LANDSBERG CURVATURE, S-CURVATURE AND RIEMANN CURVATURE

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Contents

1. Introduction

Roughly speaking, Finsler metrics on a manifold are regular (but not necessarily reversible) distance functions. In 1854, B. Riemann attempted to study a special class of Finsler metrics—Riemannian metrics, and introduced the so-called Riemann curvature. This infinitesimal quantity faithfully reveals the local geometry of a Riemannian manifold and becomes the central concept of Riemannian geometry. It is a natural problem to understand general regular distance functions by introducing suitable infinitesimal quantities. For more than half a century, there had been no essential progress until P. Finsler studied the variational problem in a Finsler manifold. However, it was L. Berwald who first successfully extended the notion of Riemann curvature to Finsler metrics by introducing the so-called Berwald connection. Berwald also introduced some non-Riemannian quantities via his connection [Berwald 1926], [Berwald 1928]. Since then, Finsler geometry has been developed gradually.

The Riemann curvature is defined using the induced spray which is independent of any well-known connection in Finsler geometry. It measures the shape of the space. The Cartan torsion and the distortion are two primary geometric quantities which describe the geometric properties of the Minkowski norm in each tangent space. Differentiating them along geodesics gives rise to the Landsberg curvature and the S-curvature. These quantities describe the rates of change of the "color/pattern" on the space.

In this article, I am going to discuss the geometric meaning of the Landsberg curvature, the S-curvature, the Riemann curvature, and their relationship. I will give detailed proofs for several important local and global results.

2. Finsler Metrics

By definition, a Finsler metric on a manifold is a family of Minkowski norms on the tangent spaces. A *Minkowski norm* on a vector space V is a nonnegative function $F : \mathrm{V} \to [0, \infty)$ with the following properties:

(i) F is positively y-homogeneous of degree one, i.e., for any $y \in \mathrm{V}$ and any $\lambda > 0$,

$$F(\lambda y) = \lambda F(y);$$

(ii) F is C^∞ on $\mathrm{V}\setminus\{0\}$ and for any tangent vector $y \in \mathrm{V}\setminus\{0\}$, the following bilinear symmetric form $\mathbf{g}_y : V \times V \to \mathrm{R}$ is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]|_{s=t=0}.$$

A Minkowski norm is said to be *reversible* if $F(-y) = F(y)$, $y \in \mathrm{V}$. In this article, Minkowski norms are not assumed to be reversible. By (i) and (ii), one can show that $F(y) > 0$ for $y \neq 0$ and $F(u + v) \leq F(u) + F(v)$ for $u, v \in \mathrm{V}$. See [Bao et al. 2000] for a proof.

Let $\langle\ ,\ \rangle$ denote the standard inner product on R^n, defined by $\langle u, v\rangle := \sum_{i=1}^n u^i v^i$. Then $|y| := \sqrt{\langle y, y\rangle}$ is called the standard Euclidean norm on R^n. Let $b \in \mathrm{R}^n$ with $|b| < 1$, then $F = |y| + \langle b, y\rangle$ is a Minkowski norm on R^n, which is called a *Randers norm*.

Let M be an n-dimensional C^∞ connected manifold. Denote by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M, where T_xM is the tangent space at $x \in M$. We denote a point in TM by (x, y) where $y \in T_xM$. A Finsler metric on M is a function $F : TM \to [0, \infty)$ with the following properties:

(a) F is C^∞ on $TM_o := TM \setminus \{0\}$;
(b) At each point $x \in M$, the restriction $F_x := F|_{T_xM}$ is a Minkowski norm on T_xM.

The pair (M, F) is called a *Finsler manifold*.

Let (M, F) be a Finsler manifold. Let (x^i, y^i) be a standard local coordinate system in TM, i.e., y^i's are determined by $y = y^i \frac{\partial}{\partial x^i}|_x$. For a vector $y =$

$y^i \frac{\partial}{\partial x^i}|_x \neq 0$, let $g_{ij}(x,y) := \frac{1}{2}[F^2]_{y^i y^j}(x,y)$. The induced inner product $\mathbf{g}_y$ is given by

$$\mathbf{g}_y(u,v) = g_{ij}(x,y)u^i v^j,$$

where $u = u^i \frac{\partial}{\partial x^i}|_x$ and $v = v^i \frac{\partial}{\partial x^i}|_x$. By the homogeneity of F, we have

$$F(x,y) = \sqrt{\mathbf{g}_y(y,y)} = \sqrt{g_{ij}(x,y)y^i y^j}.$$

A Finsler metric $F = F(x,y)$ is called a *Riemannian metric* if $g_{ij} = g_{ij}(x)$ are functions of $x \in M$ only.

There are three special Riemannian metrics.

Example 2.1. (Euclidean Metric) The simplest metric is the Euclidean metric $\alpha_0 = \alpha_0(x,y)$ on R^n, which is defined by

$$\alpha_0(x,y) := |y|, \qquad y = (y^i) \in T_x\mathrm{R}^n \cong \mathrm{R}^n. \tag{1}$$

We will simply denote (R^n, α_0) by R^n, which is called *Euclidean space.*

Example 2.2. (Spherical Metric) Let $\mathrm{S}^n := \{x \in \mathrm{R}^{n+1},\ |x| = 1\}$ denote the standard unit sphere in R^{n+1}. Every tangent vector $y \in T_x\mathrm{S}^n$ can be identified with a vector in R^{n+1} in a natural way. The induced metric α_{+1} on S^n is defined by $\alpha_{+1} = \|y\|_x$, $y \in T_x\mathrm{S}^n \subset \mathrm{R}^{n+1}$, where $\|\cdot\|_x$ denotes the induced Euclidean norm on $T_x\mathrm{S}^n$. Let $\varphi : \mathrm{R}^n \to \mathrm{S}^n \subset \mathrm{R}^{n+1}$ be defined by

$$\varphi(x) := \Big(\frac{x}{\sqrt{1+|x|^2}},\ \frac{\pm 1}{\sqrt{1+|x|^2}}\Big). \tag{2}$$

Then φ pulls back α_{+1} on the upper (lower) hemisphere to a Riemannian metric on R^n, which is given by

$$\alpha_{+1} = \frac{\sqrt{|y|^2 + (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1+|x|^2}, \qquad y \in T_x\mathrm{R}^n \cong \mathrm{R}^n. \tag{3}$$

Example 2.3. (Hyperbolic Metric) Let B^n denote the unit ball in R^n. Define

$$\alpha_{-1} := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1-|x|^2}, \qquad y \in T_x\mathrm{B}^n \cong \mathrm{R}^n. \tag{4}$$

We call α_{-1} the *Klein metric* and denote $(\mathrm{B}^n, \alpha_{-1})$ by H^n.

