik|j} + a_{ij} s^{m}_{\ km} - a_{ik} s^{m}_{\ jm} \quad (21)$$

Subtracting (21) from (20), we get

$$s_{ik|j} - s_{jk|i} = \frac{2}{n+1} s_{ij|k} + \frac{1}{n+1}\Big\{ a_{ik} s^{m}_{\ jm} - a_{jk} s^{m}_{\ im} \Big\}. \quad (22)$$

Plugging (22) back into (19) yields

$$s_{ij|k} = \frac{1}{n-1}\Big\{ a_{ik} s^{m}_{\ jm} - a_{jk} s^{m}_{\ im} \Big\}.$$

We are done. Q.E.D.

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