# On a Class of Projectively Flat Finsler Metrics with Constant Flag Curvature

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**Abstract**

In this paper, we study a class of Finsler metrics defined by a Riemannian metric and a 1-form. We classify those projectively flat with constant flag curvature.

## 1 Introduction

One of important problems in Finsler geometry is to study and characterize Finsler metrics of constant flag curvature. Another problem is to study and characterize projectively flat Finsler metrics on an open domain in $R^n$. The later is the famous Hilbert's Fourth Problem in the regular case. In Riemannian geometry, these two problems are essentially same. The Beltrami theorem tells us that a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. However, there are locally projectively flat Finsler metrics which are not of constant flag curvature; and there are Finsler metrics of constant flag curvature which are not locally projectively flat. In [7], we have given the Taylor extensions at the origin $0 \in R^n$ for $x$-analytic projectively flat metrics $F = F(x,y)$ of constant flag curvature $K$. For $K \leq 0$, we construct such metrics nearby the origin in $R^n$ using algebraic equations for any given data $F|_{x=0} = \psi(y)$ and $F_{x^k}y^k/(2F)|_{x=0} = \varphi(y)$. In particular, for $K = -\frac{1}{4}$, $\psi = |y|$ and $\varphi = \frac{1}{2}|y|$, we get the Funk metric $\Theta$ on the unit ball $B^n \subset R^n$:

$$\Theta = \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x,y\rangle^2}}{1-|x|^2} + \frac{\langle x,y\rangle}{1-|x|^2}, \tag{1.1}$$

where $y \in T_x\mathrm{B}^n \approx R^n$. For $K = 0$, $\varphi = |y|$ and $\psi = |y|$, we get Berwald's metric

$$B = \frac{(\sqrt{(1-|x|^2)|y|^2 + \langle x,y\rangle^2} + \langle x,y\rangle)^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)|y|^2 + \langle x,y\rangle^2}},$$

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where $y \in T_x\mathrm{B}^n \approx R^n$. The Funk metric and Berwald's metric are related and they can be expressed in the form

$$\Theta = \bar{\alpha} + \bar{\beta}, \qquad B = \frac{(\tilde{\alpha} + \tilde{\beta})^2}{\tilde{\alpha}}, \tag{1.2}$$

where

$$\bar{\alpha} := \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}{1-|x|^2}, \qquad \bar{\beta} := \frac{\langle x, y\rangle}{1-|x|^2},$$

$$\tilde{\alpha} := \lambda\bar{\alpha}, \qquad \tilde{\beta} := \lambda\bar{\beta}, \qquad \lambda := \frac{1}{1-|x|^2},$$

In [6], [3] and [10], we have classified projectively flat metrics in the form $F = \alpha + \beta$ or $F = (\alpha+\beta)^2/\alpha$ with constant flag curvature .

The above discussion leads to the study of the following function $F$ defined by a Riemannian metric $\alpha$ and a 1-form $\beta$,

$$F = \alpha\phi(s), \qquad s = \frac{\beta}{\alpha}, \tag{1.3}$$

where $\phi = \phi(s)$ satisfies certain condition such that $F$ is a (positive definite regular) Finsler metric. Finsler metrics in the form (1.3) are called $(\alpha, \beta)$-metrics.

Therefore it is a natural problem to classify $(\alpha, \beta)$-metrics of constant flag curvature.

**Theorem 1.1** *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on an open subset $\mathcal{U}$ in the n-dimensional Euclidean space $R^n$ $(n \geq 3)$, where $\alpha = \sqrt{a_{ij}y^iy^j}$ and $\beta = b_iy^i \neq 0$. Suppose that $db \neq 0$ everywhere or $b = constant$ on $\mathcal{U}$. Then $F$ is projectively flat with constant flag curvature $K$ if and only if one of the following holds*

*(i) $\alpha$ is projectively flat and $\beta$ is parallel with respect to $\alpha$;*

*(ii) $F = \sqrt{\alpha^2 + k\beta^2} + \epsilon\beta$ is projectively flat with constant flag curvature $K < 0$, where $k$ and $\epsilon \neq 0$ are constants;*

*(iii) $F = (\sqrt{\alpha^2 + k\beta^2} + \epsilon\beta)^2/\sqrt{\alpha^2 + k\beta^2}$ is projectively flat with $K = 0$, where $k$ and $\epsilon \neq 0$ are constants.*

It is a trivial fact that if $\alpha$ is locally projectively flat and $\beta$ is parallel, then $F = \alpha\phi(\beta/\alpha)$ is a projectively flat Berwald metric. Further, if the flag curvature $K = constant$, then it is either Riemannian ($K \neq 0$) or locally Minkowskian ($K = 0$). See [4].