The above metrics in (1), (3) and (4) can be combined into one formula:

$$\alpha_\mu = \frac{\sqrt{(1+\mu|x|^2)|y|^2 - \mu\langle x,y\rangle^2}}{1+\mu|x|^2}. \tag{5}$$

Of course, there are many non-Riemannian Finsler metrics on R^n with special geometric properties. We just list some of them below and discuss their geometric properties later.

Example 2.4. (Funk Metric) Let

$$\Theta := \frac{\sqrt{|y|^2-(|x|^2|y|^2-\langle x,y\rangle^2)}+\langle x,y\rangle}{1-|x|^2}, \qquad y\in T_x\mathrm{B}^n\cong \mathrm{R}^n. \tag{6}$$

$\Theta=\Theta(x,y)$ is a Finsler metric on B^n. We call Θ the *Funk metric* on B^n.

For an arbitrary constant vector $a\in\mathrm{R}^n$ with $|a|<1$, let

$$\Theta_a := \frac{\sqrt{|y|^2-(|x|^2|y|^2-\langle x,y\rangle^2)}+\langle x,y\rangle}{1-|x|^2}+\frac{\langle a,y\rangle}{1+\langle a,x\rangle}. \tag{7}$$

where $y\in T_x\mathrm{B}^n\cong\mathrm{R}^n$. $\Theta_a=\Theta_a(x,y)$ is a Finsler metric on B^n. Note that $\Theta_0=\Theta$ is the Funk metric on B^n. We call Θ_a the *generalized Funk metric* on B^n [Shen 2003a].

Example 2.5. ([Shen 2003b]) Let δ be an arbitrary number with $\delta<1$. Let

$$\begin{aligned} F_\delta \;:=\; & \frac{\sqrt{|y|^2-(|x|^2|y|^2-\langle x,y\rangle^2)}+\langle x,y\rangle}{2(1-|x|^2)} \\ & -\delta\frac{\sqrt{|y|^2-\delta^2(|x|^2|y|^2-\langle x,y\rangle^2)}+\delta\langle x,y\rangle}{2(1-\delta^2|x|^2)}, \end{aligned} \tag{8}$$

where $y\in T_x\mathrm{B}^n\cong\mathrm{R}^n$. F_δ is a Finsler metric on B^n. Note that $F_{-1}=\alpha_{-1}$ is the Klein metric on B^n. Let Θ be the Funk metric on B^n defined in (6). We can express F_δ by

$$F_\delta=\frac{\Theta(x,y)-\delta\Theta(\delta x,y)}{2}.$$

Example 2.6. ([Berwald 1929b]) Let

$$B := \frac{\Big(\sqrt{|y|^2-(|x|^2|y|^2-\langle x,y\rangle^2)}+\langle x,y\rangle\Big)^2}{(1-|x|^2)^2\sqrt{|y|^2-(|x|^2|y|^2-\langle x,y\rangle^2)}}, \tag{9}$$

where $y\in T_x\mathrm{B}^n\cong\mathrm{R}^n$. $B=B(x,y)$ is a Finsler metric on B^n.

Example 2.7. Let ϵ be an arbitrary number with $|\epsilon|<1$. Let

$$F_\epsilon := \frac{\sqrt{\Psi\Big[\frac{\sqrt{\Phi^2+(1-\epsilon^2)|y|^4}+\Phi}{2}\Big]+(1-\epsilon^2)\langle x,y\rangle^2}+\sqrt{1-\epsilon^2}\langle x,y\rangle}{\Psi}, \tag{10}$$

where

$$\Phi:=\epsilon|y|^2+(|x|^2|y|^2-\langle x,y\rangle^2), \qquad \Psi:=1+2\epsilon|x|^2+|x|^4.$$

$F_\epsilon=F_\epsilon(x,y)$ is a Finsler metric on R^n. Note that if $\epsilon=1$, then $F_1=\alpha_{+1}$ is the spherical metric on R^n.

In [Bryant 1996], [Bryant 1997], R. Bryant classifies Finsler metrics on the standard unit sphere S^2 with constant flag curvature equal to $+1$ and geodesics being great circles. The Finsler metrics F_ϵ in (10) is a special family of Bryant's metrics expressed in a local coordinate system. See Example 12.7 below for further discussion.

The above mentioned Finsler metrics all have special geometric properties. They are locally projectively flat with constant flag curvature. Some of them belong to the class of the so-called (α, β)-*metrics* in the following form:

$$F = \alpha\,\phi\Big(\frac{\beta}{\alpha}\Big), \tag{11}$$

where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric on M and $\beta = b_i(x)y^i$ is a 1-form on M. Here the function $\phi = \phi(s)$ is a C^∞ positive function on some symmetric open interval $I = [-r, r]$. It is easy to see that any function F expressed in the form (11) is positively homogeneous of degree one.

Let $F := \alpha\phi(s)$ where $s = \beta/\alpha$ and $\phi = \phi(s)$ is a C^∞ function on some symmetric interval $I = [-r, r]$. Assume that

$$\|\beta\|_x := \sup_{y\in T_xM} \frac{\beta_x(y)}{\alpha_x(y)} \le r, \qquad x \in M.$$

Using a Maple program, we compute the Hessian $g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}$ as follows,

$$g_{ij} \;=\; \rho a_{ij} + \rho_0 b_ib_j + \rho_1(b_i\alpha_j + b_j\alpha_i) + \rho_2\alpha_i\alpha_j,$$

where $\alpha_i = \alpha_{y^i}$ and

$$\rho = \phi^2 - s\phi\phi', \qquad \rho_0 = \phi\phi'' + \phi'\phi',$$

$$\rho_1 = -s(\phi\phi'' + \phi'\phi') + \phi\phi', \qquad \rho_2 = s^2(\phi\phi'' + \phi'\phi') - s\phi\phi',$$

where $s := \beta/\alpha$ with $|s| \le \|\beta\|_x \le r$. Then $\det(g_{ij})$ is given by

$$\det(g_{ij}) = \phi^{n+1}\,(\phi - s\phi')^{n-2}\Big[(\phi - s\phi') + (\|\beta\|_x^2 - s^2)\phi''\Big]\det(a_{ij}).$$

If $\phi = \phi(s)$ satisfies the following conditions:

$$\phi(s) > 0, \qquad \phi(s) - s\phi'(s) > 0, \qquad \phi''(s) \ge 0, \qquad \forall |s| \le r, \tag{12}$$

then (g_{ij}) is positive definite, hence F is a Finsler metric.

Sabau–Shimada have classified (α, β)-metrics into several classes and they have also computed the Hessian g_{ij} for each class [Sabau–Shimada 2001]. Below are some special (α, β)-metrics.

