A Comparison Theorem on the Ricci Curvature in Projective Geometry*

Xinyue Chen†and Zhongmin Shen

Revised on June 15, 2002

Abstract

We show that if two Riemannian metrics $\tilde{g}$ and g are pointwise projectively equivalent and the Ricci curvatures satisfy $\widetilde{\mathbf{Ric}} \le \mathbf{Ric}$, then the projective equivalence is trivial provided that g is complete. In this case, $\tilde{g}$ is parallel with respect to g and the Riemann curvatures of g and $\tilde{g}$ are equal. The Ricci curvature condition can be weakened when the manifold is compact. This rigidity theorem actually holds for more general geometric structures, such as Finsler metrics and sprays. In this paper, we will also discuss several examples and show that the completeness of g can not be dropped.

1 Introduction

One of the important problems in projective geometry is to study the relationship among geometric structures with common geodesics (as point sets). More general, given a geometric structure $\mathcal{G}$, one would like to characterize all geometric structure $\tilde{\mathcal{G}}$ of same nature such that the geodesics of $\mathcal{G}$ are the geodesics of $\tilde{\mathcal{G}}$ as point sets. Here, a geometric structure, $\mathcal{G}$, under our consideration is a Riemannian metric, a Finsler metric or a spray. In the case when $\mathcal{G}$ is the Euclidean metric on R^n, it is the Hilbert Fourth Problem in the smooth case to study and characterize Finsler metrics on an open subset in R^n such that straight lines are geodesics [1], [2], [5], [8], [15]. Such Finsler metrics are said to be *projective*. Beltrami shows that a Riemannian metric on an open subset in R^n is projective if and only if it has constant sectional curvature. However, there are lots of projective Finsler metrics whose flag curvature are not constant.

Two geometric structures are said to be *projectively equivalent* if $c(t)$ is a geodesic of one geometric structure, then after a suitable reparametrization, $t = t(s)$, the curve $c(s) := c(t(s))$ is a geodesic of another. The projective equivalence is said to be *trivial* if the required reparametrization is always affine, i.e., $t = a\, s + b$, or equivalently, the corresponding sprays are equal. We should

*1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 53C60, 53C20
†supported in part by the National Natural Science Foundation of China (10171117)

point out that if the geometric structures under consideration are Riemannian metrics or Finsler metrics, then they are not necessarily isometric. That is, geodesics do not completely determine metric structures.

The Ricci curvature plays an important role in the projective geometry of Riemannian/Finsler/Spray manifolds. It is proved [9] that for two pointwise projectively equivalent Einstein metrics g and $\tilde{g}$ on an n-dimensional compact manifold M, their Einstein constants have the same sign. In addition, if their Einstein constants are negative and equal, then $g = \tilde{g}$. In this paper, we continue to study projectively equivalent geometric structures. In particular, we will prove the following

Theorem 1.1 *Let (M, g) be a complete Riemannian manifold and $\tilde{g}$ be another Riemannian metric which is pointwise projectively equivalent to g. Suppose that their Ricci curvatures $\mathbf{Ric}$ and $\widetilde{\mathbf{Ric}}$ satisfy*

$$\widetilde{\mathbf{Ric}} \leq \mathbf{Ric}. \tag{1}$$

Then the projective equivalence is trivial. Hence the Riemann curvatures are equal, $\tilde{\mathbf{R}} = \mathbf{R}$. Further, $\tilde{g}$ is parallel with respect to g, and $dV_{\tilde{g}}$ is proportional to dV_g.

We immediately obtain the following corollary.

Corollary 1.2 *Let g be a complete Riemannian metric on an open convex subset $\mathcal{U} \subset \mathrm{R}^n$ with $\mathbf{Ric} \geq 0$. Suppose that the geodesics of g are straight lines, then it is an Euclidean metric on $\mathcal{U}$.*

Theorem 1.1 is a special case of Corollary 4.3 below. The conclusion in Theorem 1.1 (except for the last statement on $\tilde{g}$) also holds for sprays. We will prove a generalized theorem on projectively equivalent sprays in §2 (Theorem 2.3).

If we reverse the inequality (1), then the conclusion in Theorem 1.1 does not hold. See Example 2.4 below. However, if the manifold is compact, the inequality (1) can be reversed. More precisely, we have the following

Theorem 1.3 *Let g and $\tilde{g}$ be pointwise projectively equivalent Riemannian metrics on a compact manifold M. Assume that one of the following conditions is satisfied,*

(a) $\mathrm{tr}_g \widetilde{\mathbf{Ric}} \leq \mathbf{s}_g$,

(b) $\mathrm{tr}_g \widetilde{\mathbf{Ric}} \geq \mathbf{s}_g$,

then the same conclusion as in Theorem 1.1 holds.

Here $\mathrm{tr}_g \widetilde{\mathbf{Ric}}$ denotes the trace of the Ricci curvature $\widetilde{\mathbf{Ric}}$ of $\tilde{g}$ with respect to g and $\mathbf{s}_g := \mathrm{tr}_g \mathbf{Ric}$ denotes the trace of the Ricci curvature $\mathbf{Ric}$ of g with respect to g. The function $\mathbf{s}_g$ is called the scalar curvature of g. See (25) for definition. The proof of Theorem 1.3 will be given at the end of Section 5.

In a recent paper [6], Seontag Kim proves that for pointwise projectively equivalent Riemannian metrics g and $\tilde{g}$ on a manifold, if (a) g is complete, (b) $\int_M \mathbf{s}_g dV_g \geq 0$, (c) $\widetilde{\mathbf{Ric}} \leq 0$, and (d) $\mathrm{Vol}_g(B(p,r)) = o(r^2)$, then the projective equivalence is trivial. Note that if the manifold is compact, (a) and (d) are always satisfied. In Theorem 1.1, we assume a stronger curvature condition and drop the volume growth condition in Kim's theorem.