The Finsler metric in Theorem 1.1 (ii) is of Randers type, i.e., $F = \bar{\alpha} + \bar{\beta}$, where $\bar{\alpha} := \sqrt{\alpha^2 + k\beta^2}$ and $\bar{\beta} := \epsilon\beta$. In [6], it is proved that a Finsler metric in the form $F = \bar{\alpha} + \bar{\beta}$ is projectively flat with constant flag curvature if and only if it is locally Minkowskian or it is locally isometric to a generalized Funk metric $F = c(\bar{\alpha} + \bar{\beta})$ on the unit ball $B^n \subset R^n$, where $c > 0$ is a constant, and

$$\bar{\alpha} : = \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}{1-|x|^2} \tag{1.4}$$

$$\bar{\beta} : = \pm\Big\{\frac{\langle x, y\rangle}{1-|x|^2} + \frac{\langle a, y\rangle}{1+\langle a, x\rangle}\Big\}, \tag{1.5}$$

where $a \in R^n$ is a constant vector.

The Finsler metric in Theorem 1.1 (iii) is in the form $F = (\tilde{\alpha} + \tilde{\beta})^2/\tilde{\alpha}$, where $\tilde{\alpha} := \sqrt{\alpha^2 + k\beta^2}$ and $\tilde{\beta} := \epsilon\beta$. In [3] and [10], it is proved that a non-Minkowkian metric $F = (\tilde{\alpha} + \tilde{\beta})^2/\tilde{\alpha}$ is projectively flat with $K = 0$ if and only if it is, after scaling on $x$, locally isometric to a metric $F = c(\tilde{\alpha} + \tilde{\beta})^2/\tilde{\alpha}$ on the unit ball $B^n \subset R^n$, where $c =$ *constant*, $\tilde{\alpha} = \lambda\bar{\alpha}$ and $\tilde{\beta} = \lambda\bar{\beta}$, where $\bar{\alpha}$ and $\bar{\beta}$ are given in (1.4) and (1.5), and

$$\lambda := \frac{(1+\langle a, x\rangle)^2}{1-|x|^2}.$$

Theorem 1.1 tells us that there is no other types of $(\alpha, \beta)$-metrics which are locally projectively flat with constant flag curvature. However, the following problem is still open:

*Is there any metric $F = (\alpha + \beta)^2/\alpha$ of constant flag curvature which is not locally projectively flat?*

## 2 Preliminaries

A Finsler metric $F = F(x, y)$ on an open domain $\mathcal{U} \subset R^n$ is said to be *projectively flat* in $\mathcal{U}$ if all geodesics are straight lines. This is equivalent to $G^i = P(x, y)y^i$, where $G^i = G^i(x, y)$ are the geodesic coefficients of $F$, which are given by

$$G^i = \frac{1}{4}g^{il}\Big\{[F^i]_{x^m y^l}y^m - [F^2]_{x^l}\Big\}.$$

In this case the flag curvature $K$ is a scalar function on $T\mathcal{U}$ given by

$$K = \frac{P^2 - P_{x^m}y^m}{F^2}. \tag{2.1}$$

By definition, an ($\alpha$,$\beta$)-metric is a Finsler metric expressed in the following form,

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha},$$

where $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form with $\|\beta_x\|_\alpha < b_o$, $x \in M$. The function $\phi = \phi(s)$ is a $C^\infty$ positive function on an open interval $(-b_o, b_o)$ satisfying

$$\phi(0) = 1, \quad \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \le b < b_o$$

that guarantees that $F = \alpha\phi(\beta/\alpha)$ is a regular positive definite Finsler metric [5]. $(\alpha, \beta)$-metrics are "computable" Finsler metrics as we expressed above although the computation is sometimes not easy. Let $G^i_\alpha$ denote the spray coefficients of $\alpha$ given by

$$G^i_\alpha = \frac{a^{il}}{4}\Big\{[\alpha^2]_{x^k y^l} y^k - [\alpha^2]_{x^l}\Big\},$$

where $(a^{ij}) := (a_{ij})^{-1}$. We have the following formula for $G^i$.