(a) $\phi(s) = 1 + s$. The defined function $F = \alpha + \beta$ is a Finsler metric if and only if the norm of β with respect to α is less than one at any point,

$$\|\beta\|_x := \sqrt{a^{ij}(x)b_i(x)b_j(x)} < 1, \qquad x \in M.$$

A Finsler metric in this form is called a *Randers metric*. The Finsler metrics in Example 2.4 are Randers metrics. The Finsler metrics in Example 2.5 is the sum of two Randers metrics.

(b) $\phi(s) = (1 + s)^2$. The defined function $F = (\alpha + \beta)^2/\alpha$ is a Finsler metric if and only if $\|\beta\|_x < 1$ at any point $x \in M$. The Finsler metric in Example 2.6 are in this form.

By a *Finsler structure* on a manifold M, we usually mean a Finsler metric. Sometimes, we also define a Finsler structure as a scalar function F^* on T^*M such that F^* is C^∞ on $T^*M \setminus \{0\}$ and $F^*_x := F^*|_{T^*_xM}$ is a Minkowski norm on T^*_xM for $x \in M$. Such a function F^* is called a *co-Finsler metric*. Given a co-Finsler metric, one can always define a Finsler metric via the standard duality which is defined below.

Let $F^* = F^*(x,\xi)$ be a co-Finsler metric on a manifold M. Define a non-negative scalar function $F = F(x,y)$ on TM by

$$F(x,y) := \sup_{\xi \in T^*_xM} \frac{\xi(y)}{F^*(x,\xi)}.$$

Then $F = F(x,y)$ is a Finsler metric on M. F is said to be *dual to* F^*. $F^* = F^*(x,\xi)$ is also dual to F in the same sense that

$$F^*(x,\xi) = \sup_{y \in T_xM} \frac{\xi(y)}{F(x,y)}.$$

Every vector $y \in T_xM \setminus \{0\}$ uniquely determines a co-vector $\xi \in T^*_xM \setminus \{0\}$ by

$$\xi(w) := \frac{1}{2}\frac{d}{dt}\Big[F^2(x, y + tw)\Big]_{|t=0}, \qquad w \in T_xM.$$

The resulting map $\ell_x : y \in T_xM \to \xi \in T^*_xM$ is called the *Legendre transformation* at x. Similarly, every co-vector $\xi \in T^*_xM \setminus \{0\}$ uniquely determines a vector $y \in T_xM \setminus \{0\}$ by

$$\eta(y) := \frac{1}{2}\frac{d}{dt}\Big[F^{*2}(x, \xi + t\eta)\Big]_{|t=0}, \qquad \eta \in T^*_xM.$$

The resulting map $\ell^*_x : \xi \in T^*_xM \to y \in T_xM$ is called the inverse *Legendre transformation* at x. Indeed, ℓ_x and ℓ^*_x are inverses of each other. Moreover, they preserve the Minkowski norms, i.e.,

$$F(x,y) = F^*(x, \ell_x(y)), \qquad F^*(x,\xi) = F(x, \ell^*_x(\xi)). \tag{13}$$

Let $\Phi = \Phi(x,y)$ be a Finsler metric on a manifold M and $\Phi^* = \Phi^*(x,\xi)$ be the co-Finsler metric dual to Φ. By the above formulas, one can easily show that if $y \in T_xM \setminus \{0\}$ and $\xi \in T^*_xM \setminus \{0\}$ satisfy

$$\frac{d}{dt}\Big[\Phi^*(x, \xi + t\eta)\Big]_{|t=0} = \eta(y), \qquad \eta \in T^*_xM.$$

Then

$$\Phi(x,y) = 1. \tag{14}$$

Let V be a vector field on M with $\Phi(x, -V_x) < 1$ and $V^* : T^*M \to [0,\infty)$ denote the 1-form dual to V, which is defined by

$$V^*_x(\xi) = \xi(V_x), \qquad \xi \in T^*_xM.$$

We have $\Phi^*(x, -V_x^*) = \Phi(x, -V_x) < 1$. Thus $F^* := \Phi^* + V^*$ is a co-Finsler metric on M. Define $F = F(x,y)$ by

$$F(x,y) := \sup_{\xi \in T_x^* M} \frac{\xi(y)}{F^*(x,\xi)}, \qquad y \in T_x M. \tag{15}$$

F is a Finsler metric on M, which is called the *Finsler metric generated from* the pair (Φ, V). One can also define F in a different way without using the duality.

Lemma 2.8. *Let $\Phi = \Phi(x,y)$ be a Finsler metric on M and V be a vector field on M with $\Phi(x, -V_x) < 1$ for all $x \in M$. Then $F = F(x,y)$ defined in (15) satisfies the following*

$$\Phi\Big(x, \ \frac{y}{F(x,y)} - V_x\Big) = 1, \qquad y \in T_x M. \tag{16}$$

Conversely, if $F = F(x,y)$ is defined by (16), then it is dual to the co-Finsler metric $F^* := \Phi^* + V^*$ as defined in (15).

Proof: For the co-Finsler metric $F^* = \Phi^* + V^*$, let $F = F(x,y)$ be defined in (15). Fix an arbitrary non-zero vector $y \in T_x M$. There is a co-vector $\xi \in T_x^* M$ such that

$$F(x,y) = \frac{\xi(y)}{F^*(x,\xi)}. \tag{17}$$

Let $\eta \in T_x^* M$ be an arbitrary co-vector. Consider the function

$$h(t) := \frac{\xi(y) + t\eta(y)}{\Phi^*(x, \xi + t\eta) + \xi(V_x) + t\eta(V_x)}.$$

Then $h(t) \le h(0) = F(x,y)$. Thus $h'(0) = 0$, namely,

$$\eta(y) F^*(x,\xi) - \xi(y) \Big\{ \frac{d}{dt}\Big[\Phi^*(x, \xi + t\eta)\Big]|_{t=0} + \eta(V_x) \Big\} = 0.$$

By (17), $\xi(y) = F(x,y) F^*(x,\xi)$, one obtains

$$\eta(y) - F(x,y) \Big\{ \frac{d}{dt}\Big[\Phi^*(x, \xi + t\eta)\Big]|_{t=0} + \eta(V_x) \Big\} = 0.$$

From the above identity it follows that

$$\frac{d}{dt}\Big[\Phi^*(x, \xi + t\eta)\Big]|_{t=0} = \eta\Big(\frac{y}{F(x,y)} - V_x\Big), \qquad \eta \in T_x^* M.$$

Thus $F(x,y)$ satisfies (16) as we have explained in (14).

