On Randers Metrics of Quadratic Riemann Curvature[*]

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Abstract

In this paper, we study Randers metrics with quadratic Riemann curvature as in the Riemannian case. We find equations that characterize R-quadratic Randers metrics. In particular, we show that R-quadratic Randers metrics must have constant S-curvature.

1 Introduction

Finsler metrics are Riemann metrics without quadratic restriction. For a Finsler metric $F = F(x,y)$, its locally minimizing curves are characterized by a system of differential equations:

$$\frac{d^2x^i}{dt^2} + 2G^i\Big(x, \frac{dx}{dt}\Big) = 0,$$

where the local functions $G^i = G^i(x,y)$ are called the *spray coefficients*. If $F = \sqrt{g_{ij}(x)y^iy^j}$ is Riemannian, then $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$ are quadratic in $y \in T_xM$. This quadratic property is crucial in the regularity of the exponential map $\exp_x : T_xM \to M$ at the origin of T_xM. Namely, $\exp_x$ is C^∞ at the origin $0 \in T_xM$ at any point x if and only if the spray coefficients of F are quadratic in $y \in T_xM$ at any point x. There are non-Riemannian metrics whose spray coefficients still have this quadratic property. Finsler metrics with this property are called *Berwald metrics*. It is known that every Berwald metric has the same geodesics as a Riemannian metric [11]. Thus Berwald metrics can be identified with Riemannian metrics at geodesic level. The Riemann curvature is a family of linear maps $\mathbf{R}_y = R^i{}_k \frac{\partial}{\partial x^i} \otimes dx^k : T_xM \to T_xM$, given by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - y^j\frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

From the above formula, one can see that if F is a Berwald metric, then $R^i{}_k = R^i{}_k(x,y)$ are quadratic in $y \in T_xM$. Finsler metrics with such curvature

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property are called *R-quadratic metrics*. The notion of R-quadratic metrics is weaker than that of Berwald metrics. If a Finsler metric of zero flag curvature, then it is R-quadratic. If in addition, it is not locally Minkowskian, then it is not Berwaldian. See Example 1.1 below.

Our goal is try to understand Finsler metrics whose Ricci/Riemann curvature has the quadratic property as Riemannian metrics. In general, it is a quite difficult problem. Thus we start with Randers metrics.

A Randers metric on a manifold is a Finsler metric in the form $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form with $\|\beta_x\|_\alpha := \sqrt{a^{ij}(x)b_i(x)b_j(x)} < 1$. By the main theorem in [7], any R-quadratic Finsler metric on a closed manifold must be a Landsberg metric. On the other hand, a Randers metric $F = \alpha + \beta$ is a Landsberg metric if and only if β is parallel. In this case, it must be a Berwald metric. Thus we get the following

Theorem 1.1 *Let $F = \alpha + \beta$ be a positively complete Randers metric on a manifold. Then F is R-quadratic if and only if it is a Berwald metric.*

Let $\nabla\beta = b_{i|j}y^i dx^j$ denote the covariant derivatives of β with respect to α.

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \qquad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \qquad s_j := b^i s_{ij}.$$

We denote ${s^i}_j := a^{ik}s_{kj}$. For a tensor with coefficients $T_{\dots ij}$ in local coordinates, we denote $T_{\dots i0} := T_{\dots ij}y^j$ and $T_{\dots 00} := T_{\dots ij}y^i y^j$, etc. We have the following

Theorem 1.2 *A Randers metric $F = \alpha + \beta$ on a manifold is R-quadratic if and only if*

$$r_{00} + 2s_0\beta = 2c(\alpha^2 - \beta^2), \tag{1}$$

$$2\alpha^2 {\Phi^i}_{0k} - \alpha^2 {\Phi^i}_{k0} - {\Phi^i}_{00}a_{jk}y^j = 0, \tag{2}$$

where ${\Phi^i}_{jk} := {s^i}_{j|k} - (2cs_j + c^2 b_j + s_m {s^m}_j)\delta^i_k$ *and* $c = constant$.