2 Projectively Equivalent Sprays

In this section we will prove a generalized version of Theorem 1.1 for sprays. Roughly speaking, a spray manifold is manifold with a collection of parametrized curves (called geodesics). See [10] for a detailed discussion on sprays.

Let M be an n-dimensional manifold and $\pi : TM \to M$ denote the natural projection from the tangent bundle TM to the base manifold M. A spray $\mathbf{G}$ on M is a vector field on TM such that in any standard local coordinate system (x^i, y^i) in an open domain $\pi^{-1}(U) \subset TM$,

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(\mathbf{y}) \frac{\partial}{\partial y^i},$$

where $G^i(\mathbf{y})$ are local functions of $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_x \in \pi^{-1}(U)$ with the following properties:

(a) $G^i(\lambda \mathbf{y}) = \lambda^2 G^i(\mathbf{y})$, $\forall \lambda > 0$ and $\mathbf{y} \in \pi^{-1}(U)$;

(b) $G^i(\mathbf{y})$ are C^∞ on $\pi^{-1}(U) \setminus \{0\}$.

Geodesics of $\mathbf{G}$ in M are defined as the projections of integral curves of $\mathbf{G}$ under the projection $\pi : TM \setminus \{0\} \to M$. Geodesics $c(t)$ in M are locally characterized by

$$\ddot{c}^i(t) + 2G^i(\dot{c}(t)) = 0. \tag{2}$$

$\mathbf{G}$ is said to be *complete* (resp. *positively complete*) if any geodesic defined on (a, b) can be extended to a geodesic defined on $(-\infty, \infty)$ (resp. (a, ∞)).

Every Finsler metric $g = g_{ij}(\mathbf{y}) y^i y^j$ induces a spray $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(\mathbf{y}) \frac{\partial}{\partial y^i}$ by

$$G^i(\mathbf{y}) = \frac{1}{4} g^{il}(\mathbf{y}) \Big\{ 2\frac{\partial g_{jl}}{\partial x^k}(\mathbf{y}) - \frac{\partial g_{jk}}{\partial x^l}(\mathbf{y}) \Big\} y^j y^k,$$

where $g_{ij} := \frac{1}{2} g_{y^i y^j}(\mathbf{y})$ and $(g^{ij}) := (g_{ij})^{-1}$. The geodesics of g are just those of $\mathbf{G}$. Sprays are more general geometric structures than Finsler metrics/Riemannian metrics.

For a spray $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(\mathbf{y}) \frac{\partial}{\partial y^i}$ on an n-manifold M, the Riemann curvature is a family of endomorphisms $\mathbf{R_y} : T_x M \to T_x M$, $\mathbf{y} \in T_x M \setminus \{0\}$. Express

$$\mathbf{R_y}(\mathbf{u}) = R^i{}_k(\mathbf{y}) u^k \frac{\partial}{\partial x^i}|_x, \qquad \mathbf{u} = u^k \frac{\partial}{\partial x^k}|_x \in T_x M.$$

Then

$$R^i{}_k := 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2\frac{\partial^2 G^i}{\partial y^j \partial y^k} G^j - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{3}$$

The Ricci curvature $\mathbf{Ric}(\mathbf{y})$ is defined to be the trace of $\mathbf{R_y}$ on each tangent space T_xM,

$$\mathbf{Ric}(\mathbf{y}) := R^i{}_i(\mathbf{y}) = 2\frac{\partial G^i}{\partial x^i} - \frac{\partial^2 G^i}{\partial x^j \partial y^i} y^j + 2\frac{\partial^2 G^i}{\partial y^j \partial y^i} G^j - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^i}. \tag{4}$$

In general, the Ricci curvature $\mathbf{Ric}$ is a positively homogeneous function of degree two on TM, i.e., $\mathbf{Ric}(\lambda\mathbf{y}) = \lambda^2\mathbf{Ric}(\mathbf{y})$, $\lambda > 0$. If $\mathbf{G}$ is induced by a Riemannian metric g, then

$$R^i{}_k(\mathbf{y}) = R_j{}^i{}_{kl}(x) y^j y^l,$$

where $R_j{}^i{}_{kl}(x)$ denote the coefficients of the Riemannian curvature tensor on M. In the case, $\mathbf{Ric}(\mathbf{y}) = R_j{}^k{}_{kl}(x) y^j y^l$ is quadratic in $\mathbf{y} \in T_xM$. See [10] for more details.

Two sprays/Finsler metrics are said to be *pointwise projectively equivalent* if they have the same geodesics as point sets. We have the following trivial lemma.

Lemma 2.1 *Let* $\mathbf{G}$ *and* $\tilde{\mathbf{G}}$ *be two sprays on a manifold* M*.* $\mathbf{G}$ *and* $\tilde{\mathbf{G}}$ *are pointwise projectively equivalent if and only if there is a scalar function* P *on* TM *such that*

$$\tilde{\mathbf{G}} = \mathbf{G} - 2P\,\mathbf{Y},$$

where $\mathbf{Y} = y^i\frac{\partial}{\partial y^i}$ *denotes the canonical vertical tangent vector field on* TM*.*

Given two projectively equivalent sprays $\mathbf{G}$ and $\tilde{\mathbf{G}} := \mathbf{G} - 2P\,\mathbf{Y}$, the projective equivalence is said to be *trivial* if they have the same geodesics, or equivalently, $P = 0$. In this case, by (3), their Riemann curvatures are equal, $\mathbf{R} = \tilde{\mathbf{R}}$.

