

# On a Class of Weakly Landsberg Metrics

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## Abstract

In this paper, we discuss a class of Finsler metrics defined by a Riemannian metric and a 1-form on a manifold. We characterize weakly Landsberg metrics in this class and show that there exist weakly Landsberg metrics which are not Landsberg metrics in dimension greater than two.

## 1 Introduction

The study of Landsberg metrics in Finsler geometry has a long history. In late 20's last century, L. Berwald studied a class of Finsler metrics  $F = F(x, y)$  on a manifold  $M$ , whose geodesics are determined by second order ODEs similar to the Riemannian case. More precisely, the geodesics in local coordinates satisfy

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where  $G^i(x, y) = \frac{1}{2}\Gamma_{jk}^i(x)y^jy^k$  are quadratic in  $y = y^i\frac{\partial}{\partial x^i}|_x \in T_xM$ . Finsler metrics with this property are called *Berwald metrics*. It can be shown that Berwald manifolds are modeled on a single norm space, i.e., all the tangent spaces  $T_xM$  with the induced norm  $F_x$  are linearly isometric to each other. Intuitively speaking, if one assigns a color to each tangent norm space  $(T_xM, F_x)$ , depending on the geometric shape of the unit tangent sphere  $S_xM$ , then a Berwald manifold has uniform color.

There is a weaker notion of metrics defined by a non-Riemannian quantity,  $\mathbf{L} = L_{ijk}dx^i \otimes dx^j \otimes dx^k$ , on the slit tangent bundle  $TM \setminus \{0\}$ , where

$$L_{ijk} := -\frac{1}{2}FF_{y^m}[G^m]_{y^iy^jy^k}.$$

Finsler metrics with  $L_{ijk} = 0$  are called *Landsberg metrics*. Clearly, any Berwald metric is a Landsberg metric. It is known that on a Landsberg manifold, the tangent spaces  $T_xM$  with the induced Riemannian metric  $\hat{g}_x := g_{ij}(x, y)dy^i \otimes dy^j$ , where  $g_{ij}(x, y) := \frac{1}{2}[F^2]_{y^iy^j}(x, y)$ , are all isometric. It might be possible that

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two tangent Riemannian spaces  $(T_x M, \hat{g}_x)$  and  $(T_{x'} M, \hat{g}_{x'})$  are isometric, but  $(T_x M, F_x)$  and  $(T_{x'} M, F_{x'})$  are not linearly isometric. Thus a natural question arises:

*Is there a Landsberg metric which is not a Berwald metric?*

This question has a satisfied answer now. First, G. Asanov proved that his metrics arising from Finslerian General Relativity are actually Landsberg metrics but not Berwald metrics ([1], [2]). Shortly after Asanov's preprints appeared, the second author made a classification of Landsberg metrics defined by a Riemannian metric and a 1-form. He obtained a two-parameter family of Landsberg metrics including Asanov's examples ([11]).

On a Landsberg manifold, the volume function  $\text{Vol}(x)$  of the unit tangent sphere  $S_x M \subset (T_x M, \hat{g}_x)$  is a constant. The constancy of  $\text{Vol}(x)$  is required to establish a Gauss-Bonnet theorem for Finsler manifolds [3]. The volume function  $\text{Vol}(x)$  is closely related to a weaker non-Riemannian quantity,  $\mathbf{J} = J_k dx^k$ , where

$$J_k := g^{ij} L_{ijk}.$$

Finsler metrics with  $\mathbf{J} = 0$  are called *weakly Landsberg metrics*. Clearly, in dimension two, any weakly Landsberg metric must be a Landsberg metric. It has been shown that on a weakly Landsberg manifold, the volume function  $\text{Vol}(x)$  is a constant ([4]). Some rigidity problems also lead to weakly Landsberg manifolds. For example, for a closed Finsler manifold of nonpositive flag curvature, if the S-curvature is a constant, then it is weakly Landsbergian [10]. Apparently, weakly Landsberg manifolds deserve further investigation. Another natural question arises:

*Is there a weakly Landsberg metric which is not a Landsberg metric?*

To find answers to the above questions, we also consider Finsler metrics defined by a Riemannian metric and a 1-form as in [11], because that it is relatively easy to compute the geometric quantities of these metrics. Such Finsler metrics are expressed in the following form

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha},$$

where  $\alpha = \sqrt{a_{ij}y^i y^j}$  is a Riemannian metric,  $\beta = b_i y^i$  is a 1-form on an  $n$ -dimensional manifold  $M$  and  $\phi = \phi(s)$  is a positive  $C^\infty$  function on  $(-b_o, b_o)$ . We assume that

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad (|s| \leq b < b_o) \quad (1.1)$$

so that  $F = \alpha\phi(s)$ , where  $s = \beta/\alpha$  with  $b(x) := \|\beta_x\|_\alpha < b_o$ , is a positive definite Finsler metric on an open subset of  $M$ . For the above function  $\phi = \phi(s)$ , if we consider a 1-form  $\beta$  with  $\|\beta_x\|_\alpha := \sqrt{a^{ij}b_i b_j} \leq b_o$ . then  $F = \alpha\phi(\beta/\alpha)$  might

be singular at a point  $x$  with  $b(x) = b_o$ . Such metrics are called *almost regular*  $(\alpha, \beta)$ -metrics.

For a Riemannian metric  $\alpha = \sqrt{a_{ij}y^i y^j}$  and a 1-form  $\beta := b_i y^i$  on a manifold  $M$ , let  $\nabla\beta = b_{i|j}y^i dx^j$  denote the covariant derivative of  $\beta$  with respect to  $\alpha$ . Set

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}),$$

$$r_j := b^i r_{ij}, \quad s_j := b^i s_{ij}.$$

For a function  $\phi = \phi(s)$  satisfying (1.1), we let

$$Q := \frac{\phi'}{\phi - s\phi'}, \quad \Delta := 1 + sQ + (b^2 - s^2)Q'.$$

In this paper we prove the following

**Theorem 1.1** *Let  $F = \alpha\phi(\beta/\alpha)$  be an almost regular  $(\alpha, \beta)$ -metric on an  $n$ -dimensional manifold  $M$  ( $n \geq 3$ ), where  $\alpha = \sqrt{a_{ij}y^i y^j}$  and  $\beta = b_i y^i$ . Suppose  $\beta$  is not parallel with respect to  $\alpha$  and  $\phi \neq k_1\sqrt{1 + k_2 s^2}$  for any constants  $k_1$  and  $k_2$ . Let  $b(x) := \|\beta_x\|_\alpha \neq 0$ . Then  $F$  is a weakly Landsberg metric if and only if one of the following holds*

(i)  $\beta$  satisfies

$$r_{ij} = k(b^2 a_{ij} - b_i b_j), \quad s_{ij} = 0, \quad (1.2)$$

where  $k = k(x)$  is a number depending on  $x$  and  $\phi = \phi(s)$  satisfies

$$(Q - sQ')\{n\Delta + 1 + sQ\} + (b^2 - s^2)(1 + sQ)Q'' = \frac{\lambda}{\sqrt{b^2 - s^2}}\Delta^{\frac{3}{2}} \quad (1.3)$$

where  $\lambda$  is a constant.

(ii)  $\beta$  satisfies

$$r_j + s_j = 0, \quad (1.4)$$

and  $\phi = \phi(s)$  satisfies

$$(Q - sQ')\{n\Delta + 1 + sQ\} + (b^2 - s^2)(1 + sQ)Q'' = 0. \quad (1.5)$$

In both cases,  $b = b_o$ .

The proof of Theorem 1.1 is proved in Propositions 3.2 and 5.1 below.