$$G^i = G^i_\alpha + \alpha Q s^i{}_0 + \alpha^{-1}\Theta\Big(-2\alpha Q s_0 + r_{00}\Big)y^i + \Psi\Big(-2\alpha Q s_0 + r_{00}\Big)b^i, \quad (2.2)$$

where

$$\begin{aligned} Q &= \frac{\phi'}{\phi - s\phi'} \\ \Theta &= \frac{\phi - s\phi'}{2\Big((\phi - s\phi') + (b^2 - s^2)\phi''\Big)} \cdot \frac{\phi'}{\phi} - s\Psi \\ \Psi &= \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}. \end{aligned}$$

The formula (2.2) is given in [5] and [8]. A different version of (2.2) is given in [1] and [2].

From (2.2), it is easy to see that if $\alpha$ is projectively flat ($G^i_\alpha = \xi y^i$) and $\beta$ is parallel with respect to $\alpha$ ($r_{ij} = 0, s_{ij} = 0$), then $G^i = G^i_\alpha = \xi y^i$. Thus $F = \alpha\phi(\beta/\alpha)$ is a projectively flat Berwald metric. If $K \neq 0$, then $F$ is Riemannian by a Numata's theorem [4]. If $K = 0$, then $F$ is locally Minkowskian.

Recently, we have proved the following

**Theorem 2.1** ([9]) *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on an open subset $\mathcal{U}$ in the $n$-dimensional Euclidean space $R^n$ ($n \geq 3$), where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i \neq 0$. Suppose that the following conditions: (a) $\beta$ is not parallel with respect to $\alpha$, (b) $F$ is not of Randers type, and (c) $db \neq 0$ everywhere or $b =$ constant on $\mathcal{U}$. Then $F$ is projectively flat on $\mathcal{U}$ if and only if the function $\phi = \phi(s)$ satisfies*

$$\Big\{1 + (k_1 + k_2 s^2)s^2 + k_3 s^2\Big\}\phi''(s) = (k_1 + k_2 s^2)\Big\{\phi(s) - s\phi'(s)\Big\}, \quad (2.3)$$

$$b_{i|j} = 2\tau\Big\{(1 + k_1 b^2)a_{ij} + (k_2 b^2 + k_3)b_i b_j\Big\}, \quad (2.4)$$

$$G^i_\alpha = \xi y^i - \tau\Big(k_1\alpha^2 + k_2\beta^2\Big)b^i, \quad (2.5)$$

*where $\tau = \tau(x)$ is a scalar function on $\mathcal{U}$ and $k_1, k_2$ and $k_3$ are constants.*

Note that if a function $\phi = \phi(s)$ satisfies (2.3) with $(k_2, k_3) = 0$, then $\phi = a_1 s + \sqrt{1 + k_1 s}$, where $a_1$ is a constant. Thus $F = \alpha\phi(\beta/\alpha)$ is of Randers type. This case is excluded in the assumption of Theorem 2.1.

More general, we have the following

**Lemma 2.2** *If $k_2 = k_1k_3$ in (2.3), then $\phi(s) = a_1 s + \sqrt{1 + k_1 s^2}$, where is a constant.*

*Proof*: Substituting $k_2 = k_1k_3$ in (2.3), we get

$$(1 + k_1 s^2)(1 + k_3 s^2)\phi''(s) = k_1(1 + k_3 s^2)\Big\{\phi(s) - s\phi'(s)\Big\}.$$

Since $1 + k_3 s^2 \neq 0$ for $s$ close to zero, we get

$$(1 + k_1 s^2)\phi''(s) = k_1\Big\{\phi(s) - s\phi'(s)\Big\}.$$

The general solution of the above equation is $\phi = c_1 s + c_2\sqrt{1 + k_1 s^2}$. Q.E.D.

The following lemma is trivial. One can verify it by a direct computation.

**Lemma 2.3** *If $k_2 = k_1k_3 + \frac{6}{25}(k_1 - k_3)^2$ in (2.3), then*

$$\phi = a_1 s + \sqrt{1 + \frac{1}{5}(3k_1 + 2k_3)s^2} + \frac{\frac{1}{5}(k_1 - k_3)s^2}{\sqrt{1 + \frac{1}{5}(3k_1 + 2k_3)s^2}}.$$

The solution of (2.3) depends only on $a_1 = \phi'(0)$ (we always assume that $\phi(0) = 1$). The coefficients of $s^k$, $k \geq 2$, in the Taylor expansion of $\phi$ at $s = 0$ can be uniquely determined by $k_1, k_2$ and $k_3$.