Conversely, let $F = F(x,y)$ be defined by (16). Then for any $\xi \in T_x^* M$,

$$\Phi^*(x,\xi) = \sup_{y \in T_x M} \eta\Big(\frac{y}{F(x,y)} - V_x\Big).$$

One obtains

$$\begin{aligned} \sup_{y \in T_x M} \frac{\xi(y)}{F(x,y)} &= \sup_{y \in T_x M} \xi\Big(\frac{y}{F(x,y)} - V_x\Big) + \xi(V_x) \\ &= \Phi^*(x,\xi) + V_x^*(\xi). \end{aligned}$$

This means that $F^* := \Phi^* + V^*$ is dual to F and hence F is dual to F^*, namely, F is given by (15). □

Let $\Phi = \sqrt{\phi_{ij}(x)y^iy^j}$ be a Riemannian metric and $V = V^i(x)\frac{\partial}{\partial x^i}$ be a vector field on a manifold M with

$$\Phi(x, -V_x) = \|V\|_x := \sqrt{\phi_{ij}(x)V^i(x)V^j(x)} < 1, \qquad x \in M.$$

Solving (16) for $F = F(x, y)$, one obtains

$$F = \frac{\sqrt{(1 - \phi_{ij}V^iV^j)\phi_{ij}y^iy^j + (\phi_{ij}y^iV^j)^2} - \phi_{ij}y^iV^j}{1 - \phi_{ij}V^iV^j}. \tag{18}$$

Clearly, F is a Randers metric. It is easy to verify that any Randers metric $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $\beta = b_i(x)y^i$, can be expressed in the form (18). According to Lemma 2.8, any Randers metric $F = \alpha+\beta$ expressed in the form (18) can be constructed in the following way. Let $\Phi^* := \sqrt{\phi^{ij}(x)\xi_i\xi_j}$ be the Riemannian metric dual to $\Phi = \sqrt{\phi_{ij}(x)y^iy^j}$ and $V^* := \xi(V_x) = V^i(x)\xi_i$ be the 1-form dual to V. Then $F^* := \Phi^*(x, \xi) + V^*(\xi) = \sqrt{\phi^{ij}(x)\xi_i\xi_j} + V^i(x)\xi_i$ is a co-Finsler metric on M. Moreover, the dual Finsler metric F of F^* is given by (18). This fact is proved by Hrimuic–Shimada [Hrimiuc–Shimada 1996].

It is discovered in [Shen 2003c], [Shen 2002] that if Φ is a Riemannian metric of constant curvature and V is a special vector field, then the generated metric F is of constant flag curvature. This discovery is simply a matter of luck. But Maple programs play an important role in the computations which lead to this discovery, and open the door for classifying Randers metrics of constant flag curvature [Bao et al. 2003].

Example 2.9. Let $\phi = \phi(y)$ be a Minkowski norm on R^n and let

$$U_\phi := \Big\{ y \in \mathrm{R}^n \ \Big| \ \phi(y) < 1 \Big\}.$$

Define

$$\Phi(x, y) := \phi(y), \qquad y \in T_x\mathrm{R}^n \cong \mathrm{R}^n.$$

$\Phi = \Phi(x, y)$ is called a *Minkowski metric* on R^n. Let $V_x := -x$, $x \in \mathrm{R}^n$. V is a radial vector field pointing toward the origin. Observe that for any $x \in U_\phi$,

$$\Phi\Big(x,\ -V_x\Big) = \phi(x) < 1.$$

The pair (Φ, V) generates a Finsler metric $\Theta = \Theta(x, y)$ on U_ϕ by (16), i.e.,

$$\Theta(x, y) = \phi\Big(y + \Theta(x, y)x\Big). \tag{19}$$

Differentiating (19) with respect to x^k and y^k respectively, one obtains

$$\Big(1 - \phi_{w^l}(w)x^l\Big)\Theta_{x^k}(x, y) = \phi_{w^k}(w)\Theta(x, y), \tag{20}$$

$$\Big(1 - \phi_{w^l}(w)x^l\Big)\Theta_{y^k}(x, y) = \phi_{w^k}(w), \tag{21}$$

where $w := y + \Theta(x, y)x$. It follows from (20) and (21) that

$$\Theta_{x^k}(x, y) = \Theta(x, y)\Theta_{y^k}(x, y). \tag{22}$$

The above argument is given by T. Okada [Okada 1983].

A domain U_ϕ in R^n defined by a Minkowski norm ϕ is called a *strongly convex domain*. A Finsler metric $\Theta = \Theta(x, y)$ defined in (19) is called the *Funk metric* on a strongly convex domain in R^n. When $\phi = |y|$ is the standard Euclidean metric on R^n, $U_\phi = \mathrm{B}^n$ is the standard unit ball and $\Theta = \Theta(x, y)$ is given by (6). Equation (22) is the key property of Θ, by which one can find other geometric properties of Θ. Therefore we make the following

Definition 2.10. A Finsler function $\Theta = \Theta(x, y)$ on an open subset in R^n is called a *Funk metric* if it satisfies (22).

Example 2.11. Let $\Phi(x, y) := |y|$ be the standard Euclidean metric on R^n and let $V = V(x)$ be a vector field on R^n defined by

$$V_x := |x|^2 a - 2\langle a, x\rangle x,$$

where $a \in \mathrm{R}^n$ is a constant vector. Note that

$$\Phi(x, -V_x) = \sqrt{\phi_{ij} V^i V^j} = |V_x| = |a||x|^2 < 1, \qquad x \in \mathrm{B}^n(1/\sqrt{|a|})$$

and

$$\phi_{ij} y^i V^j = |x|^2\langle a, y\rangle - 2\langle a, x\rangle\langle x, y\rangle.$$

Given the above pair $\{\Phi, V\}$, solving (16) for F, one obtains

$$\begin{aligned} F \quad &= \quad \frac{\sqrt{(|x|^2\langle a, y\rangle - 2\langle a, x\rangle\langle x, y\rangle)^2 + |y|^2(1 - |a|^2|x|^4)}}{1 - |a|^2|x|^4} \\ &\quad - \frac{|x|^2\langle a, y\rangle - 2\langle a, x\rangle\langle x, y\rangle}{1 - |a|^2|x|^4}. \end{aligned} \tag{23}$$

This Randers metric F has very important properties. It is of scalar curvature and isotropic S-curvature. But the flag curvature and the S-curvature are not constant. See Example 11.2 below for further discussion.