The Finsler metric defined by  $\phi = 1 + s$  is a Randers metric  $F = \alpha + \beta$ . Note that  $\phi$  does not satisfy (1.3) for any number  $\lambda$ . Thus a Randers metric  $F = \alpha + \beta$  is a weakly Landsberg metric if and only if  $\beta$  is parallel with respect to  $\alpha$ . This has been proved in [9] and [12].

Note that (1.2) implies (1.4), and the length  $b(x) := \|\beta_x\|_\alpha$  satisfies that  $db(x) = 0$  at a point  $x \in M$  if and only if (1.4) holds at  $x$ . Thus  $b = \text{constant}$  on an open set  $\mathcal{U}$  if and only if  $\beta$  satisfies (1.4) on  $\mathcal{U}$ .

It has been shown that if  $\beta$  satisfies (1.4) and  $\phi = \phi(s)$  satisfies (1.5), then  $F = \alpha\phi(\beta/\alpha)$  has vanishing S-curvature. See [6] for more details and related results.

It is shown in [11] that in dimension  $n \geq 3$ ,  $F = \alpha\phi(\beta/\alpha)$  is a Landsberg metric if and only if  $\beta$  satisfies (1.2) and  $Q := \phi'/(\phi - s\phi')$  is given by

$$Q = q_0\sqrt{1 - (s/b)^2} + q_1s, \quad (1.6)$$

where  $q_0$  and  $q_1$  are constants, provided that  $\beta$  is not parallel with respect to  $\alpha$ . This leads to the discovery of almost regular Landsberg metrics which are not Berwald metrics. The first example is given by G. Asanov from a completely different approach. See [1] and [2]. His example can be described by  $Q = q_0\sqrt{1 - (s/b)^2}$  using  $\beta$  satisfying (1.2).

Let  $\beta$  satisfy (1.2). If  $Q = Q(s)$  satisfies (1.3), then  $F = \alpha\phi(\beta/\alpha)$  is a weakly Landsberg metric by Theorem 1.1. However, if

$$\lambda \neq \frac{nq_0b}{\sqrt{1 + q_1b^2}},$$

where  $q_0 = Q(0)$  and  $q_1 = Q'(0)$ , then  $Q = Q(s)$  can not be expressed by (1.6). Thus  $F = \alpha\phi(\beta/\alpha)$  is not a Landsberg metric, according to [11].

**Example 1.1** At a point  $x = (x^1, x^2, x^3) \in R^3$  and in the direction  $y = (y^1, y^2, y^3) \in T_xR^3$ , define  $\alpha = \alpha(x, y)$  and  $\beta = \beta(x, y)$  by

$$\begin{aligned} \alpha &: = \sqrt{(y^1)^2 + e^{2x^1}((y^2)^2 + (y^3)^2)} \\ \beta &: = y^1. \end{aligned}$$

Then  $\alpha$  and  $\beta$  satisfy (1.2) with  $b = \|\beta_x\|_\alpha \equiv 1$  and  $k = 1$ . Let  $\phi = \phi(s)$  satisfy (1.3) with  $b = 1$ . Then  $F = \alpha\phi(\beta/\alpha)$  is an almost regular weakly Landsberg metric. If

$$\lambda \neq \frac{nQ(0)}{\sqrt{1 + Q'(0)}},$$

then  $F$  is not a Landsberg metric.

One can also take  $\phi = \phi(s)$  such that  $Q = \phi'/(\phi - s\phi')$  satisfies (1.5). If  $Q(0) = q_0 \neq 0$ , then  $Q$  can not be expressed as (1.6) for any constant  $q_1$ . Thus for any non-parallel  $\beta$  with  $b(x) := \|\beta_x\|_\alpha = \text{constant} \neq 0$ ,  $F = \alpha\phi(\beta/\alpha)$  is a weakly Landsberg metric, but not a Landsberg metric.

**Example 1.2** Let  $\bar{F} = \alpha + \beta$  be the family of Randers metrics on  $S^3$  constructed in [5].  $\beta$  satisfies that  $r_{ij} = 0$  and  $s_j = 0$ . Thus for any  $C^\infty$  positive function  $\phi = \phi(s)$  satisfying (1.5), the  $(\alpha, \beta)$ -metric  $F := \alpha\phi(\beta/\alpha)$  is a weakly Landsberg metric,  $\mathbf{J} = 0$ . If  $\phi'(0) \neq 0$ , then  $F$  is not a Landsberg metric. According to [6], this metric has vanishing S-curvature,  $\mathbf{S} = 0$ .

In this paper, we also discuss the two-dimensional case. We shall prove the following

**Theorem 1.2** ( $n = 2$ ) *Let  $F = \alpha\phi(\beta/\alpha)$  be an almost regular  $(\alpha, \beta)$ -metric on a surface  $M$ , where  $\phi = \phi(s)$  is defined on  $(-b_o, b_o)$  such that  $\phi \neq k_1\sqrt{1+k_2s^2}$  for any constants  $k_1$  and  $k_2$ . Let  $b(x) := \|\beta_x\|_\alpha \neq 0$ . Then  $F$  is a Landsberg metric on  $M$  if and only if either  $\beta$  is parallel with respect to  $\alpha$ , or  $\beta$  has constant length,*

$$b = b_o.$$

and  $\phi$  is given by

$$\phi(s) = \exp \left[ \int \frac{Q}{1+sQ} ds \right], \quad (1.7)$$

where

$$Q := -\frac{s}{b^2} + \sqrt{1 - (s/b)^2} \left\{ q_0 + \frac{(1/b^2 + q_1)s}{\sqrt{1 - (s/b)^2} + \left( q_0 - \frac{\lambda}{2} \sqrt{1/b^2 + q_1} \right) s} \right\}, \quad (1.8)$$

where  $q_0, q_1$  and  $\lambda$  are constants. In fact,  $F$  is a Berwald metric.

One can verify that the function  $Q = Q(s)$  in (1.8) satisfies (1.3) when  $n = 2$ .

In 1973, Hashiguchi-Hōjō-Matsumoto wrote a paper on two-dimensional Landsberg  $(\alpha, \beta)$ -metrics [7], trying to find two-dimensional Landsberg metrics. Later on, they wrote another paper with the same title [8], pointing out an error in [7]. The main result in [8] is that if a two-dimensional Randers metric or Kropina metrics is a Landsberg metric, then it must be Berwaldian. In Theorem 1.2, we give a complete characterization of two-dimensional Landsberg  $(\alpha, \beta)$ -metrics.

## 2 Preliminaries

Consider a Finsler metric  $F = F(x, y)$  on an  $n$ -dimensional manifold  $M$ . Let  $G^i = G^i(x, y)$  denote the spray coefficients of  $F$  in a local coordinate system.

$$G^i = \frac{1}{4} g^{il} \left\{ [F^2]_{x^m y^l} y^m - [F^2]_{x^l} \right\}, \quad (2.1)$$

where  $g_{ij} := \frac{1}{2} [F^2]_{y^i y^j}$  and  $(g^{ij}) := (g_{ij})^{-1}$ . The *Landsberg curvature*  $\mathbf{L} = L_{ijk}(x, y) dx^i \otimes dx^j \otimes dx^k$  is a horizontal tensor on  $TM \setminus \{0\}$ , defined by

$$L_{ijk} := -\frac{1}{2} F F_{y^m} [G^m]_{y^i y^j} y^k.$$

Clearly, if  $F$  is a Berwald metric ( $G^i = \frac{1}{2} \Gamma_{jk}^i(x) y^j y^k$ ), then  $L_{ijk} = 0$ . The *mean Landsberg curvature*  $\mathbf{J} = J_k dx^k$  is defined by

$$J_k = g^{ij} L_{ijk}.$$

It is easy to see that in dimension  $n = 2$ ,  $J_k = 0$  if and only if  $L_{ijk} = 0$ . However, this is not true in higher dimensions for almost regular  $(\alpha, \beta)$ -metrics as we see in Theorem 1.1.