**Lemma 2.4** *If*

$$\phi(s) = 1 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4 + a_5 s^5 + a_6 s^6 + a_7 s^7 + a_8 s^8 + o(s^8)$$

*satisfies (2.3), then*

$$a_3 = 0, \quad a_5 = 0, \quad a_7 = 0,$$

*and*

$$\begin{aligned}
a_2 &= \frac{k_1}{2},\\
a_4 &= \frac{1}{12}(k_2 - k_1k_3) - \frac{1}{8}k_1^2,\\
a_6 &= -\frac{11}{120}(k_1 + \frac{4}{11}k_3)(k_2 - k_1k_3) + \frac{1}{16}k_1^3\\
a_8 &= \frac{1}{56}(k_2 - k_1k_3)(\frac{61}{12}k_1^2 + k_3^2) - \frac{5}{224}k_2^2 + \frac{31}{336}k_1k_2k_3 - \frac{47}{672}k_1^2k_3^2 - \frac{5}{128}k_1^4.
\end{aligned}$$

As matter of fact, for any solution $\phi(s)$ of (2.3), the function $\phi(s) - \phi'(0)s$ is even.

## 3 Projectively flat $(\alpha,\beta)$-metrics with $K = constant$

Our proof of Theorem 1.1 is given in two steps. In this section we shall show that $K = 0$. In the following section, we shall determine the function $\phi$.

**Lemma 3.1** *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an $(\alpha,\beta)$-metric on an open subset $\mathcal{U} \subset R^n$ $(n \geq 3)$, where $\alpha = \sqrt{a_{ij}y^iy^j}$ and $\beta = b_iy^i \neq 0$. Suppose that $F$ is not of Randers type and $db \neq 0$ everywhere or $b = constant$ on $\mathcal{U}$. If $F$ is projectively flat with constant flag curvature $K$, then $K = 0$.*

*Proof*: If $\beta$ is parallel with respect to $\alpha$, then

$$G^i = G^i_\alpha. \tag{3.1}$$

Since $F$ is projectively flat,

$$G^i = \theta y^i.$$

By (3.1),

$$G^i_\alpha = \theta y^i.$$

We conclude that $\alpha$ is projectively flat. Moreover, $\xi$ is a 1-form. By Beltrami theorem, $\alpha$ is of constant sectional curvature $\kappa$,

$$\frac{\theta^2 - \theta_{x^k}y^k}{\alpha^2} = \kappa.$$

On the other hand, the flag curvature of $F$ is given by

$$K = \frac{\theta^2 - \theta_{x^k}y^k}{F^2} = \kappa\frac{\alpha^2}{F^2}.$$

If $K \neq 0$, then

$$F^2 = \frac{\kappa}{K}\alpha^2.$$

That is, $F$ is Riemannian. This case is excluded from assumption. Thus we must have that $K = 0$.

Now we assume that $\beta$ is not parallel with respect to $\alpha$. By Theorem 2.1, $\phi$, $\alpha$ and $\beta$ satisfy (2.3), (2.4) and (2.5). It is easy to get that

$$\begin{aligned} s_{ij} &= 0 \\ r_{00} &= 2\tau\Big\{1 + (k_1 + k_2s^2)b^2 + k_3s^2\Big\}\alpha^2, \\ \Psi &= \frac{k_1 + k_2s^2}{2[1 + (k_1 + k_2s^2)b^2 + k_3s^2]} \\ \Theta &= \frac{1 + (k_1 + k_2s^2)s^2 + k_3s^2}{2[1 + (k_1 + k_2s^2)b^2 + k_3s^2]}\frac{\phi'}{\phi} - s\Psi. \end{aligned}$$

By (2.2) we get

$$G^i = Py^i, \qquad P = \xi + \tau\alpha\Xi(s),$$

where $\xi = \xi_i y^i$, $\tau = \tau(x)$ and

$$\Xi := (1 + (k_1 + k_2 s^2)s^2 + k_3 s^2)\frac{\phi'}{\phi} - (k_1 + k_2 s^2)s.$$

By (2.4) and (2.5), we get

$$\begin{aligned} \alpha_{x^m} y^m &= 2\alpha\Big\{\xi - \tau\alpha(k_1 + k_2 s^2)s\Big\}, \\ s_{x^m} y^m &= 2\tau\alpha\Big\{1 + (k_1 + k_2 s^2)s^2 + k_3 s^2\Big\}, \end{aligned}$$