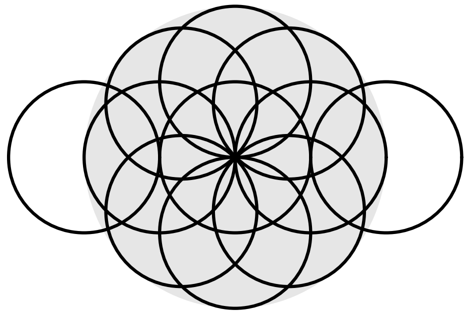
Example 2.2 ([10]) Identify $T\mathrm{R}^2 = \mathrm{R}^2 \times \mathrm{R}^2$ and identify a tangent vector at (x, y), $u\frac{\partial}{\partial x}|_{(x,y)} + v\frac{\partial}{\partial y}|_{(x,y)}$, with (x, y, u, v). Let

$$\mathbf{G} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - v\sqrt{u^2+v^2}\frac{\partial}{\partial u} + u\sqrt{u^2+v^2}\frac{\partial}{\partial v}. \tag{5}$$

$\mathbf{G}$ is a spray on R^2. Geodesics $c(t) = (x(\mathbf{y}), y(t))$ are characterized by

$$\ddot{x} + \frac{1}{2}\dot{y}\sqrt{\dot{x}^2+\dot{y}^2} = 0, \qquad \ddot{y} - \frac{1}{2}\dot{x}\sqrt{\dot{x}^2+\dot{y}^2} = 0.$$

Thus geodesics are circles of radius 1 in R^2. This implies that $\mathbf{G}$ is complete.



By a direct computation, we obtain

$$R^1{}_1 = v^2, \quad R^1{}_2 = -uv, \quad R^2{}_1 = -uv, \quad R^2{}_2 = u^2,$$

where $\mathbf{y} = (x, y, u, v) \in T_{(x,y)}\mathrm{R}^2$. This gives

$$\mathbf{Ric}(\mathbf{y}) := R^1{}_1 + R^2{}_2 = u^2 + v^2.$$

We claim that $\mathbf{G}$ is not projectively equivalent to any Finsler metric defined on the whole R^2. If a Finsler metric F induces a spray $\tilde{\mathbf{G}}$ which is projectively equivalent to $\mathbf{G}$, then F must be complete. According to the Hopf-Rinow theorem, the exponential map $\exp_p : T_p\mathrm{R}^2 \to \mathrm{R}^2$ at any point $p = (x_o, y_o) \in \mathrm{R}^2$ must be an onto map. However, the geodesics of F are circles of radius $r = 1$, thus $\exp_p$ can not be onto. This proves our claim.

At an arbitrary point $p = (x_o, y_o)$, let

$$F := \sqrt{u^2 + v^2} + (1 - k)(y - y_o)u - k(x - x_o)v \tag{6}$$

F is a Finsler metric on a neighborhood $\mathcal{U}$ of p, where

$$\mathcal{U} := \Big\{(x, y) \mid k^2(x - x_o)^2 + (1 - k)^2(y - y_o)^2 < 1\Big\}.$$

Further, the spray $\tilde{\mathbf{G}}$ of F is related to $\mathbf{G}$ by $\tilde{\mathbf{G}} = \mathbf{G} - 2P\mathbf{Y}$, where

$$P = \frac{1}{2F}\Big\{(1 - 2k)uv - \sqrt{u^2 + v^2}\Big[k(x - x_o)u + (1 - k)(y - y_o)v\Big]\Big\}.$$

Thus F is projectively equivalent to $\mathbf{G}$ on $\mathcal{U}$. Let $k = 0.5$ and $p = (0, 0)$. F is defined on the disk $\mathrm{B}^2(\rho)$ with radius $\rho = 2$ around the origin p. Thus the exponential map $\exp_p$ at p only covers $\mathrm{B}^2(\rho)$.

Theorem 2.3 *Let $(M, \mathbf{G})$ be a complete spray manifold and $\tilde{\mathbf{G}}$ another spray which is pointwise projectively equivalent to $\mathbf{G}$. Suppose that the Ricci curvatures of $\mathbf{G}$ and $\tilde{\mathbf{G}}$ satisfy*

$$\widetilde{\mathbf{Ric}} \leq \mathbf{Ric}. \tag{7}$$

Then the projective equivalence is trivial.

Proof: By Lemma 2.1, there is a scalar function P on TM such that $\tilde{\mathbf{G}} = \mathbf{G} - 2P\,\mathbf{Y}$. In local coordinates, $\tilde{G}^i = G^i + Py^i$. By (4), we obtain

$$\tilde{\mathbf{R}}_{\mathbf{y}} = \mathbf{R}_{\mathbf{y}} + \Big(P(\mathbf{y})^2 - P_{;k}(\mathbf{y})y^k\Big)\ I + \tau_{\mathbf{y}}\ \mathbf{y}, \tag{8}$$

where

$$\tau_{\mathbf{y}} := \Big\{3\Big(P_{;k}(\mathbf{y}) - P(\mathbf{y})\frac{\partial P}{\partial y^k}(\mathbf{y})\Big) + \frac{\partial}{\partial y^k}\Big(P(\mathbf{y})^2 - P_{;k}(\mathbf{y})y^k\Big)\Big\}dx^k,$$

where

$$P_{;k} := \frac{\partial P}{\partial x^k} - \frac{\partial G^l}{\partial y^k}\frac{\partial P}{\partial y^l}.$$