To understand the difference between Berwald metrics and Landsberg metrics, we consider an  $(\alpha, \beta)$ -metric,  $F = \alpha\phi(s)$ ,  $s = \beta/\alpha$ . Let  $r_{ij}, s_{ij}, r_i$  and  $s_i$  be defined as in Introduction. Let  $G^i = G^i(x, y)$  and  $\bar{G}^i = \bar{G}^i(x, y)$  denote the coefficients of  $F$  and  $\alpha$  respectively in the same coordinate system. By (2.1), we have the following identity.

$$G^i = \bar{G}^i + \alpha Q s^i_0 + \Theta \left\{ -2\alpha Q s_0 + r_{00} \right\} \left\{ \frac{y^i}{\alpha} + \frac{Q'}{Q - sQ'} b^i \right\}, \quad (2.2)$$

where

$$Q : = \frac{\phi'}{\phi - s\phi'}, \quad (2.3)$$

$$\Theta : = \frac{Q - sQ'}{2(1 + sQ + (b^2 - s^2)Q')}. \quad (2.4)$$

Using the above formula, the second author computed the Landsberg curvature in [11]. The Landsberg curvature of  $F$  is given by

$$L_{jkl} = \frac{-\rho}{6\alpha^5} \left\{ h_j h_k C_l + h_j h_l C_k + h_k h_l C_j + 3(E_j h_{kl} + E_k h_{jl} + E_l h_{jk}) \right\}, \quad (2.5)$$

where  $\rho = \phi(\phi - s\phi')$  and

$$h_j : = \alpha b_j - s y_j,$$

$$h_{jk} : = \alpha^2 a_{jk} - y_j y_k,$$

$$C_j : = (X_4 r_{00} + Y_4 \alpha s_0) h_j - 3Q'' D_j,$$

$$E_j : = (X_6 r_{00} + Y_6 \alpha s_0) h_j - (Q - sQ') D_j,$$

$$D_j : = \frac{\alpha^2}{\Delta} (\Delta s_{j0} + r_{j0} - Q \alpha s_j) - \frac{1}{\Delta} (r_{00} - Q \alpha s_0) y_j,$$

where

$$X_4 : = \frac{1}{2\Delta^2} \left\{ -2\Delta Q''' + 3[(Q - sQ') + (b^2 - s^2)Q'']Q'' \right\},$$

$$X_6 : = \frac{1}{2\Delta^2} \left\{ (Q - sQ')^2 + [2(s + b^2Q) - (b^2 - s^2)(Q - sQ')]Q'' \right\},$$

$$Y_4 : = -2Q X_4 + \frac{3Q'Q''}{\Delta},$$

$$Y_6 : = -2Q X_6 + \frac{(Q - sQ')Q'}{\Delta}.$$

**Lemma 2.1** *Let*

$$\Phi : = -(n\Delta + 1 + sQ)(Q - sQ') - (b^2 - s^2)(1 + sQ)Q''$$

$$\Psi_1 : = \sqrt{b^2 - s^2} \Delta^{1/2} \left[ \frac{\sqrt{b^2 - s^2} \Phi}{\Delta^{3/2}} \right]'$$

$$\Psi_2 : = 2(n + 1)(Q - sQ') + 3 \frac{\Phi}{\Delta}.$$

The mean Landsberg curvature is given by

$$\begin{aligned}
J_j = & \frac{-\Delta}{2\alpha^4} \left\{ \frac{2\alpha^2}{b^2 - s^2} \left[ \frac{\Phi}{\Delta} + (n+1)(Q - sQ') \right] (s_0 + r_0) h_j \right. \\
& + \frac{\alpha}{b^2 - s^2} \left[ \Psi_1 + s \frac{\Phi}{\Delta} \right] (r_{00} - 2\alpha Q s_0) h_j \\
& + \alpha \left[ -\alpha Q' s_0 h_j + \alpha Q (\alpha^2 s_j - y_j s_0) + \alpha^2 \Delta s_{j0} \right. \\
& \left. \left. + \alpha^2 (r_{j0} - 2\alpha Q s_j) - (r_{00} - 2\alpha Q s_0) y_j \right] \frac{\Phi}{\Delta} \right\}, \tag{2.6}
\end{aligned}$$

where  $s_0 := s_i y^i$ ,  $r_0 := r_i y^i$ ,  $r_{00} := r_{ij} y^i y^j$ ,  $r_{j0} := r_{jk} y^k$  and  $s_{j0} := s_{jk} y^k$ .

*Proof:* To compute the mean Landsberg curvature  $J_k := g^{ij} L_{ijk}$  using (2.5), we need the following formula for the inverse  $(g^{ij}) = (g_{ij})^{-1}$ . First we have

$$g_{kl} = \rho a_{kl} + \rho_0 b_k b_l - \alpha^{-1} \rho_1 (b_k y_l + b_l y_k) + s \alpha^{-2} \rho_1 y_k y_l,$$

where  $y_k := a_{kl} y^l$  and

$$\rho := \frac{\phi^2}{1 + sQ}, \quad \rho_0 := \frac{(Q' + Q^2)\phi^2}{(1 + sQ)^2}, \quad \rho_1 := \frac{(sQ' - Q)\phi^2}{(1 + sQ)^2}.$$

By elementary linear algebra, we get a formula for  $g^{jk}$ .

$$\begin{aligned}
g^{jk} = & \frac{1}{\rho} \left\{ a^{jk} - \frac{1}{\Upsilon} (\rho \rho_0 + s \rho_1 \rho_0 - \rho_1^2) b^j b^k \right. \\
& + \frac{1}{\Upsilon \alpha} (-s^2 \rho_0 \rho_1 + s \rho_1^2 - \rho_0 \rho) (b^j y^k + y^j b^k) \\
& \left. - \frac{1}{\Upsilon \alpha^2} (s \rho_1 (\rho + \rho_0 b^2) - \rho_1 b^2) y^j y^k \right\},
\end{aligned}$$

where  $\Upsilon := \rho(\rho + \rho_0 b^2 - s \rho_1) + \rho_1 (b^2 - s^2)(s \rho_0 - \rho_1) = \phi^4 \Delta / (1 + sQ)^2$ . Using  $L_{jkl} y^k = 0$  and  $\rho \rho_0 + s \rho_1 \rho_0 - \rho_1^2 = \phi^4 Q' / (1 + sQ)^2$  we get

$$J_l = \frac{1}{\rho} \left\{ a^{jk} - \frac{Q'}{\Delta} \right\} L_{jkl}.$$

We need

$$\begin{aligned}
a^{kl} L_{jkl} &= \frac{-\rho}{6\alpha^5} \left\{ 2\alpha C h_j + \alpha^2 (b^2 - s^2) C_j + 3(n+1)\alpha^2 E_j \right\} \\
b^k b^l L_{jkl} &= \frac{-\rho}{6\alpha^5} \left\{ \alpha^2 (b^2 - s^2)^2 C_j + 2\alpha ((b^2 - s^2) C + 3E) h_j + 3\alpha^2 (b^2 - s^2) E_j \right\}.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
J_j = & \frac{-\Delta}{6\alpha^4} \left\{ 2((1 + sQ)C - 3Q'E) h_j \right. \\
& \left. + \alpha (b^2 - s^2) (1 + sQ) C_j + 3\alpha (n\Delta + 1 + sQ) E_j \right\}, \tag{2.7}
\end{aligned}$$

where  $C := C_j b^j$  and  $E := E_j b^j$ . By further computation, one can easily express  $J_j$  by (2.6). Q.E.D.

