On a class of Douglas metrics

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Abstract

In this paper, we study a class of Finsler metrics defined by a Riemannian metric and a 1-form on a manifold. We find an equation that characterizes Douglas metrics on a manifold of dimension $n \geq 3$.

1 Introduction

In projective Finsler geometry, we study projectively equivalent Finsler metrics on a manifold, namely, geodesics are same up to a parametrization. J. Douglas introduces two projective quantities: the (projective) Douglas curvature and the (projective) Weyl curvature ([2]). The Douglas curvature always vanishes for Riemannian metrics and the Weyl curvature is an extension of the Weyl curvature in Riemannian geometry. Finsler metrics with vanishing Douglas curvature are called *Douglas metrics*. Roughly speaking, Douglas metrics are locally projectively Finsler metrics. Douglas metrics form an important class of metrics in Finsler geometry.

Consider a Rander metric $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form with $b(x) := \|\beta_x\|_\alpha < 1$. It is known that $F = \alpha + \beta$ is a Douglas metric if and only if β is closed [1]. In this case, $F = \alpha + \beta$ has the same geodesics as α. More general, we consider Finsler metrics in the form $F = \sqrt{\alpha^2 + k\beta^2} + \epsilon\beta$, where k and $\epsilon \neq 0$ are constants. Such metrics are said to be *of Randers type*. Clearly, $F = \sqrt{\alpha^2 + k\beta^2} + \epsilon\beta$ is a Douglas metric if and only if β is closed. This motivates us to consider Finsler metrics in the form $F = \alpha\phi(s)$, $s = \beta/\alpha$. Such a metric is called an *(α, β)-metric.*

Recently, the third author [8] proved a theorem that characterizes projectively flat (α, β)-metrics of dimension $n \geq 3$. Based on his result we study and characterize Douglas (α, β)-metrics.

Theorem 1.1 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an open subset $\mathcal{U} \subset R^n$ $(n \geq 3)$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i \neq 0$. Let $b := \|\beta_x\|_\alpha$. Suppose that the following conditions: (a) β is not parallel with respect to α, (b) F is not of Randers type, and (c) $db \neq 0$ everywhere or $b = constant$ on $\mathcal{U}$.*

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Then F is a Douglas metric on $\mathcal{U}$ if and only if the function $\phi = \phi(s)$ satisfies the following ODE:

$$\Big\{1 + (k_1 + k_2 s^2)s^2 + k_3 s^2\Big\}\phi''(s) = (k_1 + k_2 s^2)\Big\{\phi(s) - s\phi'(s)\Big\} \tag{1.1}$$

and the covariant derivative $\nabla\beta = b_{i|j} y^i dx^j$ of β with respect to α satisfies the following equation:

$$b_{i|j} = 2\tau\Big\{(1 + k_1 b^2)a_{ij} + (k_2 b^2 + k_3)b_i b_j\Big\}, \tag{1.2}$$

where $\tau = \tau(x)$ is a scalar function on $\mathcal{U}$ and k_1, k_2 and k_3 are constants with $(k_2, k_3) \neq (0, 0)$.

Theorem 1.1 holds good in dimension $n \geq 3$. When $n = 2$, the classification is still unknown.

Equation (1.2) implies that β is closed. There are many elementary solutions of (1.1). For example, the following functions ϕ satisfy (1.1) for some constants k_i.

$$b\phi = 1 + s, \qquad \phi = 1 + \epsilon s + s^2,$$

$$\phi = 1 + \epsilon s + s\arctan(s), \qquad \phi = 1 + \epsilon s + 2s^2 - \frac{1}{3}s^4,$$

where ϵ is a constant. We are particularly interested in the function $\phi = (1+s)^2$ which satisfies (1.1) with $k_1 = 2$, $k_2 = 0$, $k_3 = -3$. See [6], [10], [11], [13] for the related discissions on metrics defined by the above functions ϕ.

The functions $\phi = e^s + \epsilon s$ and $\phi = 1/(1-s) + \epsilon s$ do not satisfy (1.1) for any constants k_i. Thus the (α, β)-metrics defined by these functions are Douglas metrics if and only if β is parallel with respect to α (Cf. [4], [12]).