This gives that

$$\widetilde{\mathbf{Ric}}(\mathbf{y}) = \mathbf{Ric}(\mathbf{y}) + (n-1)\Big(P(\mathbf{y})^2 - P_{;k}(\mathbf{y})y^k\Big). \tag{9}$$

See [10]. Fix an arbitrary vector $\mathbf{y} \in T_xM \setminus \{0\}$ and let $c(t)$ denote the geodesic of $\mathbf{G}$ with $\dot{c}(0) = y$. By assumption, $\mathbf{G}$ is complete, hence $c(t)$ is defined for $-\infty < t < \infty$. Let

$$P(t) := P(\dot{c}(t)).$$

Observe that

$$P'(t) = P_{;k}(\dot{c}(t))\dot{c}^k(t).$$

By assumption,

$$P_{;k}y^k - P^2 = \frac{1}{n-1}\Big(\mathbf{Ric} - \widetilde{\mathbf{Ric}}\Big) \geq 0.$$

Thus

$$P'(t) - P(t)^2 \geq 0.$$

Let

$$P_0(t) := \frac{P(\mathbf{y})}{1 - P(\mathbf{y})t}.$$

$P_0(t)$ satisfies

$$P_0'(t) - P_0(t)^2 = 0.$$

To compare $P(t)$ with $P_0(t)$, define

$$h(t) := \exp\Big\{-\int_0^t [P(s) + P_0(s)]ds\Big\}\Big\{P(t) - P_0(t)\Big\}.$$

Observe that

$$h'(t) = \exp\Big\{-\int_0^t [P(s) + P_0(s)]ds\Big\}\Big\{P'(t) - P_0'(t) + P_0(t)^2 - P(t)^2\Big\} \geq 0.$$

Note that $h(0) = 0$. Thus $h(t) \geq 0$ for $t > 0$ and $h(t) < 0$ for $t < 0$. This implies that

$$P(t) \geq P_0(t), \qquad t > 0,$$

$$P(t) \le P_0(t), \qquad t < 0.$$

Assume that $P(\mathbf{y}) \neq 0$. Let $t_o = 1/P(\mathbf{y})$. If $P(\mathbf{y}) > 0$, then $t_o > 0$ and

$$P(\dot{c}(t_o)) \ge \lim_{t \to t_o^-} P_0(t) = \infty.$$

If $P(\mathbf{y}) < 0$, then $t_o < 0$ and

$$P(\dot{c}(t_o)) \le \lim_{t \to t_o^+} P_0(t) = -\infty.$$

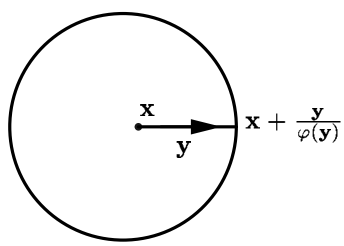
Both are impossible. Therefore, $P(\mathbf{y}) = 0$ for any $\mathbf{y} \in TM$ and $\tilde{\mathbf{G}} = \mathbf{G}$. By (8), we conclude that the Riemann curvatures are equal, $\tilde{\mathbf{R}} = \mathbf{R}$. Q.E.D.

Example 2.4 Theorem 2.3 is false if the completeness of $\mathbf{G}$ is weakened to the positive completeness. Let $|\cdot|$ and $\langle\ ,\ \rangle$ denote the standard Euclidean norm and inner product in R^n. Define

$$\begin{aligned}
\varphi(\mathbf{y}) :&= \frac{\sqrt{|\mathbf{y}|^2 - (|\mathbf{x}|^2|\mathbf{y}|^2 - \langle \mathbf{x}, \mathbf{y}\rangle^2)} + \langle \mathbf{x}, \mathbf{y}\rangle}{1 - |\mathbf{x}|^2}, \\
\bar{\varphi}(\mathbf{y}) :&= \varphi(-\mathbf{y}), \\
\tilde{\varphi}(\mathbf{y}) :&= \frac{\sqrt{|\mathbf{y}|^2 - (|\mathbf{x}|^2|\mathbf{y}|^2 - \langle \mathbf{x}, \mathbf{y}\rangle^2)}}{1 - |\mathbf{x}|^2} = \frac{1}{2}\Big\{\varphi(\mathbf{y}) + \bar{\varphi}(\mathbf{y})\Big\},
\end{aligned}$$

where $\mathbf{y} \in T_x\mathrm{B}^n(1) = \mathrm{R}^n$. $\varphi(\mathbf{y}) > 0$ is determined by the following identity,

$$\mathbf{x} + \frac{\mathbf{y}}{\varphi(\mathbf{y})} \in \partial \mathrm{B}^n(1).$$



$\varphi, \bar{\varphi}$ and $\tilde{\varphi}$ are Finsler metrics on the unit ball $\mathrm{B}^n(1) \subset \mathrm{R}^n$. φ and $\tilde{\varphi}$ are the Funk metric and the Klein metric on the unit ball $\mathrm{B}^n(1)$, respectively. We have

$$\frac{1}{2}\varphi \le \tilde{\varphi}. \tag{10}$$

φ and $\bar{\varphi}$ induce the following sprays $\mathbf{G}$ and $\tilde{\mathbf{G}}$, respectively,

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - \varphi \frac{\partial}{\partial y^i}$$

$$\tilde{\mathbf{G}} = y^i \frac{\partial}{\partial x^i} - (\varphi - \bar{\varphi}) y^i \frac{\partial}{\partial y^i} = \mathbf{G} + \bar{\varphi}\mathbf{Y}.$$

Note that $\mathbf{G}$ is only positively complete. By a direction computation, we obtain the Ricci curvatures of $\tilde{\mathbf{G}}$ and $\mathbf{G}$,

$$\widetilde{\mathbf{Ric}} = -(n-1)\bar{\varphi}^2, \qquad \mathbf{Ric} = -(n-1)\frac{1}{4}\varphi^2.$$

By (10), we see that

$$\widetilde{\mathbf{Ric}} \leq \mathbf{Ric}.$$

Thus the Ricci curvature condition (7) holds. But $\tilde{\mathbf{G}} \neq \mathbf{G}$, even $\tilde{\mathbf{G}}$ is complete.