Let

$$J := J_j b^j.$$

By Lemma 2.1, we get

**Lemma 2.2** *The quantity  $J$  is given by the following formulas*

$$J = -\frac{\Delta}{2\alpha^2} \left\{ \Psi_1(r_{00} - 2\alpha Q s_0) + \alpha \Psi_2(r_0 + s_0) \right\}. \quad (2.8)$$

We also need the following lemmas.

**Lemma 2.3** *If  $Q - sQ' = 0$  or  $Q = q_1 s$ , where  $q_1$  is a number independent of  $s$ , then*

$$\phi = k_1 \sqrt{1 + k_2 s^2},$$

where  $k_1$  and  $k_2$  are numbers independent of  $s$ .

*Proof:* Trivial. Q.E.D.

The following lemma is given in [11].

**Lemma 2.4** *Assume that  $\phi = \phi(s)$  satisfies*

$$db^2 s - (b^2 Q + \Delta s) = 0, \quad (2.9)$$

where  $d$  is a number independent of  $s$ . Then

$$Q = -\frac{q_1(b^2 - s^2)}{s} + \frac{db^2 - 1}{2s}, \quad (2.10)$$

where  $q_1$  is a number independent of  $s$ . Thus  $Q$  is either singular at  $s = 0$  or  $Q = q_1 s$ .

### 3 Sufficient Conditions

In this section, we are going to prove sufficient conditions for an almost regular  $(\alpha, \beta)$ -metric  $F = \alpha\phi(\beta/\alpha)$  to be weakly Landsbergian. These conditions are also valid in dimension two.

Let

$$\Psi_3 := \frac{s}{b^2 - s^2} \Psi_1 + \frac{b^2}{b^2 - s^2} \Psi_2.$$

We shall prove the following

**Proposition 3.1** *Let  $F = \alpha\phi(\beta/\alpha)$  be an almost regular  $(\alpha, \beta)$ -metric on a manifold. Suppose that  $\alpha = \sqrt{a_{ij}y^i y^j}$  and  $\beta = b_i y^i$  satisfy*

$$s_{ij} = 0, \quad (3.1)$$

$$r_{ij} = k(b^2 a_{ij} - b_i b_j) + c b_i b_j, \quad (3.2)$$

where  $k = k(x)$  and  $c = c(x)$  are scalar functions. Assume that  $\phi = \phi(s)$  satisfy the following ODE:

$$\Psi_1 k + s \Psi_3 c = 0 \quad (3.3)$$

Then  $F = \alpha\phi(\beta/\alpha)$  is a weakly Landsberg metric.

*Proof:* It follows from (3.1) and (3.2) that

$$\begin{aligned} r_0 &= c s b^2 \alpha \\ r_{j0} &= k(b^2 y_j - \beta b_j) + c \beta b_j \\ r_{00} &= \left\{ k(b^2 - s^2) + c s^2 \right\} \alpha^2. \end{aligned}$$

We can simplify (2.6) as follows.

$$J_j = -\frac{\Delta}{2\alpha} \left\{ \Psi_1 k + s \Psi_3 c \right\} h_j \quad (3.4)$$

Under our assumption, we always have  $J_j = 0$ .

Q.E.D.

Note that equation (3.3) involves scalar functions  $k = k(x)$  and  $c = c(x)$ . Moreover,  $b = b(x)$  might also be a scalar functions too. However, the function  $\phi = \phi(s)$  is independent of  $x$ . We shall make further simplification on these equations in Proposition 5.1 below.

There is an interesting case when  $\Phi = 0$ . In this case, the formula for  $J_j$  can be greatly simplified. Then we obtain the following sufficient and necessary condition for  $F = \alpha\phi(\beta/\alpha)$  to be weakly Landsbergian at a point.

**Proposition 3.2** *Let  $F = \alpha\phi(\beta/\alpha)$  be an almost regular  $(\alpha, \beta)$ -metric on a manifold. Suppose that  $\Phi = 0$  at a point  $x$  and  $\phi \neq k_1 \sqrt{1 + k_2 s^2}$  for any numbers  $k_1$  and  $k_2$ . Then  $F$  is a weakly Landsberg metric at a point  $x$  if and only if  $r_j + s_j = 0$  holds at  $x$ .*

*Proof:* By (2.6),  $\Phi = 0$  implies that

$$J_j = -\frac{(n+1)\Delta(Q - sQ')}{\alpha^2(b^2 - s^2)}(s_0 + r_0)h_j. \quad (3.5)$$

Notice that  $Q - sQ' = 0$  if and only if  $\phi = k_1 \sqrt{1 + k_2 s^2}$  for some numbers  $k_1$  and  $k_2$ . Thus  $Q - sQ' \neq 0$  under our assumption. Then it is obvious that  $J_j = 0$  at a point  $x$  if and only if  $s_0 + r_0 = 0$  holds at  $x$ .

Q.E.D.

## 4 Necessary Conditions

To characterize weakly Landsberg  $(\alpha, \beta)$ -metrics, one has to deal with the following terms:  $\phi(s), \phi'(s)$  and etc, where  $s = \beta/\alpha$ . In general, it is impossible to simplify  $J_j = 0$  if  $\phi = \phi(s)$  is an unknown function. To overcome this difficulty, we choose a special coordinate system at a point  $x$  as in [11]. First, we assume that

$$\alpha_x = \sqrt{\sum_{i=1}^n (y^i)^2}, \quad \beta_x = by^1,$$

where  $b = \|\beta_x\|_\alpha$ . Then take another special coordinates:  $(s, y^a) \rightarrow (\bar{y}^i)$  given by

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha}, \quad y^a = y^a,$$

where  $\bar{\alpha} := \sqrt{\sum_{a=2}^n (y^a)^2}$ . We have

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}} \bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}} \bar{\alpha}.$$

Let  $\bar{r}_{10} := r_{1a}y^a$ ,  $\bar{s}_{10} := s_{1a}y^a$ ,  $\bar{r}_{00} := r_{ab}y^a y^b$ , etc. By a direct computation using the formula (2.6), one can show that  $J_1 = 0$  is equivalent to that

$$\Psi_3 \left\{ s^2 r_{11} \bar{\alpha}^2 + (b^2 - s^2) \bar{r}_{00} \right\} - b^2 \Psi_2 \bar{r}_{00} = 0, \quad (4.1)$$

and

$$\left\{ 2s\Psi_1 + b^2\Psi_2 \right\} \bar{r}_{10} + b^2 \left\{ \Psi_2 - 2Q\Psi_1 \right\} \bar{s}_{10} = 0, \quad (4.2)$$

$J_a = 0$  ( $a = 2, \dots, n$ ) is equivalent to that

$$\begin{aligned} & \left\{ \Psi_3 \left( s^2 r_{11} \bar{\alpha}^2 + (b^2 - s^2) \bar{r}_{00} \right) - b^2 \Psi_2 \bar{r}_{00} \right\} y_a \\ & + \frac{b^2 \Phi}{\Delta} \left\{ \bar{r}_{00} y_a - (\bar{r}_{a0} + \Delta \bar{s}_{a0}) \bar{\alpha}^2 \right\} = 0, \end{aligned} \quad (4.3)$$

and

$$\begin{aligned} & s \left\{ 2s\Psi_1 + b^2\Psi_2 \right\} \bar{r}_{10} y_a + sb^2 \left\{ \Psi_2 - 2Q\Psi_1 \right\} \bar{s}_{10} y_a \\ & + \frac{b^2 \Phi}{\Delta} \left\{ s(\bar{r}_{10} y_a - r_{1a} \bar{\alpha}^2) - (b^2 Q + \Delta s)(\bar{s}_{10} y_a - s_{1a} \bar{\alpha}^2) \right\} = 0. \end{aligned} \quad (4.4)$$

Contracting (4.3) with  $y^a$  yields (4.1) and contracting (4.4) with  $y^a/s$  yields (4.2). In dimension two, (4.3) is equivalent to (4.1) and (4.4) is equivalent to (4.2). In higher dimensions, we can use (4.3) and (4.4) to simplify (4.1) and (4.2) respectively.