3. Cartan Torsion and Matsumoto Torsion

To characterize Euclidean norms, E. Cartan introduces the Cartan torsion [Cartan 1934]. Let $F = F(y)$ be a Minkowski norm on a vector space V. Fix a basis $\{\mathbf{b}_i\}$ for V. Then $F = F(y^i\mathbf{b}_i)$ is a function of (y^i). Let

$$g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}, \qquad C_{ijk} := \frac{1}{4}[F^2]_{y^iy^jy^k}(y), \qquad I_i := g^{jk}(y)C_{ijk}(y),$$

where $(g^{ij}) := (g_{ij})^{-1}$. It is easy to see that

$$I_i = \frac{\partial}{\partial y^i}\Big[\ln\sqrt{\det\Big(g_{jk}\Big)}\Big]. \tag{24}$$

For $y \in \mathrm{V} \setminus \{0\}$, set

$$\mathbf{C}_y(u,v,w) := C_{ijk}(y)u^i v^j w^k, \qquad \mathbf{I}_y(u) := I_i(y)u^i,$$

where $u := u^i\mathbf{b}_i$, $v := v^j\mathbf{b}_j$ and $w := w^k\mathbf{b}_k$. The family $\mathbf{C} := \{\mathbf{C}_y \mid y \in \mathrm{V}\setminus\{0\}\}$ is called the *Cartan torsion* and the family $\mathbf{I} := \{\mathbf{I}_y \mid y \in \mathrm{V} \setminus \{0\}\}$ is called the *mean Cartan torsion*. They are not tensors in a usual sense. In later sections, we will convert them to tensors on TM_o and call them the (mean) Cartan tensor.

We view a Minkowski norm F on a vector space V as a *color pattern*. When F is Euclidean, the color pattern is *trivial* or *Euclidean*. The Cartan torsion $\mathbf{C}_y$ describes the non-Euclidean features of the color pattern in the direction $y \in \mathrm{V} \setminus \{0\}$. And the mean Cartan torsion $\mathbf{I}_y$ is the average value of $\mathbf{C}_y$.

A trivial fact is that a Minkowski norm F on a vector space V is Euclidean if and only if $\mathbf{C}_y = 0$ for all $y \in \mathrm{V} \setminus \{0\}$. This can be improved to the following

Proposition 3.1. ([Deicke 1953]) *A Minkowski norm is Euclidean if and only if* $\mathbf{I} = 0$.

To characterize Randers norms, M. Matsumoto introduces the following quantity:

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\Big\{ I_i h_{jk} + I_j h_{ik} + I_k h_{ij}\Big\}, \tag{25}$$

where $h_{ij} := FF_{y^iy^j} = g_{ij} - g_{ip}y^p g_{jq}y^q/F^2$. For $y \in \mathrm{V} \setminus \{0\}$, set

$$\mathbf{M}_y(u,v,w) := M_{ijk}(y)u^i v^j w^k,$$

where $u = u^i\mathbf{b}_i$, $v = v^j\mathbf{b}_j$ and $w := w^k\mathbf{b}_k$. The family $\mathbf{M} := \{\mathbf{M}_y \mid y \in \mathrm{V}\setminus\{0\}\}$ is called the *Matsumoto torsion*. A Minkowski norm is said to be *C-reducible* if $\mathbf{M} = 0$.

Lemma 3.2. ([Matsumoto 1972b]) *Every Randers metric satisfies* $\mathbf{M} = 0$.

Proof: Let $F = \alpha + \beta$ be an arbitrary Randers norm on a vector space V, where $\alpha = \sqrt{a_{ij}y^iy^j}$ and $\beta = b_iy^i$ with $\|\beta\|_\alpha < 1$. By a direct computation, $g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}$ are given by

$$g_{ij} = \frac{F}{\alpha}\Big\{ a_{ij} - \frac{y_i}{\alpha}\frac{y_j}{\alpha} + \frac{\alpha}{F}\Big(b_i + \frac{y_i}{\alpha}\Big)\Big(b_j + \frac{y_j}{\alpha}\Big)\Big\}, \tag{26}$$

where $y_i := a_{ij}y^j$. Then $h_{ij} = FF_{y^iy^j} = g_{ij} - g_{ip}y^pg_{jq}y^q/F^2$ are given by

$$h_{ij} = \frac{\alpha+\beta}{\alpha}\Big(a_{ij} - \frac{y_iy_j}{\alpha^2}\Big). \tag{27}$$

The inverse matrix $(g^{ij}) = (g_{ij})^{-1}$ is given by

$$g^{ij} = \frac{\alpha}{F}\Big\{ a^{ij} - (1-\|\beta\|^2)\frac{y^i}{F}\frac{y^j}{F} + \frac{\alpha}{F}\Big[\Big(b^i - \frac{y^i}{\alpha}\Big)\Big(b^j - \frac{y^j}{F}\Big) - b^ib^j\Big]\Big\}. \tag{28}$$

The determinant $\det(g_{ij})$ is given by

$$\det(g_{ij}) = \Big(\frac{\alpha+\beta}{\alpha}\Big)^{n+1}\det(a_{ij}). \tag{29}$$

By (24) and (29), one obtains

$$I_i = \frac{n+1}{2(\alpha+\beta)} \cdot \Big(b_i - \frac{y_i}{\alpha}\frac{\beta}{\alpha}\Big). \tag{30}$$

Differentiating (26) yields

$$C_{ijk} = \frac{1}{n+1}\Big\{ I_i h_{jk} + I_j h_{ik} + I_k h_{ij} \Big\}. \tag{31}$$

This implies that $M_{ijk} = 0$. □

Later on, Matsumoto and Hōjō proved that the converse is true as well, whenever $\dim \mathrm{V} \geq 3$.

Proposition 3.3. ([Matsumoto 1972b], [Matsumoto–Hōjō 1978]) *Let F be a Minkowski norm on a vector space* V *of dimension $n \geq 3$. Then the Matsumoto torsion vanishes if and only if F is a Randers norm.*

The proof given by Matsumoto and Hōjō seems quite long and the author could not find a shorter proof which fits into this article.

4. Geodesics and Sprays

Every Finsler metric F on a manifold M defines a length structure L_F on oriented curves in M. Let $c : [a, b] \to M$ be a piecewise C^∞ curve. The *length* of c is defined by

$$L_F(c) := \int_a^b F\Big(c(t), \dot{c}(t)\Big) dt.$$

For any two points $p, q \in M$, define

$$d_F(p, q) := \inf_c L_F(c),$$

where the infimum is taken over all piecewise C^∞ curves c from p to q. The quantity $d_F = d_F(p, q)$ is a nonnegative function on $M \times M$. It has the following properties

(a) $d_F(p, q) \geq 0$ and equality holds if and only if $p = q$;
(b) $d_F(p, q) \leq d_F(p, r) + d_F(r, q)$ for any $p, q, r \in M$.

d_F is called the *distance function* induced by F. If the Finsler metric F is reversible, i.e., $F(x, -y) = F(x, y)$, $y \in T_xM$, d_F satisfies the *reversibility condition*: $d_F(p, q) = d_F(q, p)$, $p, q \in M$. In this article, the notion of distance function is weaker than the usual one.

Given an ordered pair of points $p, q \in M$, a piecewise C^∞ curve $\sigma : I = [a, b] \to M$ from $p = \sigma(a)$ to $q = \sigma(b)$ is said to be *minimizing* if

$$L_F(\sigma) = d_F(p, q).$$

Definition 4.1. A C^∞ curve $\sigma(t)$, $t \in I$, is called a *geodesic* if it has constant speed ($F(\sigma(t), \dot{\sigma}(t)) =$ *constant*) and it is locally minimizing, i.e., for any $t_o \in I$, there is a small number $\epsilon > 0$ such that c is minimizing on $[t_o - \epsilon, t_o + \epsilon] \cap I$.