The above example also shows that if the inequality in (7) is reversed, then the conclusion in Theorem 2.3 is false.

3 Projectively Equivalent Finsler Metrics

In this section, we will first discuss an interesting result by Rapcsák [16] on projectively equivalent Finsler metrics. Then we apply Theorem 2.3 to Finsler metrics and obtain a stronger result in this case.

Let $g = g_{ij}(\mathbf{y})y^i y^j$ be a Finsler metric and $\tilde{g} = \tilde{g}_{ij}(\mathbf{y})y^i y^j$ another Finsler metric on M. Warning: $g_{ij}(\mathbf{y})$ and $\tilde{g}_{ij}(\mathbf{y})$ depend on $\mathbf{y} \in T_xM \setminus \{0\}$ at each point x, unless they are Riemannian. One can easily verify that the geodesic coefficients $\tilde{G}^i = \tilde{G}^i(\mathbf{y})$ of $\tilde{g}$ are related to that of g by

$$\tilde{G}^i = G^i + \frac{1}{4}\tilde{g}^{il}\Big\{\frac{\partial \tilde{g}_{;k}}{\partial y^l} y^k - \tilde{g}_{;l}\Big\}, \tag{11}$$

where $\tilde{g}_{;k} := \tilde{g}_{ij;k} y^i y^j$ denote the covariant derivatives of $\tilde{g}$ with respect to g.

$$\tilde{g}_{;k} := \frac{\partial \tilde{g}}{\partial x^k} - \frac{\partial G^l}{\partial y^k}\frac{\partial \tilde{g}}{\partial y^l}.$$

We simply denote $\nabla\tilde{g} := \tilde{g}_{;k} dx^k$ which is a 1-form on $TM \setminus \{0\}$. By Lemma 2.1 and (11), we immediately conclude that $\tilde{g}$ is pointwise projective equivalent to g if and only if there is a scalar function P on TM such that

$$\frac{\partial \tilde{g}_{;k}}{\partial y^l} y^k - \tilde{g}_{;l} = 2P\frac{\partial \tilde{g}}{\partial y^l}. \tag{12}$$

Lemma 3.1 *$\tilde{g}$ is pointwise projectively equivalent to g if and only if there is a scalar function P on TM such that*

$$\tilde{g}_{;k} = P\frac{\partial \tilde{g}}{\partial y^k} + 2\frac{\partial P}{\partial y^k}\tilde{g}. \tag{13}$$

In this case

$$P = \frac{\tilde{g}_{;k} y^k}{4\tilde{g}}. \tag{14}$$

Suppose that the projective equivalence is trivial, $P = 0$, then $\tilde{g}$ is horizontally parallel with respect to g, $\nabla\tilde{g} = 0$.

Proof: First we assume that $\tilde{g}$ is pointwise projective to g. Then (12) holds for some scalar function P on TM. Contracting (12) with y^l yields

$$\tilde{g}_{;k}y^k = 4P\tilde{g}. \tag{15}$$

By (12) and (15), we obtain

$$\begin{aligned} 2P\frac{\partial \tilde{g}}{\partial y^l} &= \frac{\partial}{\partial y^l}\Big[g_{;k}y^k\Big] - 2\tilde{g}_{;l} \\ &= \frac{\partial}{\partial y^l}\Big[4P\tilde{g}\Big] - 2\tilde{g}_{;l}. \end{aligned}$$

This gives (13). Conversely, if (13) holds, then

$$\begin{aligned} \frac{\partial \tilde{g}_{;k}}{\partial y^l}y^k - \tilde{g}_{;l} &= y^k\frac{\partial}{\partial y^l}\Big\{P\frac{\partial \tilde{g}}{\partial y^k} + 2\frac{\partial P}{\partial y^k}\tilde{g}\Big\} - \Big\{P\frac{\partial \tilde{g}}{\partial y^l} + 2\frac{\partial P}{\partial y^l}\tilde{g}\Big\} \\ &= 2\frac{\partial P}{\partial y^l}\tilde{g} + P\frac{\partial \tilde{g}}{\partial y^l} + 2P\frac{\partial \tilde{g}}{\partial y^l} - P\frac{\partial \tilde{g}}{\partial y^l} - 2\frac{\partial P}{\partial y^l}\tilde{g} \\ &= 2P\frac{\partial \tilde{g}}{\partial y^l}. \end{aligned}$$

This gives (12). Q.E.D.

By Lemma 3.1, we obtain an additional conclusion to Theorem 2.3 for Finsler metrics.

Corollary 3.2 *Let (M, g) be a complete Finsler manifold and $\tilde{g}$ another Finsler metric on M, which is pointwise projectively equivalent to g. Suppose that*

$$\widetilde{\mathbf{Ric}} \leq \mathbf{Ric}.$$

Then the projective equivalence is trivial. Hence $\tilde{g}$ is horizontal parallel with respect to g.