**Lemma 4.1** ( $n \geq 3$ ) Assume that  $\phi = \phi(s)$  satisfies  $\Phi \neq 0$  at a point  $x$ . If  $J_j = 0$  at  $x$ , then we have,

$$s_{ab} = 0, \quad (4.5)$$

$$r_{ab} = kb^2\delta_{ab}, \quad (4.6)$$

$$r_{11} = cb^2, \quad (4.7)$$

where  $k$  and  $c$  are numbers, and  $\phi = \phi(s)$  satisfies (3.3).

*Proof:* It follows from (4.1) and (4.3) that

$$\bar{r}_{00}y_a - (\bar{r}_{a0} + \Delta\bar{s}_{a0})\bar{\alpha}^2 = 0. \quad (4.8)$$

Since  $n \geq 3$ , (4.8) implies (4.5) and (4.6). Letting  $c := r_{11}/b^2$ , we obtain (4.7).

Plugging (4.6) and (4.7) into (4.1) yields (3.3). Q.E.D.

**Lemma 4.2** ( $n \geq 3$ ) Assume that  $\phi$  satisfies  $\Phi \neq 0$  and  $\phi \neq k_1\sqrt{1+k_2s^2}$  for any numbers  $k_1$  and  $k_2$ . If  $J_j = 0$  at a point  $x$ , then

$$s_{1a} = 0, \quad r_{1a} = 0.$$

*Proof:* It follows from (4.2) and (4.4) that

$$\left\{s\bar{r}_{10} - (b^2Q + \Delta s)\bar{s}_{10}\right\}y_a - \left\{r_{1a} - (b^2Q + \Delta s)s_{1a}\right\}\bar{\alpha}^2 = 0. \quad (4.9)$$

Since  $n \geq 3$ , we obtain from (4.9) that

$$sr_{1a} - (b^2Q + \Delta s)s_{1a} = 0. \quad (4.10)$$

We claim that  $s_0 = 0$ . If not,  $s_{1a_o} \neq 0$  for some  $a_o$ , let  $d := r_{1a_o}/(b^2s_{1a_o})$ . Plugging it into (4.10) with  $a = a_o$  yields

$$db^2s - (b^2Q + \Delta s) = 0. \quad (4.11)$$

By Lemma 2.4, we get  $Q = q_1s$  for some number  $q_1$ . Then  $\phi = k_1\sqrt{1+k_2s^2}$  for some numbers  $k_1$  and  $k_2$ . This contradicts the assumption. Thus  $s_{1a} = 0$ , hence from (4.10), we conclude that  $r_{1a} = 0$ . Q.E.D.

Let us summarize what we have proved.

**Proposition 4.3** ( $n \geq 3$ ) Let  $F = \alpha\phi(\beta/\alpha)$  be an almost regular  $(\alpha, \beta)$ -metric on an  $n$ -dimensional manifold. Suppose that  $\phi \neq k_1\sqrt{1+k_2s^2}$  for any numbers  $k_1$  and  $k_2$ . Suppose that  $F = \alpha\phi(\beta/\alpha)$  is a weakly Landsberg metric at a point  $x$ . Then

(i) If  $\Phi \neq 0$ , then  $\beta$  satisfies (3.1) and (3.2) and  $\phi$  satisfies (3.3).

(ii) If  $\Phi = 0$ , then  $r_0 + s_0 = 0$ .

Proposition 4.3 (ii) is already included in Proposition 3.2.

## 5 The Case: $\Phi \neq 0$

In the previous two sections we obtain necessary and sufficient conditions for an  $(\alpha, \beta)$ -metrics to be weakly Landsbergian. In this section, we shall make further simplification on these equations in the case when  $\Phi \neq 0$ .

We prove the following

**Proposition 5.1** *Let  $F = \alpha\phi(\beta/\alpha)$  be an  $(\alpha, \beta)$ -metric on a manifold  $M$  of dimension  $n \geq 3$ . Suppose  $\beta$  is not parallel with respect to  $\alpha$ ,  $\Phi \neq 0$  and  $\phi \neq k_1\sqrt{1+k_2s^2}$  for any constants  $k_1$  and  $k_2$ . Then  $F$  is a weakly Landsberg metric if and only if  $\beta$  satisfies the following equations*

$$s_{ij} = 0 \quad (5.1)$$

$$r_{ij} = k(b^2a_{ij} - b_ib_j), \quad (5.2)$$

(hence  $b = \text{constant}$ ) and  $\phi = \phi(s)$  satisfies the following ODE:

$$\Phi = -\frac{\lambda}{\sqrt{b^2 - s^2}}\Delta^{3/2}, \quad (5.3)$$

where  $\lambda$  is a constant.

*Proof:* It suffices to prove the necessary condition on  $\beta$  and  $\phi = \phi(s)$ . By Proposition 4.3,  $\beta$  satisfies (3.1) and (3.2) and  $\phi = \phi(s)$  satisfies

$$\Psi_1k + s\Psi_3c = 0. \quad (5.4)$$

By assumption,  $\beta$  is not parallel with respect to  $\alpha$ , we see that  $(k, c) \neq (0, 0)$ . The proof of Proposition 5.1 is carried out in the following lemmas.

**Lemma 5.2** ( $k = 0, c \neq 0$ ) *Assume that*

$$s_{ij} = 0, \quad r_{ij} = cb_ib_j.$$

*Then  $b \neq \text{constant}$  and  $Q = q_1s$ , where  $q_1$  is a constant.*

*Proof:* Note that

$$[b^2]_{x^i}y^i = 2cb^2\beta \neq 0.$$

Thus  $b \neq \text{constant}$ . Since  $k = 0$ , (5.4) is reduced to

$$\Psi_3 = 0.$$

Thus  $\Delta^2\Psi_3 = 0$ . We can express it in terms of  $b$  as follows

$$\Xi_4b^4 + \Xi_2b^2 + \Xi_0 = 0, \quad (5.5)$$

where  $\Xi_0, \Xi_2$  and  $\Xi_4$  depend on  $Q'''$ ,  $Q''$ ,  $Q'$ ,  $Q$  and  $s$ , but not on  $b$ . Since (5.5) holds for the variable  $b$ , we have

$$\Xi_0 = 0, \quad \Xi_2 = 0, \quad \Xi_4 = 0.$$

We get

$$s^4\Xi_4 + s^2\Xi_2 + \Xi_0 = 0. \quad (5.6)$$

This equation is  $\Delta^2\Psi_3 = 0$  with  $b$  replaced by  $s$ . Reformulating (5.6) we obtain

$$(n+1)s(1+sQ)\left\{2(1+sQ)\left\{s^2Q'' + (sQ' - Q)\right\} + s(sQ' - Q)^2\right\} = 0.$$

Thus

$$2(1+sQ)\left\{s^2Q'' + (sQ' - Q)\right\} + s(sQ' - Q)^2 = 0. \quad (5.7)$$

Let  $h := s(sQ' - Q)$  and  $f = -(sQ' - Q)/(2(1+sQ))$ . Then (5.7) can be written as

$$h' = fh.$$

We obtain

$$h(s) = \exp\left(\int_0^s f(t)dt\right)h(0).$$

Since  $h(0) = 0$ , we conclude that  $h(s) = 0$ . The

$$sQ' - Q = 0.$$

It is easy to get  $Q = Q'(0)s$ .

Q.E.D.

By Lemma 5.2, the case  $k = 0$  and  $c \neq 0$  does not occur under our assumption in Proposition 5.1.