Using the above identities and (2.1), we obtain the following equation:

$$K\alpha^2\phi^2 = \xi^2 - \xi_{x_m} y^m + \tau^2\alpha^2\Xi - \tau_{x_m} y^m \Xi^2 + 2\tau^2\alpha^2\Gamma, \tag{3.2}$$

where

$$\Gamma := (k_1 + k_2 s^2)s\Xi - \Big\{1 + (k_1 + k_2 s^2)s^2 + k_3 s^2\Big\}\Xi_s.$$

Note that the function $\Xi$ and $\phi$ in (3.2) are functions of $s = \beta/\alpha$. Since the dependence of $\Xi$ and $\phi$ on $s$ is unclear, it is difficult to get further information on $K$ or $\phi$ directly from (3.2). To overcome this problem, we choose a special coordinate system at a point as in [9]. Fix an arbitrary point $x_o \in \mathcal{U} \subset R^n$. Make a change of coordinates: $(s, y^a) \to (y^i)$ by

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad y^a = y^a,$$

where $\bar{\alpha} := \sqrt{\sum_{a=2}^n (y^a)^2}$. Then

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}}\bar{\alpha}.$$

And

$$\xi = \frac{s\xi_1}{\sqrt{b^2 - s^2}}\bar{\alpha} + \bar{\xi}_0, \quad \tau_{x^m} y^m = \frac{s\tau_1}{\sqrt{b^2 - s^2}}\bar{\alpha} + \bar{\tau}_0,$$

where $\bar{\xi}_0 := \xi_a y^a$, $\bar{\tau}_0 := \tau_{x^a} y^a$. Let

$$\xi_{ij} := \frac{1}{2}(\frac{\partial \xi_i}{\partial x^j} + \frac{\partial \xi_j}{\partial x^i}).$$

Then

$$\xi_{x^m} y^m = \xi_{ij} y^i y^j = \frac{s^2 \xi_{11}}{b^2 - s^2}\bar{\alpha}^2 + \frac{2s\bar{\xi}_{10}}{\sqrt{b^2 - s^2}}\bar{\alpha} + \bar{\xi}_{00}.$$

where $\bar{\xi}_{10} := \xi_{1a} y^a$, and $\bar{\xi}_{00} := \xi_{ab} y^a y^b$. By above identities, we obtain from (3.2) that

$$\begin{aligned} \frac{Kb^2}{b^2 - s^2}\bar{\alpha}^2\phi^2 \quad = \quad & \frac{1}{\sqrt{b^2 - s^2}}\Big\{2s\xi_1\bar{\xi}_0 - 2s\bar{\xi}_{10} - \bar{\tau}_0 b\Xi\Big\}\bar{\alpha} + \bar{\xi}_0^2 - \bar{\xi}_{00} \\ & + \frac{1}{b^2 - s^2}\Big\{s^2(\xi_1^2 - \xi_{11}) + \tau^2 b^2 \Xi^2 - \tau_1 bs\Xi + 2\tau^2 b^2 \Gamma\Big\}\bar{\alpha}^2. \end{aligned}$$

It is equivalent to the two following equations

$$2s(\xi_1\bar{\xi}_0 - \bar{\xi}_{10}) - \bar{\tau}_0 b\Xi = 0, \tag{3.3}$$

$$\begin{aligned}&\frac{1}{b^2-s^2}\Big\{s^2(\xi_1^2 - \xi_{11}) + \tau^2 b^2\Xi^2 - \tau_1 bs\Xi + 2\tau^2 b^2\Gamma - Kb^2\phi^2\Big\}\bar{\alpha}^2 \\ &+ \bar{\xi}_0^2 - \bar{\xi}_{00} = 0.\end{aligned} \tag{3.4}$$

By (3.3), one can show that $\bar{\tau}_0 = 0$ and $\xi_1\bar{\xi}_0 - \bar{\xi}_{10} = 0$. But we do not need this fact here. So the proof is omitted.

