

On a Class of Locally Dually Flat Finsler Metrics

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Abstract

Locally dually flat Finsler metrics arise from Information Geometry. Such metrics have special geometric properties and will play an important role in Finsler geometry. In this paper, we are going to study a class of locally dually flat Finsler metrics which are defined as the sum of a Riemannian metric and 1-form. We classify those with almost isotropic flag curvature.

Key words: Finsler metric, locally dually flat Randers metric, locally projectively flat Randers metric, flag curvature

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1 Introduction

The notion of dually flat metrics was first introduced by S.-I. Amari and H. Nagaoka ([2]) when they study the information geometry on Riemannian spaces. Later on, the second author extends the notion of dually flatness to Finsler metrics [10]. For a Finsler metric $F = F(x, y)$ on a manifold M , the geodesics $c = c(t)$ of F in local coordinates (x^i) are characterized by

$$\frac{d^2 x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where $(x^i(t))$ are the coordinates of $c(t)$ and $G^i = G^i(x, y)$ are defined by

$$G^i = \frac{g^{il}}{4} \left\{ [F^2]_{x^k y^l} y^k - [F^2]_{x^l} \right\},$$

where $g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}$ and $(g^{ij}) := (g_{ij})^{-1}$. The local functions $G^i = G^i(x, y)$ define a global vector field $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$ on TM . G is called the *spray* of F and G^i are called the *spray coefficients*.

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A Finsler metric $F = F(x, y)$ on a manifold is *locally dually flat* if at every point there is a coordinate system (x^i) in which the spray coefficients are in the following form

$$G^i = -\frac{1}{2}g^{ij}H_{y^j}, \quad (1)$$

where $H = H(x, y)$ is a local scalar function. Such a coordinate system is called an adapted coordinate system. Locally dually flat Finsler metrics are studied in Finsler information geometry ([10]).

It is known that a Riemannian metric $F = \sqrt{g_{ij}(x)y^i y^j}$ is locally dually flat if and only if in an adapted coordinate system,

$$g_{ij}(x) = \frac{\partial^2 \psi}{\partial x^i \partial x^j}(x),$$

where $\psi = \psi(x)$ is a C^∞ function [1][2].

The first example of non-Riemannian dually flat metrics is given in [10] as follows.

$$F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y \rangle^2)}}{1 - |x|^2} \pm \frac{\langle x, y \rangle}{1 - |x|^2}. \quad (2)$$

This metric is defined on the unit ball $B^n \subset R^n$.

The Finsler metric in (2) is of Randers type. A Randers metric on a manifold M is a Finsler metric expressed in the following form:

$$F = \alpha + \beta,$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M with $b := \|\beta\|_\alpha(x) < 1$. Randers metrics were first introduced by physicist G. Randers in 1941 from the standpoint of general relativity. Later on, these metrics were applied to the theory of electron microscope by R. S. Ingarden in 1957, who first named them Randers metrics.

The curvature properties of Randers metrics have been studied extensively (see [5][3][6][7]). In particular, the second author has classified projectively flat Randers metrics with constant flag curvature ([12]).

In this paper, our main aim is to characterize locally dually flat Randers metrics. First let us introduce our notations. Let $F = \alpha + \beta$ be a Randers metric on a manifold M . Define $b_{i|j}$ by

$$b_{i|j}\theta^j := db_i - b_j\theta_i^j,$$

where $\theta^i := dx^i$ and $\theta_i^j := \bar{\Gamma}_{ik}^j dx^k$ denote the Levi-Civita connection forms of α . Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}).$$

Clearly, β is closed if and only if $s_{ij} = 0$. We denote $r_{00} := r_{ij}y^i y^j$ and $s_{k0} := s_{km}y^m$. We first obtain the following theorem.