**Lemma 5.3** ( $k \neq 0, c \neq 0$ ). *Assume that*

$$s_{ij} = 0,$$

$$r_{ij} = k(b^2a_{ij} - b_ib_j) + cb_ib_j.$$

*Then  $Q = q_1s$  where  $q_1$  is a constant.*

*Proof:* It's easy to get that  $b \neq \text{constant}$ . Let  $s = 0$  in (5.4). Then we get that  $\Psi_1|_{s=0} = 0$ . We write  $2b^{-2}(1+b^2q_1)^2\Psi_1|_{s=0} = 0$  as follows.

$$\begin{aligned} & \left\{12(q_2^2 - q_1q_3) + 2(n-2)q_0q_1q_2\right\}b^4 \\ & + \left\{12(q_0q_2 - q_3) + (n-2)q_0(2q_2 + q_0q_1)\right\}b^2 + (n+1)q_0^2 = 0. \end{aligned}$$

By assumption  $b \neq \text{constant}$ , we obtain  $q_0 = 0$ ,  $q_2 = 0$  and  $q_3 = 0$ . Obviously  $Q = q_1s$  is a solution of (5.4). Note that (5.4) is a third order ODE with leading coefficient  $(b^2 - s^2)k + s^2c$  which is not equal to zero when  $s = 0$ . Thus this ODE is regular at  $s = 0$ . By uniqueness,  $Q = q_1s$  is the unique regular solution of (5.4) with  $Q'(0) = q_1$ . Q.E.D.

By Lemma 5.3, the case  $k \neq 0, c \neq 0$  does not occur under our assumption in Proposition 5.1. Therefore, the only case left is that  $k \neq 0$  and  $c = 0$ . In this case,

$$r_j = 0, \quad s_j = 0.$$

Thus  $b = \text{constant}$ . Moreover, (5.4) is reduced to

$$\Psi_1 = \sqrt{b^2 - s^2} \Delta^{1/2} \left[ \frac{\sqrt{b^2 - s^2} \Phi}{\Delta^{3/2}} \right]' = 0.$$

We can easily get

$$\Phi = -\frac{\lambda}{\sqrt{b^2 - s^2}} \Delta^{3/2}.$$

Substituting it for  $\Phi$  into the equation  $\Psi_1 = 0$  yields

$$\lambda = \frac{nq_0 b}{\sqrt{1 + q_1 b^2}}.$$

We see that  $\lambda$  must be a constant. This proves Proposition 5.1.

Q.E.D.

## 6 Two-dimensional Case

In this section we are going to study and characterize Landsberg  $(\alpha, \beta)$ -metrics in dimension two.

First we need the following

**Lemma 6.1** ( $n = 2$ ) *Let  $F = \alpha\phi(\beta/\alpha)$  be a positively almost regular  $(\alpha, \beta)$ -metric on a surface, where  $\alpha = \sqrt{a_{ij}y^i y^j}$  is a Riemannian metric and  $\beta = b_i y^i \neq 0$  is a 1-form. Then  $F$  is a Landsberg metric if and only if  $\beta$  satisfies*

$$s_{ij} = \frac{1}{b^2} \{b_i s_j - b_j s_i\}, \quad (6.1)$$

$$r_{ij} = k(b^2 a_{ij} - b_i b_j) + c b_i b_j + d(b_i s_j + b_j s_i), \quad (6.2)$$

where  $k = k(x)$ ,  $c = c(x)$  and  $d = d(x)$  are scalar functions, and  $\phi = \phi(s)$  satisfies the following ODEs:

$$\{k(b^2 - s^2) + c s^2\} \Psi_1 + c b^2 s \Psi_2 = 0. \quad (6.3)$$

If  $s_0 \neq 0$ ,  $\phi = \phi(s)$  satisfies the following additional ODE:

$$2(ds - Q)\Psi_1 + (db^2 + 1)\Psi_2 = 0. \quad (6.4)$$

*Proof:* ( $\Leftarrow$ ). Assume that  $\beta$  satisfies (6.1) and (6.2). Then

$$\begin{aligned} r_{00} &= \{k(b^2 - s^2) + c s^2\} \alpha^2 + 2ds\alpha s_0 \\ r_0 &= c s b^2 \alpha + d b^2 s_0. \end{aligned}$$

We get

$$\begin{aligned} \Psi_1(r_{00} - 2\alpha Q s_0) + \alpha \Psi_2(r_0 + s_0) &= \\ \{ (k(b^2 - s^2) + c s^2) \Psi_1 + c b^2 s \Psi_2 \} \alpha^2 &+ \\ + \{ 2(ds - Q)\Psi_1 + (db^2 + 1)\Psi_2 \} \alpha s_0. & \end{aligned}$$

Then

$$\Psi_1(r_{00} - 2\alpha Qs_0) + \alpha\Psi_2(r_0 + s_0) = 0. \quad (6.5)$$

under the remaining assumptions on  $\phi$ . By (2.8), we see that  $J = J_j b^j = 0$ . Since  $J_j y^j = 0$ , we conclude that  $J_j = 0$  in dimension two.

( $\Rightarrow$ ). By assumption  $J_j = 0$ . Thus  $J = J_j b^j = 0$ . By (2.8), we get (6.5). At a point  $x$ , we use the same special coordinate system as in the previous section. We can easily show that  $\Psi_1(r_{00} - 2\alpha Qs_0) + \alpha\Psi_2(r_0 + s_0) = 0$  is equivalent to the following two equations

$$\left( (b^2 - s^2)r_{22} + s^2 r_{11} \right) \Psi_1 + r_{11} b^2 \Psi_2 = 0, \quad (6.6)$$

$$2(sr_{12} - b^2 Qs_{12}) \Psi_1 + b^2 (r_{12} + s_{12}) \Psi_2 = 0. \quad (6.7)$$

First assume that  $s_0 = 0$ . We have

$$s_{12} = 0.$$

Let  $k := r_{22}/b^2$  and  $c := r_{11}/b^2$ . We get

$$r_{22} = kb^2, \quad r_{11} = cb^2.$$

Thus  $\beta$  satisfies (6.1) and (6.2) with  $d$  arbitrary. In this case, (6.6) implies (6.3).

Now we assume that  $s_0 \neq 0$ . Then  $s_{12} \neq 0$ . Let  $d := r_{12}/(b^2 s_{12})$ . Then

$$r_{12} = db^2 s_{12}.$$

We still let  $k := r_{22}/b^2$  and  $c = r_{11}/b^2$ . Thus  $\beta$  satisfies (6.1) and (6.2). In this case (6.7) implies (6.4). Q.E.D.

We have obtained a sufficient and necessary condition for a two-dimensional  $(\alpha, \beta)$ -metric to be Landsbergian. However, the ODEs (6.3) and (6.4) are much more complicated. Next we are going to simplify these equations in two cases,  $s_j + r_j = 0$  and  $s_j + r_j \neq 0$ .

**Lemma 6.2** *Let  $F = \alpha\phi(\beta/\alpha)$  be an almost regular  $(\alpha, \beta)$ -metric satisfying the equations in Lemma 6.1. Suppose that  $s_j + r_j = 0$  at a point  $x$ . Then  $c = 0$  and if  $s_0 \neq 0$ , then  $d = -1/b^2$ .*

*Proof:* Contracting (6.2) with  $b^i$  yields

$$s_j + r_j = cb^2 b_j + (db^2 + 1)s_j = 0. \quad (6.8)$$

Contracting (6.8) with  $b^j$  yields

$$cb^4 = 0.$$

Thus  $c = 0$  and (6.8) reduced to

$$(db^2 + 1)s_j = 0. \quad (6.9)$$

If  $s_0 \neq 0$ , then it follows from (6.9) that  $d = -1/b^2$ . Q.E.D.