From (3.4) we get

$$s^2\mu + \tau^2 b^2\Xi^2 - \tau_1 bs\Xi + 2\tau^2 b^2\Gamma - Kb^2\phi^2 + \delta(b^2 - s^2) = 0, \tag{3.5}$$

$$\bar{\xi}_0^2 - \bar{\xi}_{00} = \delta\bar{\alpha}^2, \tag{3.6}$$

where $\mu := \xi_1^2 - \xi_{11}$ and $\delta$ are numbers independent of s. Plugging the expressions of $\Xi$ and $\Gamma$ into (3.5), we get

$$\begin{aligned}&\Big\{4(1 + (k_1 + k_2s^2)s^2 + k_3s^2)\tau^2 b^2 k_2 s^2 + s^2\mu + (b^2 - s^2)\delta \\ &- ((k_1 + k_2s^2)\tau^2 b^2 - \tau_1 b)s^2(k_1 + k_2s^2)\Big\} + \Big\{ - s\tau_1 b \\ &- 2\tau^2 b^2(s(k_1 + k_2s^2) + 2s^3k_2 + 2k_3 s)\Big\}(1 + (k_1 + k_2s^2)s^2 + k_3s^2)\phi'\phi(s) \\ &+ 3\tau^2 b^2(1 + (k_1 + k_2s^2)s^2 + k_3s^2)^2(\phi')^2 - Kb^2\phi^4 = 0.\end{aligned} \tag{3.7}$$

By Lemma 2.4, we can express $\phi$ as follows,

$$\phi = 1 + a_1 s + a_2 s^2 + a_4 s^4 + a_6 s^6 + a_8 s^8 + o(s^8),$$

where $a_2$, $a_4$, $a_6$ and $a_8$ are given in Lemma 2.4. Substituting the above Taylor expansion into (3.7) and compare the coefficients of $s^k$, $k = 0, 1, 2$ we obtain

$$\begin{aligned}\delta &= K - 3\tau^2 a_1^2, \\ \tau_1 a_1 &= 2b(2(k_1 - k_3) - 3a_1^2)\tau^2 a_1 - 2ba_1 K,\end{aligned}$$

and

$$\begin{aligned}\mu = &\Big[(2a_2 + 3a_1^2)b^2 + 1\Big]K \\ &-\Big[4(k_2 - k_1k_3) + 3(2k_3 - k_1)a_1^2 + 3a_1^4\Big]\tau^2 b^2 - 3\tau^2 a_1^2.\end{aligned}$$

Substituting the above expressions back into (3.7), we obtain

$$\begin{aligned}&\Big\{ - \phi^4 + (3s^2a_1^2 - 2k_2s^4 - k_1s^2 + 1)\phi^2 + 2D(s)s\phi'\phi\Big\}K \\ &- 3\Big\{a_1^4 s^2 + (1 + 2k_3s^2 + k_1s^2 + 2k_2s^4)a_1^2 - (k_1 + k_2s^2)^2s^2\Big\}\tau^2\phi^2 \\ &+ 6\Big\{sa_1^2 - s(k_1 + k_2s^2)\Big\}D(s)\tau^2\phi'\phi + 3\tau^2 D(s)^2\phi'^2 = 0,\end{aligned} \tag{3.8}$$

where $D(s) = 1 + (k_1 + k_2 s^2)s^2 + k_3 s^2$. Substituting the Taylor expansion of $\phi$ into (3.8), we get

$$\begin{aligned}
&2a_1\Big\{\Big[a_1^2 - (k_1 - k_3)\Big]K + \Big[-3a_1^4 + 3(k_1 - k_3)a_1^2 - 2(k_2 - k_1k_3)\Big]\tau^2\Big\}s^3 \\
&+ \Big\{\Big[2a_1^4 - 2(k_1 - k_3)a_1^2 - \frac{3}{2}(k_2 - k_1k_3)\Big]K \\
&- 3a_1^2\Big[a_1^4 - (k_1 - k_3)^2 + \frac{3}{2}(k_2 - k_1k_3)\Big]\tau^2\Big\}s^4 \\
&+ a_1\Big\{\Big[k_1a_1^2 - k_1(k_1 - k_3) - 2(k_2 - k_1k_3)\Big]K + \Big[-3k_1a_1^4 \\
&+ (-4(k_2 - k_1k_3) + 3k_1(k_1 - k_3))a_1^2 + \frac{2}{5}(3k_1 - 8k_3)(k_2 - k_1k_3)\Big]\tau^2\Big\}s^5 \\
&- \frac{1}{6}(k_2 - k_1k_3)\Big\{\Big[3a_1^2 + 2(k_1 - k_3)\Big]K + \Big[3a_1^4 - 8(k_2 - k_1k_3)\Big]\tau^2\Big\}s^6 \\
&+ o(s^7) = 0.
\end{aligned} \tag{3.9}$$

By assumption, $F$ is not of Randers type. Thus $k_2 - k_1k_3 \neq 0$ by Lemma 2.2.