Theorem 1.1 *Let $F = \alpha + \beta$ be a Randers metric on a manifold M . F is locally dually flat if and only if in an adapted coordinate system, β and α satisfy*

$$r_{00} = \frac{2}{3}\theta\beta - \frac{5}{3}\tau\beta^2 + \left[\tau + \frac{2}{3}(\tau b^2 - b_m\theta^m)\right]\alpha^2, \quad (3)$$

$$s_{k0} = -\frac{\theta b_k - \beta\theta_k}{3}, \quad (4)$$

$$G_\alpha^m = \frac{1}{3}(2\theta + \tau\beta)y^m - \frac{1}{3}(\tau b^m - \theta^m)\alpha^2, \quad (5)$$

where $\tau = \tau(x)$ is a scalar function and $\theta = \theta_k y^k$ is a 1-form on M and $\theta^m := a^{im}\theta_i$.

The flag curvature in Finsler geometry is the analogue of the sectional curvature in Riemann geometry. A Finsler metric F on a manifold M is said to be of *scalar flag curvature* if the flag curvature $\mathbf{K}(P, y) = \mathbf{K}(x, y)$ is a scalar function on $TM \setminus \{0\}$. It is said to be of *almost isotropic flag curvature* if $\mathbf{K}(P, y) = 3c_x y^m / F + \sigma$, where $c = c(x)$ and $\sigma = \sigma(x)$ are scalar functions on M . If $c = 0$ and $\sigma = \text{constant}$, then F is said to be of *constant flag curvature*. See [6][8][11].

If a locally dually flat Randers metric is of almost isotropic flag curvature, then it can be completely determined.

Theorem 1.2 *Let $F = \alpha + \beta$ be a Randers metric on a manifold M . F is locally dually flat with almost isotropic flag curvature if and only if one of the following holds*

- (i) F is locally Minkowskian.
- (ii) α locally satisfies Hamel's projective flatness equation: $\alpha_{x^m y^k} y^m = \alpha_{x^k}$ with constant sectional curvature $\mathbf{K}_\alpha = -C^2 < 0$ and $\beta = \frac{\alpha_{x^m} y^m}{2C\alpha}$. In this case, $F = \alpha + \beta$ is dually flat and locally projectively flat with constant flag curvature $\mathbf{K} = -\frac{1}{4}C^2$.

For a given constant $C \neq 0$, there might be many forms for α satisfying Hamel's projective flatness equation with constant sectional curvature $\mathbf{K}_\alpha = -C^2$ and $\beta = \frac{\alpha_{x^m} y^m}{2C\alpha}$. Note that if we take $C = \pm 1$ and

$$\alpha = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y \rangle^2)}}{1 - |x|^2},$$

then

$$\beta = \pm \frac{\langle x, y \rangle}{1 + |x|^2}.$$

In this case, F is the Funk metric on the unit ball $B^n \subset R^n$ given in (2).

2 Preliminaries

A *Finsler metric* on a manifold M is a C^∞ function $F : TM \setminus \{0\} \rightarrow [0, \infty)$ satisfying the following conditions:

- (1) *Regularity*: F is C^∞ on $TM \setminus \{0\}$.
- (2) *Positive homogeneity*: $F(x, \lambda y) = \lambda F(x, y)$, $\lambda > 0$.
- (3) *Strong convexity*: the fundamental tensor $g_{ij}(x, y)$ is positive definite for all $(x, y) \in TM \setminus \{0\}$, where $g_{ij}(x, y) := \frac{1}{2} [F^2]_{y^i y^j}(x, y)$.

By the homogeneity of F , we have $F(x, y) = \sqrt{g_{ij}(x, y)y^i y^j}$. An important class of Finsler metrics are Riemann metrics, which are in the form of $F(x, y) = \sqrt{g_{ij}(x)y^i y^j}$. Another important class of Finsler metrics are Minkowski metrics, which are in the form of $F(x, y) = \sqrt{g_{ij}(y)y^i y^j}$.

Dually flat Finsler metrics on an open subset in R^n can be characterized by a simple PDE.

Lemma 2.1 ([10]) *A Finsler metric $F = F(x, y)$ on an open subset $\mathcal{U} \subset R^n$ is dually flat if and only if it satisfies the following equations:*

$$[F^2]_{x^k y^l} y^k - 2[F^2]_{x^l} = 0. \quad (6)$$

In this case, $H = H(x, y)$ in (1) is given by $H = \frac{1}{6} [F^2]_{x^m} y^m$.