2 Preliminaries

In local coordinates, the geodesics of a Finsler metric $F = F(x, y)$ are characterized by

$$\frac{d^2 x^i}{dt^2} + 2G^i\Big(x, \frac{dx}{dt}\Big) = 0,$$

where

$$G^i := \frac{1}{4} g^{il}\Big\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\Big\}. \tag{2.1}$$

The local functions $G^i = G^i(x, y)$ define a global vector field $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ on $TM \setminus \{0\}$, which is called the *spray*.

Douglas metrics can be characterized by

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x) y^j y^k + P(x, y) y^i, \tag{2.2}$$

where $\Gamma^i_{jk}(x)$ are local functions on M and $P(x, y)$ is a local positively homogeneous function of degree one.

By definition, an (α,β)-metric is a Finsler metric expressed in the following form,

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha},$$

where $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form with $\|\beta_x\|_\alpha < b_o$, $x \in M$. The function $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_o, b_o)$ satisfying

$$\phi(s) - s\phi'(s) + (\rho^2 - s^2)\phi''(s) > 0, \qquad |s| \leq \rho < b_o$$

that guarantees that $F = \alpha\phi(\beta/\alpha)$ is a regular positive definite Finsler metric [7]. By (2.1), the spray coefficients G^i_α of α are given by

$$G^i_\alpha = \frac{a^{il}}{4}\Big\{[\alpha^2]_{x^ky^l}y^k - [\alpha^2]_{x^l}\Big\}, \tag{2.3}$$

where $(a^{ij}) := (a_{ij})^{-1}$. By (2.1) again, we obtain the following formula for the spray coefficients G^i of F.

$$G^i = G^i_\alpha + \alpha Q s^i{}_0 + \alpha^{-1}\Theta\Big(-2\alpha Q s_0 + r_{00}\Big)y^i + \Psi\Big(-2\alpha Q s_0 + r_{00}\Big)b^i, \tag{2.4}$$

where

$$\begin{aligned} Q &= \frac{\phi'}{\phi - s\phi'} \\ \Theta &= \frac{\phi - s\phi'}{2\Big((\phi - s\phi') + (b^2 - s^2)\phi''\Big)} \cdot \frac{\phi'}{\phi} - s\Psi \\ \Psi &= \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}. \end{aligned}$$

The formula (2.4) is given in [7] and [9]. A different version of (2.4) is given in [3] and [5]. The reason why we are particularly interested in (α,β)-metrics is that the curvatures of an (α,β)-metric can be expressed in terms of the curvature of α, the covariant derivatives of β, and the derivatives of ϕ.

Assume that $\phi = \phi(s)$ satisfies (1.1) and $\beta = b_iy^i$ satisfies (1.2). Then the spray coefficients G^i of $F = \alpha\phi(\beta/\alpha)$ are given by

$$G^i = \hat{G}^i + Py^i,$$

where

$$\begin{aligned} \hat{G}^i &= G^i_\alpha + \tau\Big\{k_1\alpha^2 + k_2\beta^2\Big\}b^i \\ P &= \tau\Big\{(1 + (k_1 + k_2s^2)s^2 + k_3s^2)\frac{\phi'}{\phi} - s(k_1 + k_2s^2)\Big\}\alpha. \end{aligned}$$

Note that $\hat{G}^i$ are quadratic in $y \in T_xM$. Thus F is a Douglas metric. We shall prove that the converse is true too.

It is known that Douglas metrics can be also characterized by the following equations ([1]):

$$G^i y^j - G^j y^i = \frac{1}{2}(\Gamma^i_{kl} y^j - \Gamma^j_{kl} y^i) y^k y^j.$$

Using (2.4), we can easily show that an (α, β)-metric is a Douglas metric if and only if

$$\begin{aligned} &\alpha Q(s^i{}_0 y^j - s^j{}_0 y^i) + \Psi\Big(-2\alpha Q s_0 + r_{00}\Big)(b^i y^j - b^j y^i) \\ &= \frac{1}{2}(G^i_{kl} y^j - G^j_{kl} y^i) y^k y^l, \end{aligned} \tag{2.5}$$

where $G^i_{kl} := \Gamma^i_{kl} - \gamma^i_{kl}$ and $\gamma^i_{kl} := \frac{\partial^2 G^i_\alpha}{\partial y^k \partial y^l}$. To prove the necessary condition in Theorem 1.1, we shall find a way to simplify (2.5). In the simplification, we have to avoid the cases when $Q = k_1 s$ and $\Psi =$ *constant*.