If $a_1 = 0$, then (3.9) is reduced to

$$\begin{aligned}
&-\frac{3}{2}(k_2 - k_1k_3)Ks^4 \\
&-\frac{1}{3}(k_2 - k_1k_3)\Big\{(k_1 - k_3)K - 4(k_2 - k_1k_3)\tau^2\Big\}s^6 + o(s^7) = 0.
\end{aligned}$$

Then $(k_2 - k_1k_3)K = 0$. We conclude that $K = 0$.

If $a_1 \neq 0$, then from (3.9) we obtain

$$\Big[a_1^2 - (k_1 - k_3)\Big]K + \Big[-3a_1^4 + 3(k_1 - k_3)a_1^2 - 2(k_2 - k_1k_3)\Big]\tau^2 = 0, \tag{3.10}$$

$$\begin{aligned}
&\Big[2a_1^4 - 2(k_1 - k_3)a_1^2 - \frac{3}{2}(k_2 - k_1k_3)\Big]K \\
&- 3a_1^2\Big[a_1^4 - (k_1 - k_3)^2 + \frac{3}{2}(k_2 - k_1k_3)\Big]\tau^2 = 0,
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
&\Big[k_1a_1^2 - k_1(k_1 - k_3) - 2(k_2 - k_1k_3)\Big]K + \Big[-3k_1a_1^4 + (-4(k_2 - k_1k_3) \\
&+ 3k_1(k_1 - k_3))a_1^2 + \frac{2}{5}(3k_1 - 8k_3)(k_2 - k_1k_3)\Big]\tau^2 = 0,
\end{aligned} \tag{3.12}$$

$$\Big[3a_1^2 + 2(k_1 - k_3)\Big]K + \Big[3a_1^4 - 8(k_2 - k_1k_3)\Big]\tau^2 = 0. \tag{3.13}$$

By (3.10)×$k_1$−(3.12), we get

$$2(k_2 - k_1k_3)K + 4(k_2 - k_1k_3)\Big\{a_1^2 - \frac{4}{5}(k_1 - k_3)\Big\} = 0.$$

Then for $k_2 - k_1k_3 \neq 0$

$$K = 2\Big\{-a_1^2 + \frac{4}{5}(k_1 - k_3)\Big\}\tau^2. \tag{3.14}$$

By (3.10)×4−(3.13), we get

$$\Big\{a_1^2-6(k_1-k_3)\Big\}K+3a_1^2\Big\{-5a_1^2+4(k_1-k_3)\Big\}\tau^2=0. \tag{3.15}$$

Substitute (3.14) into (3.15), we can obtain $a_1^2=\frac{4}{5}(k_1-k_3)$ or $a_1^2=\frac{12}{17}(k_1-k_3)$.

If $a_1^2=\frac{4}{5}(k_1-k_3)$, from (3.14) we get $K=0$.

If $a_1^2=\frac{12}{17}(k_1-k_3)$, we get that $K=\frac{4}{15}a_1^2\tau^2$. Then from (3.10) and (3.11) we can get

$$(k_2-k_1k_3)K=\frac{41}{72}a_1^4K,$$

and

$$(k_2-k_1k_3)K=\frac{2015}{3528}a_1^4K$$

respectively. Thus $K=0$ for $k_2-k_1k_3\neq 0$. Q.E.D.

## 4 Projectively flat $(\alpha,\beta)$-metrics with $K=0$

In this section, we are going to determine the function $\phi$ under the assumotion that $K=0$.

**Lemma 4.1** *Let $F=\alpha\phi(s)$, $s=\beta/\alpha$, be an $(\alpha,\beta)$-metric on an open subset $\mathcal{U}\subset R^n$ $(n\geq 3)$, where $\alpha=\sqrt{a_{ij}y^iy^j}$ and $\beta=b_iy^i\neq 0$. Suppose that $F$ is not of Randers type, $\beta$ is not parallel with respect to $\alpha$ and $db\neq 0$ everywhere or $b=constant$ on $\mathcal{U}$. If $F$ is projectively flat with $K=0$, then*

$$\phi=\frac{(\sqrt{1+ks^2}+\epsilon s)^2}{\sqrt{1+ks^2}},$$

*where $k=\frac{1}{5}(3k_1+2k_3)$ and $\epsilon=\pm\frac{1}{\sqrt{5}}\sqrt{k_1-k_3}$.*