There is another important notion in Finsler geometry, that is locally projectively flat Finsler metrics. A Finsler metric $F = F(x, y)$ is *locally projectively flat* if at every point there is a coordinate system (x^i) in which all geodesics are straight lines, or equivalently, the spray coefficients are in the following form

$$G^i = P y^i, \quad (7)$$

where $P = P(x, y)$ is a local scalar function satisfying $P(x, \lambda y) = \lambda P(x, y)$ for all $\lambda > 0$.

Projectively flat metrics on an open subset in R^n can be characterized by a simple PDE.

Lemma 2.2 ([9]) *A Finsler metric $F = F(x, y)$ on an open subset $\mathcal{U} \subset R^n$ is projectively flat if and only if it satisfies the following equations:*

$$F_{x^k y^l} y^k - F_{x^l} = 0. \quad (8)$$

In this case, local function $P = P(x, y)$ in (7) is given by $P = F_{x^m} y^m / (2F)$.

It is easy to show that any locally projectively flat Finsler metric $F = F(x, y)$ is of scalar flag curvature. Moreover, if $G^i = P y^i$ in a local coordinate system, then the flag curvature is given by

$$\mathbf{K} = \frac{P^2 - P_{x^m} y^m}{F^2}. \quad (9)$$

Particularly, Beltrami's theorem says that a Riemann metric is locally projectively flat if and only if it is of constant sectional curvature.

We have the following

Theorem 2.3 ([12]) *Let $F = \alpha + \beta$ be a locally projectively flat Randers metric on a manifold. If it is of constant flag curvature, then one of the following holds:*

- (i) *F is locally isometric to the Randers metric $F = |y| + by^1$ on R^n , where $0 \leq b < 1$ is a constant.*
- (ii) *After a normalization, F is locally isometric to the following Randers metric on a unit ball $B^n \subset R^n$:*

$$F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y \rangle^2)}}{1 - |x|^2} \pm \frac{\langle x, y \rangle}{1 - |x|^2} \pm \frac{\langle a, y \rangle}{1 + \langle a, x \rangle}, \quad (10)$$

where $a \in R^n$ is a constant vector with $|a| < 1$.

A Finsler metric is said to be *dually flat and projectively flat* on an open subset $\mathcal{U} \subset R^n$ if the spray coefficients G^i satisfy (1) and (7) in \mathcal{U} . There are Finsler metrics on an open subset in R^n which are dually flat and projectively flat.

Example 2.4 Let $\mathcal{U} \subset R^n$ be a strongly convex domain, namely, there is a Minkowski norm $\phi(y)$ on R^n such that

$$\mathcal{U} := \{y \in R^n \mid \phi(y) < 1\}.$$

Define $F = F(x, y) > 0$, $y \neq 0$ by

$$x + \frac{y}{F} \in \partial\mathcal{U}, \quad y \in T_x\mathcal{U} = R^n.$$

It is easy to show that F is a Finsler metric satisfying

$$F_{x^k} = FF_{y^k}. \quad (11)$$

Using (11), one can easily verify that $F = F(x, y)$ satisfies (6) and (8). Thus it is dually flat and projectively flat on \mathcal{U} . F is called the *Funk metric* on \mathcal{U} .

In fact, every dually flat and projectively flat metric on an open subset in R^n must be either a Minkowski metric or a Funk metric satisfying (11) after a normalization.

Theorem 2.5 *Let $F = F(x, y)$ be a Finsler metric on an open subset $\mathcal{U} \subset R^n$. F is dually flat and projectively flat on \mathcal{U} if and only if*

$$F_{x^k} = CF_{y^k} \quad (12)$$

where C is a constant.