The following lemmas are trivial.

Lemma 2.1 *If* $\phi(0) = 1$ *and* $Q = k_1 s$, *where* k_1 *is independent of* s, *then* $\phi = \sqrt{1 + k_1 s^2}$.

Lemma 2.2 *If* $\phi(0) = 1$ *and* $2\Psi = \frac{k_1}{1+k_1 b^2}$, *where* k_1 *is a number independent of* s, *then*

$$\phi = \epsilon s + \sqrt{1 + k_1 s^2},$$

where ϵ *is a number independent of* s.

From the above lemmas, we see that if $Q = k_1 s$, then $F = \sqrt{\alpha^2 + k_1 \beta^2}$ is of Riemannian type, and if $\Psi =$ *constant*, then $F = \sqrt{\alpha^2 + k_1 \beta^2} + \epsilon\beta$ is of Randers type.

3 β is closed

In this section, we are going to show that β is closed if it satisfies (2.5) as long as $Q/s \neq$ *constant*.

Note that the functions Q and Ψ in (2.5) are functions of $s = \beta/\alpha$. Since the dependence of Q and Ψ on s is unclear, it is difficult to get further information on ϕ directly from (2.5). To overcome this problem, we choose a special coordinate system at a point as in [8]. Fix an arbitrary point $x_o \in \mathcal{U} \subset R^n$. Make a change of coordinates: $(s, y^a) \to (y^i)$ by

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha}, \quad y^a = y^a,$$

where $\bar{\alpha} := \sqrt{\sum_{a=2}^n (y^a)^2}$. Then

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}} \bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}} \bar{\alpha}.$$

Let

$$\bar{r}_{10} := \sum_{a=2}^{n} r_{1a} y^a, \quad \bar{r}_{00} := \sum_{a,b=2}^{n} y^a y^b,$$

$$\bar{s}^1{}_0 := \sum_{a=2}^{n} s^1{}_a y^a, \quad \bar{s}_0 := \sum_{a=2}^{n} s_a y^a.$$

Note that $s_1 = bs^1{}_1 = 0$, $\bar{s}_0 = b\bar{s}_{10}$. Plugging the above identities into (2.5) we get a system of equations in the following form

$$\Phi^{ij} + \bar{\alpha}\Psi^{ij} = 0,$$

where Φ^{ij} and Ψ^{ij} are polynomials in y^a. We must have

$$\Phi^{ij} = 0, \quad \Psi^{ij} = 0.$$

Let

$$\bar{G}^a_{10} = G^a_{1b} y^b, \quad \bar{G}^a_{01} = G^a_{b1} y^b, \quad \bar{G}^a_{00} = G^a_{bc} y^b y^c.$$

For $i = 1$, $j = a$, by (2.5) we get

$$\begin{aligned} &\frac{bQ\bar{s}^a{}_0 s - \Psi r_{11} s^2 b y^a}{b^2 - s^2}\bar{\alpha}^2 - \Psi \bar{r}_{00} b y^a \\ &= \frac{1}{2(b^2 - s^2)}((\bar{G}^a_{10} + \bar{G}^a_{01})s^2 - G^1_{11} s^2 y^a)\bar{\alpha}^2 - \frac{1}{2}\bar{G}^1_{00} y^a, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\frac{bQs^2 s^a{}_1}{b^2 - s^2}\bar{\alpha}^2 + (-2\Psi s \bar{r}_{10} + 2\Psi Q b^2 s^1{}_0 - Q s^1{}_0) b y^a \\ &= \frac{G^a_{11} s^3}{2(b^2 - s^2)}\bar{\alpha}^2 + \frac{1}{2}(\bar{G}^a_{00} - (\bar{G}^1_{10} + \bar{G}^1_{01}) y^a) s. \end{aligned} \tag{3.2}$$

For $i = a$, $j = b$ by (2.5) we get

$$\begin{aligned} \frac{bs}{b^2 - s^2}(s^a{}_1 y^b - s^b{}_1 y^a) Q \bar{\alpha}^2 = &\frac{s^2}{2(b^2 - s^2)}(G^a_{11} y^b - G^b_{11} y^a)\bar{\alpha}^2 \\ &+ \frac{1}{2}(\bar{G}^a_{00} y^b - \bar{G}^b_{00} y^a), \end{aligned} \tag{3.3}$$

$$(\bar{s}^a{}_0 y^b - \bar{s}^b{}_0 y^a) bQ = \frac{s}{2}\Big\{(\bar{G}^a_{10} + \bar{G}^a_{01}) y^b - (\bar{G}^b_{10} + \bar{G}^b_{01}) y^a\Big\}. \tag{3.4}$$