If a Randers metric $F = \alpha + \beta$ satisfies (1) and (2), then the Riemann curvature of F is given by

$${R^i}_k = {\bar{R}^i}_k + 3{s^i}_0 s_{k0} - ({s^i}_m {s^m}_k \alpha^2 - {s^i}_m {s^m}_0 a_{jk}y^j) + \Psi_0\delta^i_k - \Psi_k y^i \tag{3}$$

where $\Psi_k := 3c^2 a_{jk}y^j - \beta c^2 b_k + s_0 s_k + 2s_{0|k} - s_{k|0} - 6cs_{k0}$. Clearly, ${R^i}_k$ are quadratic in $y \in T_xM$.

In [1], Bacso-Matsumoto also study R-quadratic Randers metrics. But Theorem 1 in [1] is inconsist with Theorem 1.2 above. Bacso-Matsumoto's arguments are based on Matsumoto's paper on Randers metrics of constant curvature [5]. However, there is a gap in Matsumoto's arguments and the conclusion in [5] is wrong. The mistake is corrected in [2]. However, the sufficient condition in Theorem 1 in [1] is still true. See Corollary 1.3 below.

Corollary 1.3 *Let $F = \alpha + \beta$ be a Randers metric. If*

$$r_{ij} = 0, \qquad s_j = 0, \qquad s_{ij|k} = 0, \tag{4}$$

then F is R-quadratic.

It is difficult to solve the differential equations (1) and (2). Equation (1) is equivalent to that F has constant S-curvature, $\mathbf{S} = (n+1)cF$ [4]. We can express a Randers metric using navigation data as follows [8],

$$F = \frac{\sqrt{|y|_h^2 - (|W|_h^2|y|_h^2 - \langle y, W\rangle_h^2)}}{1 - |W|_h^2} - \frac{\langle y, W\rangle_h}{1 - |W|_h^2}, \tag{5}$$

where $h = \sqrt{h_{ij}(x)y^iy^j}$ is a Riemannian metric and $W = W^i\frac{\partial}{\partial x^i}$ is a vector field; $\|\cdot\|_h$ and $\langle\ ,\ \rangle_h$ denote the norm and inner product on T_xM induced by h. According to [10], the S-curvature condition $\mathbf{S} = (n+1)cF$ is equivalent to that W is homothetic, i.e., $W_{i;j} + W_{j;i} = -4ch_{ij}$, where $W_i := h_{ij}W^j$ and $W_{i;j}$ denotes the covariant derivative with respect to h. By taking a Killing vector field $W = xQ + b$ on the Euclidean space with $h = |y|$, we obtain a Randers metric with vanishing flag curvature and vanishing S-curvature. In particular, it is R-quadratic.

Example 1.1 *Consider the following Randers metric defined nearby the origin in R^n.*

$$F := \frac{\sqrt{|y|^2 - (|xQ|^2|y|^2 - \langle y, xQ\rangle^2)}}{1 - |xQ|^2} - \frac{\langle y, xQ\rangle}{1 - |xQ|^2},$$

where $Q = ({q_j}^i)$ is an anti-symmetric matrix. Then F satisfies (1) and (2) with $c = 0$. In fact, ${R^i}_k = 0$ but F is not a Berwald metric when $Q \neq 0$. This is a special example in the class of Randers metrics of zero flag curvature [3].

There are two weaker notions than R-quadratic metrics. The first one is the notion of Ricci-quadratic. We obtain a similar theorem that characterizes Ricci-quadratic Randers metrics. See Theorem 2.2 below. The second one is the notion of W-quadratic. We also obtain a similar theorem that characterizes W-quadratic Randers metrics. See Theorem 4.2 below.