Proof. Assume that F is dually flat and projectively flat. Then it satisfies (6) and (8). Rewrite (6) as follows

$$F_{x^k}y^kF_{y^l} + FF_{x^k y^l}y^k - 2FF_{x^l} = 0. \quad (13)$$

Plugging (8) into (13) yields

$$F_{x^k} = 2PF_{y^k} \quad (14)$$

where $P := F_{x^m}y^m/(2F)$. Plugging (14) into (8) we get

$$F_{x^k} = 2FP_{y^k}. \quad (15)$$

Then it follows from (14) and (15) that

$$PF_{y^k} - P_{y^k}F = 0. \quad (16)$$

By (16), we have

$$\left[\frac{P}{F}\right]_{y^k} = 0.$$

Thus

$$P = \frac{1}{2}CF$$

where $C = C(x)$ is a scalar function. Plugging $P = \frac{1}{2}CF$ into (14), we obtain

$$F_{x^k} = CFF_{y^k}. \quad (17)$$

From (17), it is easy to see that $F_{x^k}y^l = F_{x^l}y^k$ and $F_{x^k}F_{y^l} = F_{x^l}F_{y^k}$. Further, differentiating (17) with respect to x^l yields

$$C_{x^k}F_{y^l} = C_{x^l}F_{y^k}. \quad (18)$$

Suppose that $(C_{x^1}, \dots, C_{x^n}) \neq 0$. Without loss of generality, we assume that $C_{x^1} \neq 0$. For a non-zero vector $y = y^i \frac{\partial}{\partial x^i}$ with $C_{x^l}(x)y^l = 0$, we can obtain from (18) that $C_{x^1}F(x, y) = 0$, which implies that $F(x, y) = 0$. This contradicts the strong convexity of F . Thus $C = \text{constant}$.

The converse is trivial.

Q.E.D.

Proposition 2.6 *Let F be a Finsler metric on an open subset $\mathcal{U} \subset R^n$. If it is dually flat and projectively flat, then it is of constant flag curvature.*

Proof: Assume that F is dually flat and projectively flat on \mathcal{U} . By Theorem 2.5, F satisfies (12). Then $P := F_{x^k}y^k/(2F)$ is given by

$$P = \frac{1}{2}CF.$$

Then

$$P_{x^k}y^k = \frac{1}{2}CF_{x^k}y^k = \frac{1}{2}C^2F^2.$$

Since F is projectively flat, the flag curvature is given by

$$\mathbf{K} = \frac{P^2 - P_{x^k}y^k}{F^2}.$$

We obtain

$$\mathbf{K} = -\frac{1}{4}C^2.$$

Namely, F is projectively flat with constant flag curvature $\mathbf{K} = -\frac{1}{4}C^2$. Q.E.D.

The Randers metric in (2) satisfies (12) with $C = \pm 1$. Thus it is dually flat and projectively flat with $\mathbf{K} = -1/4$.

3 Locally dually flat Randers metrics

In this section, we are going to prove Theorem 1.1. It is straight forward to verify the sufficient condition. Thus we shall only prove the necessary condition.

Assume that $F = \alpha + \beta$ is dually flat on an open subset $\mathcal{U} \subset R^n$. First we have the following identities:

$$\alpha_{x^k} = \frac{y_m}{\alpha} \frac{\partial G_\alpha^m}{\partial y^k}, \quad \beta_{x^k} = b_{m|k} y^m + b_m \frac{\partial G_\alpha^m}{\partial y^k}, \quad s_{y^k} = \frac{\alpha b_k - s y_k}{\alpha^2}, \quad (19)$$

where $s := \beta/\alpha$ and $y_k := a_{jk} y^j$. By a direct computation, one obtains

$$\begin{aligned} [F^2]_{x^k} &= 2(1+s) \left[(y_m + \alpha b_m) \frac{\partial G_\alpha^m}{\partial y^k} + \alpha b_{m|k} y^m \right], \quad (20) \\ [F^2]_{x^l y^k y^l} &= \frac{2(\alpha b_k - s y_k)}{\alpha^2} [2(y_m + \alpha b_m) G_\alpha^m + \alpha r_{00}] \\ &\quad + 2(1+s) \left[2(a_{mk} + \frac{y_k}{\alpha} b_m) G_\alpha^m + (y_m + \alpha b_m) \frac{\partial G_\alpha^m}{\partial y^k} \right. \\ &\quad \left. + \frac{r_{00}}{\alpha} y_k + \alpha b_{k|0} \right]. \quad (21) \end{aligned}$$