By the calculus of variation, one can show that geodesics are characterized by a system of 2nd order ordinary differential equations (see e.g. [Shen 2001a], [Shen 2001b]).

Lemma 4.2. *A C^∞ curve $\sigma(t)$ in a Finsler manifold (M, F) is a geodesic if and only if $\sigma(t)$ satisfies the following system of 2nd order ordinary differential equations*

$$\ddot{\sigma}^i(t) + 2G^i(\sigma(t), \dot{\sigma}(t)) = 0, \tag{32}$$

where $G^i = G^i(x, y)$ are local functions on TM defined by

$$G^i := \frac{1}{4} g^{il}(x, y) \Big\{ [F^2]_{x^k y^l}(x, y) y^k - [F^2]_{x^l}(x, y) \Big\}. \tag{33}$$

Let $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$ denote the natural local frame on TM in a standard local coordinate system, and define

$$G := y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}, \tag{34}$$

where $G^i = G^i(x, y)$ are given in (33) with the homogeneity property:

$$G^i(x, \lambda y) = \lambda^2 G^i(x, y), \qquad \lambda > 0. \tag{35}$$

G is a well-defined vector field on TM. Any vector field G on TM in the above form (34) with homogeneity (35) is called a *spray* on M, where G^i are called the *spray coefficients*. Let $N^i_j = \frac{\partial G^i}{\partial y^j}$ and $\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^i_j \frac{\partial}{\partial y^j}$. Then $HTM := \mathrm{span}\{\frac{\delta}{\delta x^i}\}$ and $VTM := \mathrm{span}\{\frac{\partial}{\partial y^i}\}$ are well-defined such that $T(TM_o) = HTM \oplus VTM$. That is, every spray naturally determines a decomposition of $T(TM_o)$.

For a Finsler metric on a manifold M and its spray G, a C^∞ curve $\sigma(t)$ in M is a geodesic of F if and only if the canonical lift $\gamma(t) := \dot{\sigma}(t)$ in TM is an integral curve of G. One can use this fact to define the notion of geodesics for sprays.

It is usually difficult to compute the spray coefficients of a Finsler metric in general. However, for an (α, β)-metric, the computation is relatively simple using a Maple program. Let F be an (α, β)-metric on a manifold M, which is a Finsler metric in the following form,

$$F = \alpha\, \phi\Big(\frac{\beta}{\alpha}\Big),$$

where $\phi = \phi(s)$ is a C^∞ function on some interval $[-r, r]$ satisfying (12), $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form satisfying $\|\beta\|_x = \sqrt{a^{ij}(x) b_i(x) b_j(x)} < r$ for all $x \in M$. Let $\bar{G}^i = \bar{G}^i(x, y)$ denote the spray coefficients of α. By (33), $\bar{G}^i = \frac{1}{2} \bar{\Gamma}^i_{jk}(x) y^j y^k$ are given by

$$\bar{\Gamma}^i_{jk} = \frac{a^{il}}{2} \Big\{ \frac{\partial a_{jl}}{\partial x^k} + \frac{\partial a_{kl}}{\partial x^j} - \frac{\partial a_{jk}}{\partial x^l} \Big\}.$$

$\bar{\Gamma}^i_{jk}$ are called the *Christoffel symbols* of α. To find a formula for the spray coefficients $G^i = G^i(x, y)$ of F in terms of α and β, we need to introduce the *covariant derivatives* of β with respect to α. Let $\theta^i := dx^i$ and $\theta_j{}^i := \bar{\Gamma}^i_{jk} dx^k$. We have

$$d\theta^i = \theta^j \wedge \theta_j{}^i, \qquad da_{ij} = a_{kj}\theta_i{}^k + a_{ik}\theta_j{}^k.$$

Define $b_{i;j}$ by

$$b_{i;j}\theta^j := db_i - b_j\theta_i{}^j.$$

Let

$$r_{ij} := \frac{1}{2}\Big(b_{i;j} + b_{j;i}\Big), \qquad s_{ij} := \frac{1}{2}\Big(b_{i;j} - b_{j;i}\Big), \tag{36}$$

$$s^i{}_j := a^{ih}s_{hj}, \qquad s_j := b_i s^i{}_j, \qquad e_{ij} := r_{ij} + b_i s_j + b_j s_i. \tag{37}$$

By (33) and using a Maple program, one obtains the following relationship between G^i and $\bar{G}^i$.

Lemma 4.3. *The geodesic coefficients G^i are related to $\bar{G}^i$ by*

$$\begin{aligned} G^i &= \bar{G}^i + \frac{\alpha\phi'}{\phi - s\phi'} s^i{}_0 + \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi\Big((\phi - s\phi') + (b^2 - s^2)\phi''\Big)} \times \\ &\quad \times \Big\{ \frac{-2\alpha\phi'}{\phi - s\phi'} s_0 + r_{00} \Big\} \Big\{ \frac{y^i}{\alpha} + \frac{\phi\phi''}{\phi\phi' - s(\phi\phi'' + \phi'\phi')} b^i \Big\}, \end{aligned} \tag{38}$$

where $s = \beta/\alpha$, $s^i{}_0 = s^i{}_j y^j$, $s_0 := s_i y^i$, $r_{00} = r_{ij}y^iy^j$ and $b^2 := a^{ij}b_ib_j$.

Consider the following metric

$$F = \frac{(\alpha + \beta)^2}{\alpha},$$

where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form with $\|\beta\|_x < 1$ for every $x \in M$. By (38), we obtain a formula for the spray coefficients of F,

$$\begin{aligned} G^i &= \bar{G}^i + \frac{2\alpha}{\alpha - \beta} \alpha s^i{}_0 + \frac{\alpha(\alpha - 2\beta)}{2\alpha^2 b^2 + \alpha^2 - 3\beta^2} \times \\ &\quad \times \Big\{ \frac{-4\alpha}{\alpha - \beta} \alpha s_0 + r_{00} \Big\} \Big\{ \frac{y^i}{\alpha} + \frac{\alpha}{\alpha - 2\beta} b^i \Big\}, \end{aligned}$$

where $b = \|\beta\|_x$.