According to Corollary 3.2, if two Ricci-flat Finsler metrics are pointwise projectively equivalent and one of them is complete, then the projective equivalence is trivial and the Riemann curvatures are equal. One can show if two negative Ricci-constant Finsler metrics are pointwise projectively equivalent and one of them is complete, then they are isometric up to a scaling. These facts are proved in [9]. In the positive Ricci-constant case, we have the following

Corollary 3.3 *Let g and $\tilde{g}$ be pointwise projectively equivalent Einstein metrics on a compact n-manifold with $\mathbf{Ric} = (n-1)g$ and $\widetilde{\mathbf{Ric}} = (n-1)\tilde{g}$. Suppose that $\tilde{g} \leq g$, then $\tilde{g} = g$.*

Example 3.4 ([4][13]) Let

$$g = \frac{\Big(\sqrt{|\mathbf{y}|^2 - (|\mathbf{x}|^2|\mathbf{y}|^2 - \langle \mathbf{x}, \mathbf{y}\rangle^2)} + \langle \mathbf{x}, \mathbf{y}\rangle\Big)^4}{(1-|\mathbf{x}|^2)^4\Big(|\mathbf{y}|^2 - (|\mathbf{x}|^2|\mathbf{y}|^2 - \langle \mathbf{x}, \mathbf{y}\rangle^2)\Big)}, \qquad \mathbf{y} \in T_{\mathbf{x}}\mathrm{B}^n(1).$$

g has the following properties:

(a) g is positively complete,

(b) g is pointwise projectively equivalent to the Euclidean metric on $\mathrm{B}^n(1)$,

(c) the flag curvature of g vanishes, $\mathbf{K} = 0$.

However, the projective equivalence is not trivial since g is incomplete.

4 The Role of the S-curvature

Now we take a look at the S-curvature and the role it plays in the projective geometry of Finsler manifolds.

First, recall the definition of the S-curvature. Let $g = g_{ij}(\mathbf{y})y^iy^j$ be a Finsler metric on a manifold M. The Busemann-Hausdorff volume form $dV_g = \sigma(x)dx^1\cdots dx^n$ is defined by

$$\sigma(x) := \frac{\mathrm{Vol}(\mathrm{B}^n(1))}{\mathrm{Vol}\Big\{(y^i) \in \mathrm{R}^n \ \Big|\ g_{ij}(\mathbf{y})y^iy^j < 1\Big\}}. \tag{16}$$

For a vector $\mathbf{y} \in T_xM \setminus \{0\}$, the S-curvature $\mathbf{S}(\mathbf{y})$ is defined by

$$\mathbf{S}(\mathbf{y}) = \frac{\partial G^i}{\partial y^i}(\mathbf{y}) - y^i\frac{\partial}{\partial x^i}\Big(\ln\sigma(x)\Big), \tag{17}$$

See [10] for detailed discussion on the S-curvature. If g is a Riemannian metric, then $\sigma(x) = \sqrt{\det(g_{ij}(x))}$. Thus $\mathbf{S} = 0$. In this sense, the S-curvature is a non-Riemannian quantity. There are many non-trivial Finsler metrics with vanishing S-curvature.

Example 4.1 Let $\{\zeta^1, \zeta^2, \zeta^3\}$ be the canonical left-invariant co-frame on the Lie group $\mathrm{Sp}(1) = \mathrm{S}^3$ satisfying

$$d\zeta^1 = 2\zeta^2\wedge\zeta^3, \quad d\zeta^2 = 2\zeta^3\wedge\zeta^1, \quad d\zeta^3 = 2\zeta^1\wedge\zeta^2. \tag{18}$$

Consider $g = [\alpha+\beta]^2$, where

$$\alpha(\mathbf{y}) := \sqrt{\kappa^2[\zeta^1(\mathbf{y})]^2 + \lambda^2[\zeta^2(\mathbf{y})]^2 + \lambda^2[\zeta^3(\mathbf{y})]^2}, \qquad \beta(\mathbf{y}) := \varepsilon\,\zeta^1(\mathbf{y}),$$

where $\kappa > 0$, $\lambda > 0$ and $\varepsilon > 0$. Assume that

$$\|\beta\|_\alpha = \left|\frac{\varepsilon}{\kappa}\right| < 1.$$

Then g is a Finsler metric with $\mathbf{S} = 0$. This fact is verified in [Sh2, Example 5.2.3]. It is proved in [3] that for $\lambda = \sqrt{\kappa}$ and $\varepsilon = \sqrt{\kappa^2 - \kappa}$ with $\kappa \geq 1$, the resulting metric $g_\kappa = [\alpha + \beta]^2$ satisfies

$$\mathbf{Ric} = 2g_\kappa.$$

Examples of constant curvature constructed in [11], [12] all have zero S-curvature, $\mathbf{S} = 0$.

Lemma 4.2 *Let g and $\tilde{g}$ be Finsler metrics on an n-manifold M. Suppose that $\tilde{g}$ is pointwise projective equivalent to g. Then the projective factor P is given by*

$$P = \frac{1}{n+1}\Big(\tilde{\mathbf{S}} - \mathbf{S}\Big) - y^i \frac{\partial}{\partial x^i}\Big[\ln f\Big], \tag{19}$$

where $f = f(x)$ is a scalar function on M determined by $dV_{\tilde{g}} = \frac{1}{f^{n+1}}\, dV_g$.