Plugging (20) and (21) into (6), we obtain

$$\begin{aligned} &\frac{\alpha^2 b_k - \beta y_k}{\alpha^3} [2(y_m + \alpha b_m) G_\alpha^m + \alpha r_{00}] \\ &+ (1+s) \left\{ 2(a_{mk} + \frac{y_k}{\alpha} b_m) G_\alpha^m - (y_m + \alpha b_m) \frac{\partial G_\alpha^m}{\partial y^k} \right. \\ &\left. + \frac{r_{00}}{\alpha} y_k + \alpha(3s_{k0} - r_{k0}) \right\} = 0. \quad (22) \end{aligned}$$

Multiplying (22) by α^3 yields

$$\begin{aligned} &(b_k \alpha^2 - \beta y_k) [2(y_m + \alpha b_m) G_\alpha^m + \alpha r_{00}] + (\alpha + \beta) \alpha [2(a_{mk} \alpha + y_k b_m) G_\alpha^m \\ &- (\alpha y_m + \alpha^2 b_m) \frac{\partial G_\alpha^m}{\partial y^k} + r_{00} y_k + \alpha^2 (3s_{k0} - r_{k0})] = 0. \quad (23) \end{aligned}$$

Rewriting (23) as a polynomial in α , we have

$$\left(-b_m \frac{\partial G_\alpha^m}{\partial y^k} + 3s_{k0} - r_{k0} \right) \alpha^4 + [2b_k b_m G_\alpha^m + b_k r_{00} + 2a_{mk} G_\alpha^m$$

$$\begin{aligned}
& -y_m \frac{\partial G_\alpha^m}{\partial y^k} - \beta b_m \frac{\partial G_\alpha^m}{\partial y^k} + \beta(3s_{k0} - r_{k0})] \alpha^3 + (2b_k y_m G_\alpha^m + 2y_k b_m G_\alpha^m \\
& + r_{00} y_k + 2\beta a_{mk} G_\alpha^m - \beta y_m \frac{\partial G_\alpha^m}{\partial y^k}) \alpha^2 - 2\beta y_k y_m G_\alpha^m = 0. \tag{24}
\end{aligned}$$

From (24) we know that the coefficients of α are zero. Hence the coefficients of α^3 must be zero too. Thus we have

$$2b_k b_m G_\alpha^m + b_k r_{00} + 2a_{mk} G_\alpha^m - y_m \frac{\partial G_\alpha^m}{\partial y^k} - \beta b_m \frac{\partial G_\alpha^m}{\partial y^k} + \beta(3s_{k0} - r_{k0}) = 0, \tag{25}$$

$$\begin{aligned}
& (-b_m \frac{\partial G_\alpha^m}{\partial y^k} + 3s_{k0} - r_{k0}) \alpha^4 + (2b_k y_m G_\alpha^m + 2y_k b_m G_\alpha^m \\
& + r_{00} y_k + 2\beta a_{mk} G_\alpha^m - \beta y_m \frac{\partial G_\alpha^m}{\partial y^k}) \alpha^2 - 2\beta y_k y_m G_\alpha^m = 0. \tag{26}
\end{aligned}$$

Proof of Theorem 1.1. The sufficiency is clear because of (25) and (26). We just need to prove the necessity.