We are going to prove the following

Lemma 3.1 *Suppose that $Q/s \neq$ constant. If an (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, is a Douglas metric on an open subset in $\mathcal{U}$ in R^n ($n > 2$) and $b \neq 0$, then β is closed.*

Proof: Letting $s = 0$ in (3.3), we get $\frac{1}{2}(\bar{G}^a_{00}y^b - \bar{G}^b_{00}y^a) = 0$. Then (3.3) becomes

$$b({s^a}_1 y^b - {s^b}_1 y^a)Q = \frac{s}{2}(G^a_{11}y^b - G^b_{11}y^a \tag{3.5}$$

By assumption, $Q/s \neq \mathit{constant}$ and $b \neq 0$. We conclude that

$${s^a}_1 y^b - {s^b}_1 y^a = 0.$$

Then

$${s^a}_1 = 0. \tag{3.6}$$

Plugging (3.6) into (3.5), we get

$$G^a_{11} = 0. \tag{3.7}$$

Let

$$A^a_b := s^a_b bQ - \frac{s}{2}(G^a_{1b} + G^a_{b1}).$$

Then (3.4) can be written as

$$A^a_0 y^b = A^b_0 y^a. \tag{3.8}$$

It is easy to show that

$$A^a_b = A\delta^a_b,$$

where $A := \frac{1}{n-1}\sum_{a=2}^n A^a_a$ which might depend on s. Observe that

$$0 = A^a_b - A^b_a = 2{s^a}_b bQ - \frac{s}{2}(G^a_{1b} + G^a_{b1} - G^b_{1a} - G^b_{1a}).$$

Since $Q/s \neq \mathit{constant}$, we conclude that

$${s^a}_b = 0.$$

We also get

$$G^a_{1b} + G^a_{b1} = B\delta^b_a, \tag{3.9}$$

where $B := 2A/s = \mathit{constant}$ independent of s. Q.E.D.

4 Determining r_{ij} and ϕ

In this section, we are going to derive a formula for r_{ij}. Meanwhile, we are going to obtain an ODE satisfied by ϕ. We continue to use the special coordinate system as in the previous section. From the proof, Lemma 3.1 we get $s_{ij} = 0$, $G^a_{11} = 0$ and $G^a_{10} + G^a_{01} = By^a$. Then (3.1) and (3.2) are reduced to

$$2\Psi\Big\{r_{11}b\bar{\alpha}^2 s^2 + \bar{r}_{00}b(b^2 - s^2)\Big\} = \Big\{G^1_{11} - B\Big\}\bar{\alpha}^2 s^2 + \bar{G}^1_{00}(b^2 - s^2). \tag{4.1}$$

$$-4\Psi\bar{r}_{10}by^a = \bar{G}^a_{00} - (\bar{G}^1_{10} + \bar{G}^1_{01})y^a. \tag{4.2}$$

Note that the right side of (4.2) is independent of s. If $\bar{r}_{10} \neq 0$, then 2Ψ is independent of s. In this case, we can express 2Ψ as $2\Psi = \frac{k_1}{1+k_1 b^2}$ where k_1 is a number independent of s. By Lemma 2.2, ϕ is given by

$$\phi = \epsilon s + \sqrt{1 + k_1 s^2},$$

where ϵ is a number independent of s. This case is excluded in the theorem. Thus we conclude that $\bar{r}_{10} = 0$, that is,

$$r_{1a} = 0. \tag{4.3}$$

By assumption, β is not parallel with respect to α. Thus $(\bar{r}_{00}, r_{11}) \neq (0,0)$. By taking a special vector (y^a), we get from (4.1) that

$$2\Psi = \frac{\lambda s^2 + \mu(b^2 - s^2)}{\delta s^2 + \eta(b^2 - s^2)}, \tag{4.4}$$

where λ, μ, δ and η are numbers with $(\delta, \eta) \neq (0,0)$. In fact, δ is a multiple of r_{11}. Thus we may let

$$r_{11} = 2b^2 \delta \tau', \tag{4.5}$$

where τ' is a non-zero number.