Proof: By assumption, the geodesic coefficients of g and $\tilde{g}$ satisfy

$$\tilde{G}^i = G^i + Py^i.$$

This implies that

$$\frac{\partial \tilde{G}^i}{\partial y^i} = \frac{\partial G^i}{\partial y^i} + (n+1)P. \tag{20}$$

Express $dV_{\tilde{g}} = \tilde{\sigma}(x)dx^1 \cdots dx^n$. The S-curvature of $\tilde{g}$ is given by

$$\tilde{\mathbf{S}} = \frac{\partial \tilde{G}^i}{\partial y^i} - y^i \frac{\partial}{\partial x^i}\Big(\ln \tilde{\sigma}(x)\Big). \tag{21}$$

Let

$$f(x) := \Big(\frac{\sigma(x)}{\tilde{\sigma}(x)}\Big)^{\frac{1}{n+1}}. \tag{22}$$

$f(x)$ is a well-defined function on M, although $\sigma(x)$ and $\tilde{\sigma}(x)$ depend on the local coordinates. The volume forms of g and $\tilde{g}$ are related by

$$dV_{\tilde{g}} = \tilde{\sigma}(x)dx^1 \cdots dx^n = \frac{1}{f(x)^{n+1}}\sigma(x)dx^1 \cdots dx^n = \frac{1}{f(x)^{n+1}}\, dV_g.$$

It follows from (17), (20) and (21) that

$$\begin{aligned} P &= \frac{1}{n+1}\Big[\tilde{\mathbf{S}} - \mathbf{S} + y^i \frac{\partial}{\partial x^i}\Big(\ln \tilde{\sigma}\Big) - y^i \frac{\partial}{\partial x^i}\Big(\ln \sigma\Big)\Big] \\ &= \frac{1}{n+1}\Big(\tilde{\mathbf{S}} - \mathbf{S}\Big) - y^i \frac{\partial}{\partial x^i}\Big(\ln f\Big). \end{aligned}$$

This proves Lemma 4.2. Q.E.D.

By Lemma 4.2, we obtain an additional conclusion to Corollary 3.2 for projectively equivalent Finsler metrics with the same S-curvatures.

Corollary 4.3 *Let (M, g) be a complete Finsler manifold and $\tilde{g}$ another Finsler metric on M, which is pointwise projectively equivalent to g. Suppose that both g and $\tilde{g}$ satisfy*

$$\widetilde{\mathbf{Ric}} \le \mathbf{Ric}, \qquad \tilde{\mathbf{S}} = \mathbf{S}.$$

Then the conclusion in Corollary 3.2 holds and $dV_{\tilde{g}}$ is proportional to dV_g.

Theorem 1.1 is just a special case of Corollary 4.3.

Example 4.4 Let $(\bar{M}, h)$ be an arbitrary Riemannian surface and $I = (a, b)$ an arbitrary interval. For an arbitrary number $\varepsilon \ge 0$, define a Finsler metric g_ε on $M = I \times \bar{M}$ by

$$g_\varepsilon(\mathbf{y}) = u^2 + h(\bar{\mathbf{y}}) + \varepsilon\sqrt{u^4 + h(\bar{\mathbf{y}})^2}, \tag{23}$$

where $\mathbf{y} = u\frac{\partial}{\partial x} \oplus \bar{\mathbf{y}} \in T_{(x,p)}(I \times \bar{M}) \equiv \mathrm{R} \oplus T_p\bar{M}$. The metric g_1 is constructed by Z.I. Szabo [14] in his classification of Berwald metrics.

Express h in a local coordinate system by

$$h = e^{\rho(y,z)}(v^2 + w^2).$$

Then for $\mathbf{y} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \in T_{(x,p)}(I \times \bar{M})$,

$$g_\varepsilon(\mathbf{y}) = u^2 + e^{\rho(y,z)}(v^2 + w^2) + \varepsilon\sqrt{u^4 + e^{2\rho(y,z)}(v^2 + w^2)^2}.$$

By a direct computation, we obtain the geodesic coefficients of g_ε,

$$\begin{aligned} G^1 &= 0 \\ G^2 &= \frac{1}{4}\Big\{\rho_y\, v^2 - \rho_y\, w^2 + 2\rho_z\, vw\Big\} \\ G^3 &= \frac{1}{4}\Big\{\rho_z\, w^2 - \rho_z\, v^2 + 2\rho_y\, vw\Big\} \end{aligned}$$

We see that the spray of g_ε is independent of $\varepsilon \ge 0$ and hence all g_ε in this family are pointwise projectively related. The Ricci curvature of g_ε is given by

$$\mathbf{Ric}(\mathbf{y}) = \bar{\mathbf{K}}(p)\; h(\bar{\mathbf{y}}), \qquad \mathbf{y} = u\frac{\partial}{\partial u} \oplus \bar{\mathbf{y}} \in T_{(x,p)}(I \times \bar{M}),$$

where $\bar{\mathbf{K}}(p)$ denotes the Gauss curvature of h at $p \in \bar{M}$. We see that the Ricci curvature of g_ε is independent of ε. Note that the volume form $dV_{g_\varepsilon} = \sigma(x, y, z)dxdydz$ of g_ε is given by

$$\sigma(x, y, z) = e^{\rho(y,z)}.$$

This gives

$$\mathbf{S} = (G^1)_u + (G^2)_v + (G^3)_w - u\frac{\sigma_x}{\sigma} - v\frac{\sigma_y}{\sigma} - w\frac{\sigma_z}{\sigma} = 0.$$

From the above example, we see that trivially pointwise projectively equivalent metrics are not necessarily isometric, even if the Ricci curvatures of the metrics are equal.