**Lemma 6.3** *Let  $F = \alpha\phi(\beta/\alpha)$  be an almost regular  $(\alpha, \beta)$ -metric on a surface  $M$ . Suppose that  $s_j + r_j = 0$  at a point  $x$ , but  $(s_{ij}, r_{ij}) \neq (0, 0)$  at  $x$ . Further, suppose that  $\phi \neq k_1\sqrt{1+k_2s^2}$  for any numbers  $k_1$  and  $k_2$ . Then  $F$  is a Landsberg metric at  $x$  if and only if  $\beta$  satisfies*

$$s_{ij} = \frac{1}{b^2}(b_i s_j - b_j s_i), \quad (6.10)$$

$$r_{ij} = k(b^2 a_{ij} - b_i b_j) - \frac{1}{b^2}(b_i s_j + b_j s_i), \quad (6.11)$$

and  $\phi = \phi(s)$  satisfies

$$\Psi_1 = 0. \quad (6.12)$$

*Proof:* In virtue of Lemma 6.1, it suffices to prove the necessary condition. First, by Lemma 6.2,  $c = 0$  and if  $s_0 \neq 0$ , then  $d = -1/b^2$ . Then (6.10) and (6.11) follow from (6.1) and (6.2) respectively.

If  $k \neq 0$ , then (6.12) follows from (6.3). If  $k = 0$ , then  $s_0 \neq 0$ , otherwise,  $s_{ij} = 0$  and  $r_{ij} = 0$ . In this case, (6.1) is trivial, but  $d = -1/b^2$ . Since  $ds - Q = -(b^2 Q + s)/b^2 \neq 0$ , it follows from (6.4) that  $\Psi_1 = 0$ . Q.E.D.

The following lemma is trivial. For completeness, we state it and give a short proof here.

**Lemma 6.4** *Let  $\alpha = \sqrt{a_{ij}y^i y^j}$  be a Riemannian metric and  $\beta = b_i y^i$  be a 1-form on a manifold  $M$  with  $b(x) := \|\beta_x\|_\alpha \neq 0$ . Then at a point  $x \in M$ , then following holds*

(a)  $db = 0$  if and only if  $s_0 + r_0 = 0$ .

(b)  $(n = 2)$   $s_0 + r_0 = 0$  if and only if  $\beta$  satisfies (6.10) and (6.11).

*Proof:* (a) It is easy to get the following identity:

$$bb_{x^j} = s_j + r_j.$$

Then Lemma 6.4 follows.

(b) ( $\Rightarrow$ ). Take a special orthogonal basis at  $x$  such that  $b_1 = b$  and  $b_2 = 0$ . Then  $s_0 + r_0 = 0$  is equivalent to the following equations:

$$r_{11} = 0, \quad s_{12} + r_{12} = 0. \quad (6.13)$$

Note that in dimension two (6.10) always holds. Let  $k := r_{22}/b^2$ . Then (6.11) follows from (6.13).

( $\Leftarrow$ ). Contracting (6.11) with  $b^i$  gives  $s_j + r_j = 0$ . Q.E.D.

**Lemma 6.5** *Let  $F = \alpha\phi(\beta/\alpha)$  be an almost regular  $(\alpha, \beta)$ -metric on a surface  $M$ . Suppose that  $b_{x^i}(x) \neq 0$  at a point  $x$ . If  $F$  is a Landsberg metric on a neighborhood of  $x$ , then  $\phi = k_1\sqrt{1+k_2s^2}$  for some constants  $k_1$  and  $k_2$ .*

*Proof:* By Lemma 6.1,  $\beta$  satisfies (6.1) and (6.2) and  $\phi = \phi(s)$  satisfies (6.3). We view  $b = b(x)$  is an independent variable in (6.3), and express  $\frac{\Delta^2}{1+sQ} \times (6.3)$  as

$$H_6 b^6 + H_4 b^4 + H_2 b^2 + H_0 = 0, \quad (6.14)$$

where  $H_0, H_2, H_4$  and  $H_6$  are independent on  $b$  and we have

$$\begin{aligned} H_2 : &= c \left\{ -s^2(1+sQ)Q''' + 3s(2Q's^2 - 1)Q'' + 3s^2Q'^2 - 3sQQ' \right\} \\ &\quad + k \left\{ 2s^2(1+sQ)Q''' + 6s(-Q's^2 + 1)Q'' - \frac{3}{2}s^2Q'^2 + \frac{3}{2}Q^2 \right\} \\ &\quad + s^4 \left( c - \frac{3}{2}k \right) (2Q'''Q' - 3Q''^2) \\ H_4 : &= -\frac{s^2}{2} (c - 3k) (2Q'''Q' - 3Q''^2) + 3 \left( k(Q + sQ') - scQ' \right) Q'' \\ &\quad - k(1+sQ)Q''' \\ H_6 : &= -\frac{1}{2}k(2Q'''Q' - 3Q''^2) \end{aligned}$$

We obtain four ODEs

$$H_0 = 0, \quad H_2 = 0, \quad H_4 = 0, \quad H_6 = 0. \quad (6.15)$$

(i)  $k \neq 0$ . Then from  $H_6 = 0$ , we get

$$2Q'''Q' - 3Q''^2 = 0.$$

Then  $Q = c_0$  or  $Q = \frac{c_1}{c_2+s} + c_3$ , for any constant  $c_0, c_1, c_2 \neq 0, c_3$ . However, substituting  $Q = \frac{c_1}{c_2+s} + c_3$  into  $H_4 = 0$ , we get

$$6c_1 \frac{2kc_3s^2 + \left\{ (c_1 + 1 + 3c_3c_2)k + cc_1 \right\} s + c_2(1 + c_1 + c_3c_2)k}{(c_2 + s)^5} = 0.$$

Then we get

$$c_1c_3 = 0, \quad (6.16)$$

$$(c_1 + 1 + 3c_3c_2)k + cc_1 = 0, \quad (6.17)$$

$$c_2(1 + c_1 + c_3c_2) = 0. \quad (6.18)$$

If  $c_1 \neq 0$ , then from (6.16) we have  $c_3 = 0$ . Noting that  $c_2 \neq 0$  and from (6.18) we get  $c_1 = -1$ . Substituting it into (6.17), we get  $c = 0$ . Then substituting  $Q = \frac{-1}{c_2+s}$  and  $c = 0$  into  $H_2 = 0$ , we get

$$-\frac{3}{2} \frac{6c_2ks^2 + 5c_2^2ks - c_2^3k}{(c_2 + s)^5} = 0.$$

This means  $c_2 = 0$ . It's a contradiction to  $c_2 \neq 0$ . Thus  $c_1 = 0, Q = c_3$ .

(ii)  $k = 0, c \neq 0$ . Then it follows from (6.15) that

$$\frac{s^6 H_6 + s^4 H_4 + s^2 H_2 + H_0}{\Delta^2} (1 + sQ) = 0.$$

This equation is (6.3) with  $b$  replaced by  $s$ . Then we get

$$2(1 + sQ) \left\{ s^2 Q'' + (sQ' - Q) \right\} + s(sQ' - Q)^2 = 0.$$

Then as the discussion in Lemma 5.2, we obtain  $Q = Q'(0)s$ .

(iii)  $(k, c) = (0, 0)$ . Then (6.3) becomes trivial. In this case,  $s_0 \neq 0$ , otherwise  $s_{ij} = 0$  and  $r_{ij} = 0$ , hence  $b_{x^i}(x) = 0$ . By Lemma 6.1,  $\phi = \phi(s)$  satisfies (6.4). Then we get

$$\Delta^2 \left\{ 2(ds - Q)\Psi_1 + (db^2 + 1)\Psi_2 \right\} = 0. \quad (6.19)$$

By the definition of  $\Psi_1$  and  $\Psi_2$ , we can easily see that the left hand of the above equation is a polynomial of  $b$ . Since  $b = b(x)$  in (6.4) is a variable, as the discussion above we can replace  $b$  by  $s$  in (6.19). Then we get

$$(1 + sQ)(1 + 2sQ - s^2d)(Q - sQ') = 0.$$

Then we have

$$Q - sQ' = 0.$$

It's easy to get  $Q = Q'(0)s$ .