We claim that

$$\lambda\eta - \delta\mu \neq 0.$$

If this is not true, then $(\lambda, \mu) = k(\delta, \eta)$, then $2\Psi = k$. Let $k = k_1/(1 + k_1 b^2)$. By Lemma 2.2, ϕ is given by

$$\phi = \epsilon s + \sqrt{1 + k_1 s^2}.$$

Thus F is of Randers type. This case is excluded in our assumption of Theorem 1.1. Therefore, we conclude that $\lambda\eta - \delta\mu \neq 0$.

Plugging (4.4) into (4.1) yields

$$b\lambda r_{11} - (G^1_{11} - B)\delta = 0 \tag{4.6}$$

$$b\mu\bar{r}_{00} - \eta\bar{G}^1_{00} = 0 \tag{4.7}$$

$$b\lambda\bar{r}_{00} - \delta\bar{G}^1_{00} = \Big\{\eta(G^1_{11} - B) - \mu b r_{11}\Big\}\bar{\alpha}^2. \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$\bar{r}_{ab} = \frac{\eta[\eta(G^1_{11} - B) - b\mu r_{11}]}{b(\lambda\eta - \delta\mu)}\delta_{ab} \tag{4.9}$$

$$G^1_{ab} = \frac{\mu[\eta(G^1_{11} - B) - b\mu r_{11}]}{\lambda\eta - \delta\mu}\delta_{ab}. \tag{4.10}$$

If $\delta \neq 0$, then it follows from (4.6) that

$$G^1_{11} - B = 2b^3\lambda\tau'.$$

Plugging them into (4.9) and (4.10), we obtain

$$\bar{r}_{ab} = 2b^2\eta\tau'\bar{\alpha}^2 \tag{4.11}$$

$$G^1_{ab} = 2b^3\mu\tau'\bar{\alpha}^2. \tag{4.12}$$

If $\delta = 0$, then $r_{11} = 0$, $\eta \neq 0$ and $\lambda \neq 0$. Let τ' such that

$$G^1_{11} - B = 2b^3\lambda\tau'.$$

It follows from (4.8) that

$$\bar{r}_{00} = 2b^2\eta\bar{\alpha}^2. \tag{4.13}$$

By (4.7) and (4.13), we get

$$\bar{G}^1_{00} = 2b^3\mu\tau'\bar{\alpha}^2. \tag{4.14}$$

In virtue of (4.1), (4.5) and (4.13), we get

$$r_{ij} = 2\tau'\Big\{\delta b^2 b_i b_j + \eta(b^2 a_{ij} - b_i b_j)\Big\}. \tag{4.15}$$

Now we deal with the ODE (4.4). In [8] the third author proved the following fact

Lemma 4.1 *Assume that* $\phi = \phi(s) \neq \epsilon s + \sqrt{1 + k_1 s^2}$ *with* $\phi(0) = 1$ *and* $b \neq 0$ *satisfies*

$$2\Psi = \frac{\lambda s^2 + \mu(b^2 - s^2)}{\delta s^2 + \eta(b^2 - s^2)}.$$

Then

$$\Big\{1 + (k_1 + k_2 s^2)s^2 + k_3 s^2\Big\}\phi''(s) = (k_1 + k_2 s^2)\Big\{\phi(s) - s\phi'(s)\Big\}, \tag{4.16}$$

where $k_1 = \phi''(0)$, k_2 *and* k_3 *are constants depending on* $\phi''(0)$, $\phi^{(3)}(0)$ *and* $\phi^{(4)}(0)$. *Moreover,*

$$\mu = k_1\epsilon, \quad \eta = (1 + k_1 b^2)\epsilon, \quad \lambda = (k_1 + k_2 b^2)\epsilon,$$

$$\delta = (1 + k_1 b^2 + k_2 b^4 + k_3 b^2)\epsilon.$$

By Lemma 4.1, we can rewrite (4.15) as follows,

$$r_{ij} = 2\tau\Big\{(1 + k_1 b^2)a_{ij} + (k_2 b^2 + k_3)b_i b_j\Big\},$$

where $\tau = b^2\epsilon\tau'$ is a scalar function. This proves Theorem 1.1. Q.E.D.

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