5 Projectively Equivalent Riemannian Metrics

In this section, we will prove Theorem 1.3. Let $\tilde{g}$ and g be pointwise projectively equivalent Riemannian metrics on an n-dimensional manifold M. We know that the S-curvature of any Riemannian metric always vanishes. Thus by Lemma 4.2, the projective factor P is given by

$$P = -y^i \frac{\partial}{\partial x^i}\Big[\ln f\Big],$$

where f is a positive function on M which is defined in (22). Plugging it into (9) yields

$$\widetilde{\mathbf{Ric}}(\mathbf{y}) = \mathbf{Ric}(\mathbf{y}) + \frac{n-1}{f} D_g^2 f, \tag{24}$$

where $D_g^2 f$ denotes the Hessian of f with respect to g. Now the Ricci curvature becomes a quadratic form on each tangent space T_xM. Thus at each point $x \in M$, there is an orthonormal basis $\{e_i\}_{i=1}^n$ for (T_xM, g_x) such that

$$\widetilde{\mathbf{Ric}}(\mathbf{y}) = \sum_{i=1}^n \lambda_i \, (y^i)^2, \qquad y = y^i e_i.$$

The trace of $\widetilde{\mathbf{Ric}}$ with respect to g is defined by

$$\mathrm{tr}_g \widetilde{\mathbf{Ric}} = \sum_{i=1}^n \lambda_i, \tag{25}$$

and the trace of $\mathbf{Ric}$ with respect to g is just the scalar curvature $\mathbf{s}_g$ of g.

Taking the trace on both sides of (24) with respect to g, we obtain

$$\mathrm{tr}_g \widetilde{\mathbf{Ric}} - \mathbf{s}_g = \frac{n-1}{f}\Delta_g f. \tag{26}$$

Let $\mathbf{r} := \frac{1}{n-1}(\mathrm{tr}_g \widetilde{\mathbf{Ric}} - \mathbf{s}_g)$. Equation (26) becomes

$$\Delta_g f = \mathbf{r} \, f. \tag{27}$$

Proof of Theorem 1.3: We assume that M is compact. Integrating (27) over M, we obtain

$$\int_M \mathbf{r} \, f \, dV_g = 0.$$

Now we assume that

$$\mathbf{r} \leq 0, \qquad \text{or} \qquad \mathbf{r} \geq 0,$$

then $\mathbf{r} = 0$. Thus the function f determined by $dV_{\tilde{g}} = \frac{1}{f^{n+1}} dV_g$ satisfies

$$\Delta_g f = \mathbf{r}\ f \equiv 0.$$

Since M is compact, $f = constant$. Thus, by Lemma 4.2, $dV_{\tilde{g}}$ is proportional to dV_g and the projective equivalence is trivial. Thus the Riemann curvatures are equal, $\mathbf{R} = \tilde{\mathbf{R}}$. By Lemma 3.1, $\tilde{g}$ is horizontally parallel with respect to g too. Q.E.D.

References

[1] J.C. Álvarez, Contact topology, taut immersions, and Hilbert's fourth problem, In "*Differential and Symplectic Topology of Knots and Curves*". S. Tabachnikov (Ed.) Adv. in Math., Sciences, AMS, 1999.

[2] J.C. Álvarez, I.M. Gelfand and M. Smirnov, Crofton Densities, symplectic geometry and Hilbert's fourth problem, *Arnold-Gelfand Mathematical Seminars, Geometry and Singularity Theory, V.I.* Arnold, I.M. Gelfand, M. Smirnov, and V.S. Retakh (eds.). Birkhauser, Boston, 1997, pp. 77-92.

[3] D. Bao and Z. Shen, Finsler metrics of constant curvature on the Lie group S^3, *Journal of Math. London Soc.*, to appear.

[4] L. Berwald, *Über die n-dimensionalen Geometrien konstanter Krümmung, in denen die Geraden die kürzesten sind*, Math. Z. **30**(1929), 449-469.

[5] D. Hilbert, Mathematical Problems, it Bull. of Amer. Math. Soc. **37**(2001), 407-436. Reprinted from *Bull. Amer. Math. Soc.* **8** (July 1902), 437-479.

[6] S. Kim, Volume and projective mappings between Riemannian manifolds, preprint.

[7] J. Mikeš, On geodesic mappings of Einsteinian spaces, *Mat. Zametki* **28**(1980), no. 6, 935-939.

[8] A.V. Pogorelov, *Hilbert's Fourth Problem*, Winston & Wiley, New York, 1982.

[9] Z. Shen, Projectively Related Einstein Metrics in Riemann-Finsler Geometry, *Math. Ann.* **320**(2001), 625-647.

[10] Z. Shen, *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, Dordrecht, 2001.

[11] Z. Shen, Finsler metrics with $\mathbf{K} = 0$ and $\mathbf{S} = 0$, Canadian J. of Math., to appear.

[12] Z. Shen, Two-dimensional Finsler metrics of constant curvature, Manuscripta Mathematica to appear.

[13] Z. Shen, Projectively flat Finsler metrics of constant flag curvature, preprint, 2001.

[14] Z.I. Szabo, Positive definite Berwald spaces (Structure theorems on Berwald spaces), *Tensor, N.S.* **35**(1981), 25-39.

[15] Z.I. Szabo, Hilbert's Fourth Problem. I., *Advances in Mathematics*, **59**(1986), .

[16] A. Rapcsák, Über die bahntreuen Abbildungen metrisher Räume, *Publ. Math. Debrecen*, **8**(1961), 285-290.

Xinyue Chen

Department of Mathematics, Chongqing Institute of Technology, Chongqing, 400050, P. R. China
chenxy58@163.net

Zhongmin Shen

Department of Mathematical Sciences, Indiana Univ.-Purdue Univ. Indianapolis, IN 46202-3216, U.S.A.
zshen@math.iupui.edu