Then in either case we get  $\phi = k_1 \sqrt{1 + k_2 s^2}$  for some constants  $k_1$  and  $k_2$ .  
Q.E.D.

**Lemma 6.6** *Let  $q_0, q_1$  and  $\lambda$  be arbitrary numbers. Then the ODE,  $\Psi_1 = 0$ , with  $Q(0) = q_0$  and  $Q'(0) = q_1$  has unique solution*

$$Q = -\frac{s}{b^2} + \sqrt{1 - (s/b)^2} f, \quad (6.20)$$

where

$$f := q_0 + \frac{(1/b^2 + q_1)s}{\sqrt{1 - (s/b)^2} + \left( q_0 - \frac{\lambda}{2} \sqrt{1/b^2 + q_1} \right) s}. \quad (6.21)$$

*Proof:* First it is easy to verify that the function  $Q = Q(s)$  in (6.20) satisfies that  $\Psi_1 = 0$  with  $Q(0) = q_0$  and  $Q'(0) = q_1$ . Note that  $\Psi_1 = 0$  is a third order ODE in  $Q = Q(s)$  with leading coefficient  $-(b^2 - s^2)^2(1 + sQ + (b^2 - s^2)Q')$ . The leading coefficient is negative at  $s = 0$ . Thus this ODE is regular at  $s = 0$ . We can apply the uniqueness theorem in ODE theory and conclude that  $Q = Q(s)$  given in (6.20) is the unique solution of  $\Psi_1 = 0$ .  
Q.E.D.

*Proof of Theorem 1.2:* We first prove the necessary condition. If  $b_{x^i}(x) \neq 0$  at some point  $x$ , then by Lemma 6.5, we conclude that  $\phi = k_1 \sqrt{1 + k_2 s^2}$  for

some constants  $k_1$  and  $k_2$ . This is impossible by assumption. Thus  $b_{x^i} = 0$  on  $M$ . Then  $b = \text{constant}$ . Assume that  $\beta$  is not parallel, i.e.,  $(r_{ij}, s_{ij}) \neq 0$ . By Lemmas 6.3 and 6.4,  $\phi$  satisfies (6.12). By Lemma 6.6 and the uniqueness theorem in ODE theory,  $Q = \phi'/(\phi - s\phi')$  is given by (6.20). Since  $\phi = \phi(s)$  is  $C^\infty$  on  $(-b_o, b_o)$ , we conclude that  $b = b_o$ .

Conversely, if  $Q := \phi'/(\phi - s\phi')$  is given by (6.20), then it satisfies (6.12). By Lemma 6.4,  $b = b_o$  implies that  $\beta$  satisfies (6.10), (6.11) and  $s_j + r_j = 0$  at any point  $x$ . Thus  $F$  is a Landsberg metric on  $M$ .

Now let us take a look at the spray coefficients of  $F = \alpha\phi(\beta/\alpha)$ . We only need to consider the case  $s_j + r_j = 0$  and  $(r_{ij}, s_{ij}) \neq (0, 0)$ . By Lemma 6.3,  $\beta$  satisfies (6.10) and (6.11). Thus

$$s^i_0 = \frac{1}{b^2}(b^i s_0 - s^i \beta), \quad r_{00} - 2\alpha Q s_0 = \alpha^2 k(b^2 - s^2) - 2\alpha(Q + \frac{s}{b^2})s_0.$$

We have

$$\begin{aligned} G^i &= \bar{G}^i + \frac{\alpha Q}{b^2}(b^i s_0 - s^i \beta) \\ &\quad + \Theta \left\{ k\alpha(b^2 - s^2) - 2(Q + \frac{s}{b^2})s_0 \right\} \left\{ y^i + \frac{\alpha Q'}{Q - sQ'} b^i \right\}, \end{aligned}$$

where

$$\begin{aligned} \Theta &= \frac{b^2 f}{2(b^2 - s^2)^2 f'} - \frac{s}{2(b^2 - s^2)} \\ \frac{Q'}{Q - sQ'} &= \frac{\sqrt{1 - (s/b)^2} + (sf - (b^2 - s^2)f')}{(-b^2 f + s(b^2 - s^2)f')}, \end{aligned}$$

where  $f = f(s)$  is given in (6.21). Taking the special coordinate as in Section 4, then

$$\alpha = \sqrt{(y^1)^2 + (y^2)^2}, \quad \beta = by^1, \quad s_1 = 0.$$

Noting that

$$\sqrt{1 - \frac{s^2}{b^2}} = \frac{y^2}{\alpha},$$

by directly computation we obtain

$$\begin{aligned} G^1 &= \left\{ \left( \frac{1}{4}q_0\lambda^2 - \frac{(b^2 q_1 + 2b^2 q_0^2 + 1)\lambda}{2b\sqrt{1 + b^2 q_1}} + \frac{q_0(b^2 q_1 + b^2 q_0^2 + 1)}{1 + b^2 q_1} \right) \frac{s_2}{b} \right. \\ &\quad \left. - \frac{1}{8}bk\lambda^2 + \frac{b^2 k q_0 \lambda}{2\sqrt{1 + b^2 q_1}} - \frac{b^3 k q_0^2}{2(1 + b^2 q_1)} \right\} (y^1)^2 \\ &\quad + \frac{1}{\sqrt{1 + b^2 q_1}} \left\{ \left( \frac{-q_0 \lambda}{b} + \frac{2q_0^2}{\sqrt{1 + b^2 q_1}} \right) s_2 + \frac{1}{2}bk\lambda - \frac{b^2 k q_0}{\sqrt{1 + b^2 q_1}} \right\} y^1 y^2 \\ &\quad + \frac{1}{1 + b^2 q_1} \left\{ \frac{q_0 s_2}{b} + \frac{1}{2}b^3 k q_1 \right\} (y^2)^2, \end{aligned}$$

$$\begin{aligned}
G^2 = & \left\{ \left( -\frac{1}{4}q_0^2\lambda^2 + \frac{q_0(b^2q_1 + 1 + q_0^2b^2)\lambda}{b\sqrt{1+b^2q_1}} - \frac{q_1 + b^2q_1^2 + q_0^4b^2 + 2q_0^2 + 2b^2q_1q_0^2}{1+b^2q_1} \right) s_2 \right. \\
& + \frac{1}{8}q_0b^2k\lambda^2 - \frac{kb(b^2q_1 + 2q_0^2b^2 + 1)\lambda}{4\sqrt{1+b^2q_1}} + \left. \frac{q_0b^2k(b^2q_1 + 1 + q_0^2b^2)}{2(1+b^2q_1)} \right\} (y^1)^2 \\
& + \frac{1}{\sqrt{1+b^2q_1}} \left\{ \left( q_0^2\lambda - \frac{2q_0(b^2q_1 + 1 + q_0^2b^2)}{b\sqrt{1+b^2q_1}} \right) s_2 - \frac{1}{2}b^2kq_0\lambda + \frac{b^3kq_0^2}{\sqrt{1+b^2q_1}} \right\} y^1y^2 \\
& + \frac{1}{1+b^2q_1} \left\{ -q_0^2s_2 + \frac{1}{2}b^2kq_0 \right\} (y^2)^2.
\end{aligned}$$

Obviously,  $G^1$  and  $G^2$  are both quadratic in  $y$ . Thus  $F = \alpha\phi(\beta/\alpha)$  is always a Berwald metric. This proves Theorem 1.2. Q.E.D.

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