Note that

$$y_m \frac{\partial G_\alpha^m}{\partial y^k} = \frac{\partial(y_m G_\alpha^m)}{\partial y^k} - a_{mk} G_\alpha^m, \tag{27}$$

$$b_m \frac{\partial G_\alpha^m}{\partial y^k} = \frac{\partial(b_m G_\alpha^m)}{\partial y^k}. \tag{28}$$

Contracting (25) with b^k and by use of (27),(28), we obtain

$$\frac{\partial(y_m G_\alpha^m)}{\partial y^k} b^k + \beta \frac{\partial(b_m G_\alpha^m)}{\partial y^k} b^k = (2b^2 + 3)b_m G_\alpha^m + b^2 r_{00} + \beta(3s_{00} - r_{00}). \tag{29}$$

Contracting (26) with b^k and by use of (27),(28), we obtain

$$\begin{aligned}
& \alpha^4 \frac{\partial(b_m G_\alpha^m)}{\partial y^k} b^k + \beta \alpha^2 \frac{\partial(y_m G_\alpha^m)}{\partial y^k} b^k = (3s_{00} - r_{00}) \alpha^4 \\
& + (2b^2 y_m G_\alpha^m + 5\beta b_m G_\alpha^m + \beta r_{00}) \alpha^2 - 2\beta^2 y_m G_\alpha^m. \tag{30}
\end{aligned}$$

(29) $\times \alpha^4 - (30) \times \beta$ yields

$$\left[\frac{\partial(y_m G_\alpha^m)}{\partial y^k} b^k - 3b_m G_\alpha^m \right] \alpha^2 (\alpha^2 - \beta^2) = (2b_m G_\alpha^m \alpha^2 + r_{00} \alpha^2 - 2\beta y_m G_\alpha^m) (b^2 \alpha^2 - \beta^2). \tag{31}$$

Because $(b^2 \alpha^2 - \beta^2)$ and $(\alpha^2 - \beta^2)$ and α^2 are all irreducible polynomials of (y^i) , and $(\alpha^2 - \beta^2)$ and α^2 are relatively prime polynomials of (y^i) , we know that there is a function $\tau = \tau(x)$ on M such that

$$2b_m G_\alpha^m \alpha^2 + r_{00} \alpha^2 - 2\beta y_m G_\alpha^m = \tau \alpha^2 (\alpha^2 - \beta^2), \tag{32}$$

$$\frac{\partial(y_m G_\alpha^m)}{\partial y^k} b^k - 3b_m G_\alpha^m = \tau (b^2 \alpha^2 - \beta^2). \tag{33}$$

(32) can be reduced into

$$2\beta y_m G_\alpha^m = (2b_m G_\alpha^m + r_{00} - \tau\alpha^2 + \tau\beta^2)\alpha^2.$$

Since α^2 does not contain the factor β , we have the following

$$y_m G_\alpha^m = \theta\alpha^2, \quad (34)$$

$$b_m G_\alpha^m = \beta\theta - \frac{1}{2}r_{00} + \frac{\tau}{2}\alpha^2 - \frac{\tau}{2}\beta^2, \quad (35)$$

where $\theta := \theta_k y^k$ is a 1-form on M . Then we obtain the following

$$\frac{\partial(y_m G_\alpha^m)}{\partial y^k} = \theta_k \alpha^2 + 2\theta y_k, \quad (36)$$

$$\frac{\partial(b_m G_\alpha^m)}{\partial y^k} = \theta_k \beta + b_k \theta - r_{k0} + \tau y_k - \tau \beta b_k. \quad (37)$$

By use of (34)-(37), (25) and (26) become

$$\beta(3s_{k0} + \theta b_k - \beta\theta_k) + (\tau b_k - \theta_k)\alpha^2 + 3a_{mk} G_\alpha^m - (2\theta + \tau\beta)y_k = 0, \quad (38)$$

$$[(3s_{k0} + \theta b_k - \beta\theta_k) + (\tau b_k - \theta_k)\beta]\alpha^2 - (2\theta + \tau\beta)\beta y_k + 3\beta a_{mk} G_\alpha^m = 0. \quad (39)$$

(38) $\times \beta$ - (39) yields

$$3s_{k0} + b_k \theta - \theta_k \beta = 0. \quad (40)$$

This gives (4).