2 Ricci-quadratic metrics

First let us recall a formula for the Riemann curvature of a Randers metric $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $\beta = b_i(x)y^i$. Let

$$q_{ij} := r_{im}{s^m}_j, \quad t_{ij} := s_{im}{s^m}_j, \quad t_j := b^it_{ij} = s_m{s^m}_j.$$

Here and thereafter, we use a_{ij} to raise and lower the indices of tensors defined by b_i and $b_{i|j}$. Let $y_k := a_{jk}y^j$. Thus $y_0 = \alpha^2$. We have the following

Lemma 2.1 *$F = \alpha + \beta$ be a Randers metric on a manifold M. Then the Riemann curvature of F is given by*

$$\begin{aligned} R^i{}_k &= \bar{R}^i{}_k + Ay^iy_k + By^ib_k + C\delta^i{}_k \\ &\quad + t^i{}_0y_k - \alpha^2t^i{}_k + 3s^i{}_0s_{k0} + \frac{\alpha^2}{F}t_ky^i - \frac{3}{F}s_0s_{k0}y^i \\ &\quad + \frac{2\alpha}{F}q_{k0}y^i - \frac{\alpha}{F}q_{0k}y^i + 2\alpha s^i{}_{0|k} - \alpha s^i{}_{k|0} - \frac{1}{\alpha}s^i{}_{0|0}y_k \\ &\quad + \frac{\alpha}{F}s_{k|0}y^i - \frac{2\alpha}{F}s_{0|k}y^i + \frac{1}{F}r_{00|k}y^i - \frac{1}{F}r_{k0|0}y^i, \end{aligned} \tag{6}$$

where

$$\begin{aligned} A &:= (\frac{2\alpha}{F^2} - \frac{1}{F})t_0 - (\frac{2}{F^2} + \frac{1}{F\alpha})q_{00} \\ &\quad + \frac{1}{F\alpha}s_{0|0} - \frac{3}{4F^3\alpha}(r_{00} - 2\alpha s_0)^2 + \frac{1}{2F^2\alpha}(r_{00|0} - 2\alpha s_{0|0}), \\ B &:= \frac{2\alpha^2}{F^2}t_0 - \frac{2\alpha}{F^2}q_{00} - \frac{3}{4F^3}(r_{00} - 2\alpha s_0)^2 + \frac{1}{2F^2}(r_{00|0} - 2\alpha s_{0|0}), \\ C &:= -\frac{2\alpha^2}{F}t_0 + \frac{2\alpha}{F}q_{00} + \frac{3}{4F^2}(r_{00} - 2\alpha s_0)^2 - \frac{1}{2F}(r_{00|0} - 2\alpha s_{0|0}). \end{aligned}$$

The formula (6) is not explicitly given in [2], but they have done all the computations.

By Lemma 2.1, we can easily get a formula for the Ricci curvature $\mathbf{Ric} = R^m_{\ m}$.

$$\begin{aligned} \mathbf{Ric} =& \frac{1}{4(\beta+\alpha)^2}\Big\{ -4t^k{}_k\alpha^4 + 8(-t^k{}_k\beta + s^k{}_{0|k})\alpha^3 - 8(n-1)t_0\alpha^3 \\ &+ \Big[4(n-1)(s_{0|0} + 2q_{00} - 2t_0\beta + 3s_0^2) \\ &+ 4(-t^k{}_k\beta^2 + \overline{\mathbf{Ric}} + 4s^k{}_{0|k}\beta - 2t_{00})\Big]\alpha^2 \\ &+ \Big[(n-1)(-2r_{00|0} - 12s_0r_{00} + 4\beta s_{0|0} + 8q_{00}\beta) \\ &+ 8s^k{}_{0|k}\beta^2 + 8\overline{\mathbf{Ric}} - 16t_{00}\beta\Big]\alpha \\ &- 2(n-1)r_{00|0}\beta + 3(n-1)r_{00}^2 + 4\overline{\mathbf{Ric}}\beta^2 - 8t_{00}\beta^2\Big\}, \end{aligned} \tag{7}$$

where $\overline{\mathbf{Ric}} := \bar{R}^m_{\ m}$ denotes the Ricci curvature of α.