Given a spray G, we define the covariant derivatives of a vector field $X = X^i(t)\frac{\partial}{\partial x^i}|_{c(t)}$ along a curve c by

$$D_{\dot{c}}X(t) := \Big\{ \dot{X}^i(t) + X^j(t) N^i_j\Big(c(t), \dot{c}(t)\Big) \Big\} \frac{\partial}{\partial x^i}|_{c(t)}, \tag{39}$$

$$\nabla_{\dot{c}}X(t) := \Big\{ \dot{X}^i(t) + \dot{c}^j(t) N^i_j\Big(c(t), X(t)\Big) \Big\} \frac{\partial}{\partial x^i}|_{c(t)}. \tag{40}$$

$D_{\dot{c}}X(t)$ and $\nabla_{\dot{c}}X(t)$ are called the *linear covariant derivative* and the *covariant derivative* of $X(t)$ along c, respectively. $X(t)$ is said to be *linearly parallel (resp.*

parallel) along c if $D_{\dot c}X(t) = 0$ (resp. $\nabla_{\dot c}X(t) = 0$). It is known that for linearly parallel vector fields $X = X(t)$ and $Y = Y(t)$ along a geodesic c,

$$\mathbf{g}_{\dot c(t)}\Big(X(t), Y(t)\Big) = constant,$$

and for a parallel vector field $X = X(t)$ along a curve c,

$$F\Big(c(t), X(t)\Big) = constant.$$

5. Berwald Metrics

First let us consider a Riemannian metric $F = \sqrt{g_{ij}(x)y^iy^j}$ on a manifold M. By (33), we obtain $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$, where

$$\Gamma^i_{jk}(x) := \frac{1}{4}g^{il}(x)\Big\{\frac{\partial g_{lk}}{\partial x^j}(x) + \frac{\partial g_{jl}}{\partial x^k}(x) - \frac{\partial g_{jk}}{\partial x^l}(x)\Big\}. \tag{41}$$

In this case $G^i = G^i(x,y)$ are quadratic in $y \in T_xM$ at any point $x \in M$. A Finsler metric $F = F(x,y)$ is called a *Berwald metric* if in any standard local coordinate system, the spray coefficients $G^i = G^i(x,y)$ are quadratic in $y \in T_xM$ at any point $x \in M$.

There are many non-Riemannian Berwald metrics.

Example 5.1. Let (M_i, α_i), $i = 1, 2$, be arbitrary Riemannian manifolds and $M = M_1 \times M_2$. Let $f : [0,\infty) \times [0,\infty) \to [0,\infty)$ be an arbitrary C^∞ function satisfying

$$f(\lambda s, \lambda t) = \lambda f(s,t), \quad (\lambda > 0) \quad \text{and} \quad f(s,t) \neq 0 \text{ if } (s,t) \neq 0.$$

Define

$$F(x,y) := \sqrt{f\Big([\alpha_1(x_1,y_1)]^2,\ [\alpha_2(x_2,y_2)]^2\Big)}, \tag{42}$$

where $x = (x_1, x_2) \in M$ and $y = y_1 \oplus y_2 \in T_xM \cong T_{x_1}M_1 \oplus T_{x_2}M_2$. Now we are going to find additional condition on $f(s,t)$ under which the matrix $g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}$ is positive definite. Take standard local coordinate systems (x^a, y^a) in TM_1 and (x^α, y^α) in TM_2. Then $(x^i, y^j) := (x^a, x^\alpha, y^a, y^\alpha)$ is a standard local coordinate system in TM. Express α_1 and α_2 by

$$\alpha_1(x_1,y_1) = \sqrt{\bar g_{ab}(x_1)y^ay^b}, \qquad \alpha_2(x_2,y_2) = \sqrt{\bar g_{\alpha\beta}(x_2)y^\alpha y^\beta},$$

where $y_1 = y^a\frac{\partial}{\partial x^a}$ and $y_2 = y^\alpha\frac{\partial}{\partial x^\alpha}$. We obtain

$$g_{ab} = 2f_{ss}\bar y_a\bar y_b + f_s\bar g_{ab}, \quad g_{a\beta} = 2f_{st}\bar y_a\bar y_\beta, \quad g_{\alpha\beta} = 2f_{tt}\bar y_\alpha\bar y_\beta + f_t\bar g_{\alpha\beta}, \tag{43}$$

where $\bar y_a := \bar g_{ab}y^b$ and $\bar y_\alpha := \bar g_{\alpha\beta}y^\beta$. By an elementary argument, one can show that (g_{ij}) is positive definite if and only if $f(s,t)$ satisfies the following conditions:

$$f_s > 0, \quad f_t > 0, \quad f_s + 2sf_{ss} > 0, \quad f_t + 2tf_{tt} > 0,$$

and

$$f_sf_t - 2ff_{st} > 0.$$

In this case,

$$\det(g_{ij}) = h\Big([\alpha_1]^2,\ [\alpha_2]^2\Big)\det(\bar{g}_{ab})\det(\bar{g}_{\alpha\beta}), \tag{44}$$

where

$$h := (f_s)^{n_1-1}(f_t)^{n_2-1}\Big\{f_s f_t - 2ff_{st}\Big\}, \tag{45}$$

where $n_1 := \dim M_1$ and $n_2 := \dim M_2$.

By a direct computation, one can show that the spray coefficients of F split as the direct sum of the spray coefficients of α_1 and α_2, that is,

$$G^a(x,y) = \bar{G}^a(x_1,y_1), \qquad G^\alpha(x,y) = \bar{G}^\alpha(x_1,y_1), \tag{46}$$

where $G^a = \bar{G}^a(x_1,y_1)$ and $G^\alpha = \bar{G}^\alpha(x_2,y_2)$ are the spray coefficients of α_1 and α_2 respectively. From (46), one can see that the spray of F is independent of the choice of a particular function $f(s,t)$. In particular, $G^i(x,y)$ are quadratic in $y \in T_xM$. Thus F is a Berwald metric.

A typical example of $f = f_{\epsilon,k}$ is given by

$$f_{\epsilon,k} = s + t + \epsilon\sqrt[k]{s^k + t^k},$$

where ϵ is a nonnegative real number and k is a positive integer. The resulting Berwald metric using $f_{\epsilon,k}$ is discussed in [Szabó 1981].

Let (M,F) be a Berwald manifold and $p,q \in M$ be an arbitrary pair of points in M. Let $c : [0,1] \to M$ be a geodesic emanating from $p = c(0)$ to $q = c(1)$. Define a linear isomorphism $T : T_pM \to T_qM$ by $T(X(0)) := X(1)$, where $X(t)$ is a linearly parallel vector field along c, i.e., $D_{\dot c}X(t) = 0$. Since F is a Berwald metric, the linear covariant derivative $\nabla_{\dot c}$ coincides with the covariant derivative $D_{\dot c}$ along c. See (39) and (40). Thus $X(t)$ is also parallel along c, i.e., $\nabla_{\dot c}X(t) = 0$. Therefore, $F(c(t),X(t)) = \mathit{constant}$. This implies that $T : (T_pM,F_p) \to (T_qM,F_q)$ preserves the Minkowski norms. We have proved the following well-known result.