Contracting (38) with a^{lk} yields

$$(3s^l_0 + \theta b^l - \beta\theta^l)\beta + (\tau b^l - \theta^l)\alpha^2 + 3G^l_\alpha - (2\theta + \tau\beta)y^l = 0. \quad (41)$$

Contracting (40) with a^{lk} yields $3s^l_0 + \theta b^l - \beta\theta^l = 0$. Then, from (41), we obtain (5).

Substituting (5) into (35), we obtain (3). This completes the proof of Theorem 1.1. Q.E.D.

4 Dually flat and projectively flat Randers metrics

In this section, we are going to prove Theorem 1.2. We need the following

Lemma 4.1 *Let $F = \alpha + \beta$ be a locally dually flat Randers metric on a manifold M . Suppose that β satisfies the following equation:*

$$r_{00} = c(\alpha^2 - \beta^2) - 2\beta s_0, \quad (42)$$

where $c = c(x)$ is a scalar function on M . Then F is locally projectively flat in adapted coordinate systems with $G^i = \frac{1}{2}cFy^i$.

Proof: First recall the formula for the spray coefficients G^i of F ,

$$G^i = G_\alpha^i + \frac{r_{00} + 2\beta s_0}{2F} y^i - s_0 y^i + \alpha s^i_0, \quad (43)$$

where G_α^i denote the spray coefficients of α . We shall prove that α is projectively flat in the adapted coordinate system, i.e., $G_\alpha^i = P_\alpha y^i$, and β is closed, i.e., $s_{ij} = 0$.

By Theorem 1.1, α and β satisfy (3)-(5). By (3) and (42) we obtain

$$\left\{ c - \tau - \frac{2}{3}(\tau b^2 - b_m \theta^m) \right\} \alpha^2 = \left\{ 2s_0 + \frac{2}{3}\theta + \left(c - \frac{5}{3}\tau \right) \beta \right\} \beta.$$

Since α^2 is irreducible polynomial of (y^i) , we conclude that

$$c - \tau - \frac{2}{3}(\tau b^2 - b_m \theta^m) = 0 \quad (44)$$

$$s_0 = \frac{1}{2} \left(\frac{5}{3}\tau - c \right) \beta - \frac{1}{3}\theta. \quad (45)$$

It follows from (4) that

$$s_0 = -\frac{1}{3}(\theta b^2 - \beta b_m \theta^m). \quad (46)$$

Plugging (46) into (45), we obtain

$$\frac{2}{3}(1 - b^2)\theta = \frac{2}{3}(1 - b^2)\tau\beta + \left\{ \tau - c + \frac{2}{3}(\tau b^2 - b_m \theta^m) \right\} \beta. \quad (47)$$

Then it follows from (44) and (47) that

$$\theta = \tau\beta.$$

By (44) we see that $\tau = c$. Plugging $\theta = \tau\beta$ into (4) yields that $s_{ij} = 0$. Thus β is closed ! Then

$$r_{00} = c(\alpha^2 - \beta^2).$$

Plugging $\theta = \tau\beta$ into (5) yields

$$G_\alpha^i = \tau\beta y^i = c\beta y^i.$$

Thus α is projectively flat in the adapted coordinate system. By (43), we get

$$G^i = G_\alpha^i + \frac{r_{00}}{2F} y^i = \frac{c}{2} F y^i. \quad (48)$$

Therefore $F = \alpha + \beta$ is projectively flat in adapted coordinate systems. Q.E.D.

Remark 4.2 The S-curvature \mathbf{S} is an important non-Riemannian quantity in Finsler geometry ([6], [8], [11]). A Finsler metric is said to be of *isotropic S-curvature* if $\mathbf{S} = (n+1)c(x)F$. It is shown that a Randers metric $F = \alpha + \beta$ is of isotropic S-curvature $\mathbf{S} = (n+1)c(x)F$ if and only if it satisfies (42). See [7].