Let

$$\begin{aligned} A_k : &= 2cs_k + c^2b_k + t_k + \frac{1}{2}c_k \\ \Psi_k : &= 3c^2y_k - c^2\beta b_k + 2\beta c_k - c_0b_k + s_0s_k + 2s_{0|k} - s_{k|0} - 6cs_{k0}. \end{aligned}$$

Theorem 2.2 *Let $F = \alpha + \beta$ be a Randers metric on an n-manifold. Then it is Ricci-quadratic if and only if*

$$r_{00} + 2s_0\beta = 2c(\alpha^2 - \beta^2) \tag{8}$$

$$s^k{}_{0|k} = (n-1)A_0, \tag{9}$$

where $c = c(x)$ is a scalar function. In this case,

$$\mathbf{Ric} = \overline{\mathbf{Ric}} - 2t_{00} - t^k{}_k\alpha^2 + (n-1)\Psi_0. \tag{10}$$

Proof: Assume that F is Ricci-quadratic, that is, $\mathbf{Ric}$ is quadratic in $y \in T_xM$. Then (7) is equivalent to the following two equations.

$$\begin{aligned} &2\Big[s^k{}_{0|k} - t^k{}_k\beta - (n-1)t_0\Big]\alpha^2 + (n-1)\Big[-\frac{1}{2}r_{00|0} + 2q_{00}\beta - 3s_0r_{00} + \beta s_{0|0}\Big] \\ &+ 2s^k{}_{0|k}\beta^2 - 2\mathbf{Ric}\beta + 2\overline{\mathbf{Ric}}\beta - 4t_{00}\beta = 0, \end{aligned} \tag{11}$$

$$\begin{aligned} &-t^k{}_k\alpha^4 + \Big[(n-1)(s_{0|0} + 2q_{00} - 2t_0\beta + 3s_0^2) - 2t_{00} + \overline{\mathbf{Ric}} + 4s^k{}_{0|k}\beta \\ &- \mathbf{Ric} - t^k{}_k\beta^2\Big]\alpha^2 + \frac{1}{4}(n-1)(-2r_{00|0}\beta + 3r_{00}^2) \\ &+ \overline{\mathbf{Ric}}\beta^2 - 2t_{00}\beta^2 - \mathbf{Ric}\beta^2 = 0. \end{aligned} \tag{12}$$

(11) × β − (12) yields

$$\begin{aligned} &-\frac{3(n-1)}{4}(r_{00} + 2s_0\beta)^2 - (\alpha^2 - \beta^2)\Big\{-t^k{}_k\alpha^2 + 2s^k{}_{0|k}\beta \\ &+ (n-1)(2q_{00} + s_{0|0} + 3s_0^2) - 2t_{00} + \overline{\mathbf{Ric}} - \mathbf{Ric}\Big\} = 0. \end{aligned} \tag{13}$$

Thus there exists some scalar function $c = c(x)$ such that (8) holds. By (8) we have

$$r_{00|0} = -2\beta s_{0|0} + 4s_0^2\beta + 8cs_0\beta^2 + (2c_0 - 4cs_0 - 8\beta c^2)(\alpha^2 - \beta^2), \tag{14}$$

$$q_{00} = -s_0^2 - t_0\beta - 2cs_0\beta. \tag{15}$$

Then the Ricci curvature becomes

$$\begin{aligned} \mathbf{Ric} =& \overline{\mathbf{Ric}} - 2t_{00} - t^k{}_k\alpha^2 + (n-1)(3c^2\alpha^2 + s_0^2 - c^2\beta^2 + \beta c_0 + s_{0|0}) \\ &+ 2\{s^k{}_{0|k} - (n-1)A_0\}\alpha. \end{aligned} \tag{16}$$

Since $\mathbf{Ric}$ is quadratic in y, we see that the coefficient of α must be zero, that is, (9) holds. Then (16) is reduced to (10). Q.E.D.