*Proof*: By assumption that $K=0$, (3.8) is reduced to

$$\begin{aligned}&3\Big\{-a_1^4s^2-(1+2k_3s^2+k_1s^2+2k_2s^4)a_1^2+(k_1+k_2s^2)^2s^2\Big\}\tau^2\phi^2\\&+6\Big\{sa_1^2-s(k_1+k_2s^2)\Big\}D(s)\tau^2\phi'\phi+3\tau^2D(s)^2\phi'^2=0,\end{aligned} \tag{4.1}$$

By assumption that $\beta$ is not parallel with respect to $\alpha$, we can see from (2.4) that $\tau\neq 0$.

Let $f=\frac{\phi'}{\phi}$, then (4.1) can be written as

$$\begin{aligned}&3\Big\{-a_1^4s^2-(1+2k_3s^2+k_1s^2+2k_2s^4)a_1^2+(k_1+k_2s^2)^2s^2\Big\}\\&+6\Big\{sa_1^2-s(k_1+k_2s^2)\Big\}D(s)\tau^2f+3D(s)^2f^2=0.\end{aligned} \tag{4.2}$$

This is a quadratic equation in $f$. We obtain

$$f(s)=\frac{k_2s^3+(k_1-a_1^2)s\pm\sqrt{a_1^2+(-a_1^2k_1+2a_1^4+2k_3a_1^2)s^2}}{k_2s^4+(k_1+k_3)s^2+1}. \tag{4.3}$$

On the other hand, $f$ satisfies

$$f' = \frac{k_1 + k_2 s^2}{D(s)}(1 - sf) - f^2, \tag{4.4}$$

for $\phi(s)$ satisfying (2.3).

We first claim that $a_1 \neq 0$. If $a_1 = 0$, then it follows from (4.2) that

$$f = \frac{s(k_1 + k_2 s^2)}{D(s)}.$$

Plugging it into (4.4) we get

$$-2s^2 \frac{k_1 k_3 - k_2}{D(s)} = 0$$

Thus $k_2 = k_1 k_3$. Then by Lemma 2.2, $F$ is of Randers type. This case is excluded in the assumption of Theorem 2.1. Thus $a_1 \neq 0$.

It follows from (4.4) that

$$\begin{aligned} & 3a_1^4 - 3(k_1 - k_3)a_1^2 + 2(k_2 - k_1 k_3) \\ & \pm \frac{a_1^2}{\sqrt{\Delta}} \Big\{ -4a_1^4 + 6(k_1 - k_3)a_1^2 - 2(k_1 - k_3)^2 - (k_2 - k_1 k_3) \Big\} = 0, \end{aligned}$$

where $\Delta := a_1^2 - 2a_1^2(k_1/2 - k_3 - a_1^2)s^2$. Then

$$\begin{aligned} & 3a_1^4 - 3(k_1 - k_3)a_1^2 + 2(k_2 - k_1 k_3) = 0, \\ & 2a_1^4 - 3(k_1 - k_3)a_1^2 + (k_1 - k_3)^2 + \frac{1}{2}(k_2 - k_1 k_3) = 0. \end{aligned}$$

From the above equations, we get

(a) $a_1^2 = k_1 - k_3$ and $k_2 = k_1 k_3$, or

(b) $a_1^2 = \frac{4}{5}(k_1 - k_3)$ and $k_2 = \frac{6}{25}(k_1 - k_3)^2 + k_1 k_3$.

In case (a), By Lemma 2.2, we get

$$\phi = a_1 s + \sqrt{1 + s^2 k_1}.$$

This is excluded from the assumption of Theorem 1.1.

In case (b), by Lemma 2.3, we get

$$\phi = a_1 s + \sqrt{1 + \frac{1}{5}(3k_1 + 2k_3)s^2} + \frac{\frac{1}{5}(k_1 - k_3)s^2}{\sqrt{1 + \frac{1}{5}(3k_1 + 2k_3)s^2}}.$$

Since $(a_1)^2 = \frac{4}{5}(k_1 - k_3)$, we get

$$\phi = \frac{(\sqrt{1 + ks^2} + \epsilon s)^2}{\sqrt{1 + ks^2}},$$

where

$$\epsilon := \pm \frac{1}{\sqrt{5}}(k_1 - k_3), \qquad k := \frac{1}{5}(3k_1 + 2k_3).$$

Q.E.D.

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