Lemma 4.3 *Let $F = \alpha + \beta$ be a locally dually flat Randers metric on a manifold M . If it is of almost isotropic flag curvature, $\mathbf{K} = 3\tilde{c}_{x^m}(x)y^m/F + \sigma(x)$, then it is locally projectively flat in adapted coordinate systems with $G^i = \frac{1}{2}cFy^i$ where $c = c(x)$ is a scalar function such that $c(x) - \tilde{c}(x) = \text{constant}$.*

Proof: Assume that $F = \alpha + \beta$ is of almost isotropic flag curvature $\mathbf{K} = 3\tilde{c}_{x^m}(x)y^m/F + \sigma(x)$. According to Theorem 1.2 in [13], F must be of isotropic S-curvature, i.e., β satisfies (42) for a scalar function $c = c(x)$ such that $c(x) - \tilde{c}(x) = \text{constant}$. Further, because F is locally dually flat, by Lemma 4.1, F is locally projectively flat in adapted coordinate systems with spray coefficients given by (48). Q.E.D.

Proof of Theorem 1.2. Under the assumption, we conclude that $F = \alpha + \beta$ is dually flat and projectively flat in any adapted coordinate system. By Theorem 2.5, F satisfies (12) for some constant C . Thus the spray coefficients $G^i = Py^i$ are given by $P = \frac{1}{2}CF$. By Proposition 2.6, we see that the flag curvature of F is constant, $\mathbf{K} = -\frac{1}{4}C^2$.

It is well-known that if $F = \alpha + \beta$ is locally projectively flat, then α is locally projectively flat and β is closed ([4]). Actually one can conclude this by (12). Plugging $F = \alpha + \beta$ into (12), we get

$$[\alpha^2]_{x^k} - 2C\alpha^2b_k - C\beta[\alpha^2]_{y^k} + \alpha\left\{2\beta_{x^k} - C[\alpha^2]_{y^k} - 2C\beta b_k\right\} = 0.$$

This is equivalent to the following two equations:

$$\begin{aligned} [\alpha^2]_{x^k} &= 2C\alpha^2b_k + C\beta[\alpha^2]_{y^k}, \\ 2\beta_{x^k} &= C[\alpha^2]_{y^k} + 2C\beta b_k. \end{aligned}$$

The above equations can be simplified to the following equations

$$\alpha_{x^k} = C(\alpha\beta)_{y^k} \tag{49}$$

$$\beta_{x^k} = C(\beta\beta_{y^k} + \alpha\alpha_{y^k}). \tag{50}$$

If $C = 0$, then $\alpha = \alpha(y)$ and $\beta = \beta(y)$ are independent of position x . Thus $F = \alpha + \beta$ is a Minkowskian norm in the adapted coordinate neighborhood.

If $C \neq 0$, then it follows from (49) that

$$\alpha_{x^m y^k} y^m = \alpha_{x^k}. \tag{51}$$

Thus α is projectively flat with spray coefficients $G_\alpha^i = P_\alpha y^i$ where $P_\alpha = \frac{\alpha_{x^m} y^m}{2\alpha}$. By (50), it is easy to see that β is closed. By (49), we have

$$\beta = \frac{\alpha_{x^m} y^m}{2C\alpha}.$$

Thus $P_\alpha = C\beta$. By (50), the sectional curvature \mathbf{K}_α of α is given by

$$\mathbf{K}_\alpha = \frac{(P_\alpha)^2 - (P_\alpha)_{x^m} y^m}{\alpha^2} = \frac{C^2\beta^2 - C^2(\alpha^2 + \beta^2)}{\alpha^2} = -C^2. \tag{52}$$

Conversely, assume that α satisfies (51) with $\beta = \frac{\alpha_x^m y^m}{2C\alpha}$ and $\mathbf{K}_\alpha = -C^2$. Then $F = \alpha + \beta$ is locally projectively flat (i.e., F satisfies (8)) and α satisfies (49) by (51) and $\beta = \frac{\alpha_x^m y^m}{2C\alpha}$. Because of $\mathbf{K}_\alpha = -C^2$, it is easy to see that β satisfies (50). By Theorem 2.5 and Proposition 2.6, we conclude that $F = \alpha + \beta$ is locally dually flat and locally projectively flat with constant flag curvature $\mathbf{K} = -\frac{1}{4}C^2$. Q.E.D.

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