3 R-quadratic metrics

Now we can simplify the formula (6) for $R^i{}_k$ under the condition (8).

$$\begin{aligned} R^i{}_k =& \bar{R}^i{}_k + 3s^i{}_0s_{k0} - (t^i{}_k\alpha^2 - t^i{}_0y_k) + \Psi_0\delta^i_k - \Psi_ky^i \\ &+ \alpha^{-1}\Big\{(2\alpha^2\Phi^i{}_{0k} - \alpha^2\Phi^i{}_{k0} - \Phi^i{}_{00}y_k) + \frac{3}{2}(\alpha^2c_k - c_0y_k)y^i\Big\}, \end{aligned} \tag{17}$$

where ${\Phi^i}_{jk} := {s^i}_{j|k} - A_j\delta^i_k$.

By (17), we obtain the following

Lemma 3.1 *A Randers metric $F = \alpha + \beta$ is R-quadratic if and only if*

$$r_{00} + 2s_0\beta = 2c(\alpha^2 - \beta^2), \tag{18}$$

$$(2\alpha^2{\Phi^i}_{0k} - \alpha^2{\Phi^i}_{k0} - {\Phi^i}_{00}y_k) + \frac{3}{2}(\alpha^2 c_k - c_0 y_k)y^i = 0, \tag{19}$$

where $c = c(x)$ is a scalar function. In this case

$$R^i{}_k = \bar{R}^i{}_k + 3{s^i}_0 s_{k0} - ({t^i}_k\alpha^2 - {t^i}_0 y_k) + \Psi_0\delta^i_k - \Psi_k y^i, \tag{20}$$

We shall show that $c = c(x)$ in (18) is actually constant.

Lemma 3.2 *If $F = \alpha + \beta$ is R-quadratic, then $c = constant$.*

Proof: Contracting (19) with y_i yields

$$-(A_0 y_k - \alpha^2 A_k) - s_{0k|0} + \frac{3}{2}(\alpha^2 c_k - c_0 y_k) = 0. \tag{21}$$

We get

$${s^i}_{0|0} = (A_0 y^i - \alpha^2 A^i) - \frac{3}{2}(\alpha^2 c^i - c_0 y^i). \tag{22}$$

Plugging it into (20) yields

$$\alpha^2\Big\{(2{s^i}_{0|k} - {s^i}_{k|0}) - (2A_0\delta^i_k - A_k y^i - A^i y_k) + \frac{3}{2}(c_k y^i + c^i y_k)\Big\} = 3c_0 y_k y^i.$$

Clearly, $c_0 = 0$. Thus $c = constant$. Q.E.D.

4 W-quadratic metrics

Let $\mathbf{R}_y = R^i{}_k \frac{\partial}{\partial x^i} \otimes dx^k$ denote the Riemann curvature of a Finsler metric F. Let

$$A^i{}_k := R^i{}_k - R\delta^i_k, \qquad R := \frac{R^m_{\ m}}{n-1}.$$

Then the (projective) Weyl curvature $\mathbf{W}_y = W^i{}_k \frac{\partial}{\partial x^i} \otimes dx^k$ is defined by

$$W^i{}_k := A^i{}_k - \frac{1}{n+1}\frac{\partial A^m{}_k}{\partial y^m} y^i.$$

A Finsler metric with $W^i{}_k$ quadratic in y is said to be *W-quadratic*. Note that if $R^i{}_k$ are R-quadratic, then $W^i{}_k$ are quadratic in y. Namely, every R-quadratic Finsler metric must be W-quadratic. Thus it is a natural problem to study W-quadratic Finsler metrics.

First we recall a formula for the Weyl curvature.

Lemma 4.1 ([9]) *For a Randers metric $F = \alpha + \beta$,*

$$\begin{aligned} W^i{}_k &= \bar{W}^i{}_k + 3s^i{}_0 s_{k0} + t^i{}_0 y_k - \alpha^2 t^i{}_k \\ &\quad + \frac{1}{n-1}\Big\{(2t_{00} + \alpha^2 t^m{}_m)\delta^i_k - (2t_{k0} + t^m{}_m y_k) y^i\Big\} \\ &\quad + \alpha^{-1}\Big\{\Big(2\alpha^2 s^i{}_{0|k} - \alpha^2 s^i{}_{k|0} - s^i{}_{0|0} y_k\Big) \\ &\quad - \frac{1}{n-1}\Big(2\alpha^2 s^m{}_{0|m}\delta^i_k - s^m{}_{0|m} y_k y^i - \alpha^2 s^m{}_{k|m} y^i\Big)\Big\}, \end{aligned}$$

where $\bar{W}^i{}_k$ denote the Weyl curvature of α.

By Lemma 4.1, we immediately obtain

Theorem 4.2 *Let $F = \alpha + \beta$ be a Randers metric on an n-manifold. It is W-quadratic if and only if*

$$2\alpha^2 s^i{}_{0|k} - \alpha^2 s^i{}_{k|0} - s^i{}_{0|0} y_k = \frac{1}{n-1}\Big\{2\alpha^2 s^m{}_{0|m}\delta^i_k - s^m{}_{0|m} y_k y^i - \alpha^2 s^m{}_{k|m} y^i\Big\}. \quad (23)$$

We have the following corollaries.

Corollary 4.3 *Let $F = \alpha + \beta$ be a Randers metric on an n-manifold. If*

$$s_{ij|k} = \frac{1}{n-1}\Big\{a_{ik} s^m{}_{j|m} - a_{jk} s^m{}_{i|m}\Big\}, \quad (24)$$

then it is W-quadratic.

Proof: It follows from (24) that

$$\begin{aligned} s^i{}_{0|k} &= \frac{1}{n-1}\Big\{\delta^i_k s^m{}_{0|m} - y_k s^{mi}{}_{|m}\Big\} \\ s^i{}_{k|0} &= \frac{1}{n-1}\Big\{y^i s^m{}_{k|m} - y_k s^{mi}{}_{|m}\Big\} \\ s^i{}_{0|0} &= \frac{1}{n-1}\Big\{y^i s^m{}_{0|m} - \alpha^2 s^{mi}{}_{|m}\Big\} \end{aligned}$$

Then (23) holds. Q.E.D.

Corollary 4.4 *If (19) holds, then F is W-quadratic.*

Proof: It follows from (19) that

$$A_k = \frac{1}{n-1} s^m{}_{k|m}.$$

Then (19) can be written as (23). Q.E.D.

Finally we make the following

Conjecture 4.5 *Let $F = \alpha + \beta$ be a Randers metric on a closed manifold. If it is W-quadratic, then β is parallel.*

References

[1] S. Bacso and M. Matsumoto, *Randers spaces with the h-curvature tensor H dependent on position alone*, Publ. Math., Debrecen. **55** (1999),

[2] D. Bao and C. Robles, *On Randers metrics of constant curvature*, Rep. Math. Phys. **51**(2003), 9-42.

[3] D. Bao, C. Robles and Z. Shen, *Zermelo Navigation on Riemannian manifolds*, J. Diff. Geom. **66**(2004), 391-449.

[4] X. Chen and Z. Shen, *Randers metrics with special curvature properties*, Osaka J. of Math. **40**(2003), 87-101.

[5] M. Matsumoto, *Randers spaces of constant curvature*, Rep. on Math. Phys. **28** (1989), 249-261.

[6] C. Robles, *Einstein metrics of Randers type*, Ph.D. thesis, University of British Columbia, Canada, 2003.

[7] Z. Shen, *On R-quadratic Finsler spaces*, Publ. Math., Debrecen. **58** (2001), no. 1-2, 263-274.

[8] Z. Shen, *Finsler metrics with K=0 and S=0*, Canadian J. of Math. **55**(2003), no.1, 112-132.

[9] Z. Shen, *A characterization of Randers metrics of scalar flag curvature*, preprint.

[10] Z. Shen and H. Xing, *On Randers metrics of isotropic S-curvature*, Acta Mathematica Sinica, to apprear.

[11] Z. Szabó, *Ein Finslerscher Raum ist gerade dann von skalarer Krümmung, wenn seine Weyl sche Projektivkrümmüng verschwindet*, Acta Sci. Math. (Szeged) **39**(1977), 163-168.

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