Proposition 5.2. ([Ichijyō 1976]) *On a Berwald manifold (M,F), all tangent spaces (T_xM,F_x) are linearly isometric to each other.*

On a Finsler manifold (M,F), we view the Minkowski norm F_x on T_xM as an *infinitesimal color pattern* at x. As we have mentioned early in Section 3, the Cartan torsion $\mathbf{C}_y$ describes the non-Euclidean features of the pattern in the direction $y \in T_xM\setminus\{0\}$. In the case when F is a Berwald metric on a manifold M, by Proposition 5.2, all tangent spaces (T_xM,F_x) are linearly isometric to each other, and (M,F) is modeled on a single Minkowski space. More precisely, for any pair points $x,x' \in M$ and a geodesic from x to x', the (linearly) parallel translation defines a linear isometry $T : (T_xM,F_x) \to (T_{x'}M,F_{x'})$. This linear isometry T maps the infinitesimal color pattern at x to that at x'. Thus the infinitesimal color patterns do not change over the manifold. If one looks at a Berwald manifold on a large scale, with the infinitesimal color pattern at each point shrunken to a single spot of color, then one can only see a space with uniform color. The color depends on the Minkowski model.

A Finsler metric F on a manifold M is said to be *affinely equivalent* to another Finsler metric $\bar{F}$ on M if F and $\bar{F}$ induce the same sprays. By (46), one can see that the family of Berwald metrics in (42) are affinely equivalent.

Proposition 5.3. ([Szabó 1981]) *Every Berwald metric on a manifold is affinely equivalent to a Riemannian metric.*

Based on this observation, Z.I. Szabo determined the local structure of Berwald metrics [Szabó 1981].

6. Gradient, Divergence and Laplacian

Let $F = F(x, y)$ be a Finsler metric on a manifold M and let $F^* = F^*(x, \xi)$ be dual to F. Let f be a C^1 function on M. At a point $x \in M$, the differential $df_x \in T_x^*M$ is a 1-form. Define the dual vector $\nabla f_x \in T_xM$ by

$$\nabla f_x := \ell_x^*(df_x), \tag{47}$$

where $\ell_x^* : T_x^*M \to T_xM$ is the inverse Legendre transformation. By definition, ∇f_x is uniquely determined by

$$\eta(\nabla f_x) := \frac{1}{2}\frac{d}{dt}\Big[F^{*2}(x, df_x + t\eta)\Big]_{|t=0}, \qquad \eta \in T_x^*M.$$

∇f_x is called the *gradient* of f at x. We have

$$F(x, \nabla f_x) = F^*(x, df_x).$$

If f is C^k $(k \geq 1)$, then ∇f is C^{k-1} on $\{df_x \neq 0\}$ and C^0 at any point $x \in M$ with $df_x = 0$.

Given a closed subset $A \subset M$, let $\rho(x) := d(A, x) := \sup_{z\in A} d(z, x)$ or $\rho(x) := -d(x, A) := -\sup_{z\in A} d(x, z)$. Then $\rho = \rho(x)$ is a locally Lipschitz function, hence it is differentiable almost everywhere. It is easy to verify that

$$F(x, \nabla\rho_x) = F^*(x, d\rho_x) = 1 \tag{48}$$

holds almost everywhere [Shen 2001b].

A function ρ is called a *distance function* of a Finsler metric F if it satisfies (48). For a C^∞ distance function of F, $\rho = \rho(x)$, on an open subset $U \subset M$, $\nabla\rho$ induces a Riemannian metric $\hat{F} := \sqrt{\mathbf{g}_{\nabla\rho}(v, v)}$ on U. Then ρ is a distance function of $\hat{F}$ and $\nabla\rho = \hat{\nabla}\rho$ is the gradient of ρ with respect to $\hat{F}$. See [Shen 2001b].

Every Finsler metric F defines a volume form

$$dV_F := \sigma_F(x) dx^1 \cdots dx^n,$$

where

$$\sigma_F := \frac{\mathrm{Vol}(\mathrm{B}^n)}{\mathrm{Vol}\{(y^i) \in \mathrm{R}^n \mid F(x, y^i \frac{\partial}{\partial x^i}|_x) < 1\}}. \tag{49}$$

Here $\mathrm{Vol}(U)$ denotes the Euclidean volume of an open subset $U \subset \mathrm{R}^n$. It is proved by H. Busemann that if F is reversible, the Hausdorff measure of the induced distance function d_F is represented by dV_F [Busemann 1947]. When $F = \sqrt{g_{ij}(x)y^iy^j}$ is Riemannian, $\sigma_F = \sqrt{\det(g_{ij}(x))}$ and

$$dV_F = \sqrt{\det(g_{ij}(x))}dx^1 \cdots dx^n.$$

For a vector field $X = X^i(x)\frac{\partial}{\partial x^i}|_x$ on M, the *divergence* $\mathrm{div}(X)$ is defined by

$$\mathrm{div}(X) := \frac{1}{\sigma_F(x)} \frac{\partial}{\partial x^i}\Big[\sigma_F(x)X^i(x)\Big]. \tag{50}$$

The *Laplacian* Δ on C^k $(k \geq 2)$ functions is defined by

$$\Delta f := \mathrm{div}(\nabla f).$$

Δ is a non-linear elliptic operator. Since ∇f is only C^0 at points x where $df_x = 0$, Δf is only defined weakly in the sense of distributions.

For a C^∞ distance function $\rho = \rho(x)$ on an open subset $U \subset M$, the level set $N_r := \rho^{-1}(r) \subset U$ is a C^∞ hypersurface in U. The quantity $H := \Delta\rho|_{N_r}$ can be defined as the *mean curvature* of N_r with respect to the normal vector $\mathbf{n} = \nabla\rho|_{N_r}$.

7. S-Curvature

Consider an n-dimensional Finsler manifold (M, F). As we have mentioned in Section 5, we view the Minkowski norm F_x on T_xM as an *infinitesimal color pattern* at x. The Cartan torsion $\mathbf{C}_y$ describes the non-Euclidean features of the pattern in the direction $y \in T_xM \setminus \{0\}$. The mean Cartan torsion $\mathbf{I}_y$ is the average value of $\mathbf{C}_y$. Besides the (mean) Cartan torsion, there is another geometric quantity of F_x. Take an arbitrary standard local coordinate system (x^i, y^i) and let

$$\tau := \ln \frac{\sqrt{\det\Big(g_{ij}(x,y)\Big)}}{\sigma_F(x)}, \tag{51}$$

where $\sigma_F = \sigma_F(x)$ is defined in (49). τ is called the *distortion* [Shen 1997] [Shen 2001b]. Intuitively, the distortion $\tau = \tau(x, y)$ is the directional *twisting number* of the infinitesimal color pattern at x. Observe that

$$\tau_{y^i} = \frac{\partial}{\partial y^i}\Big[\ln\sqrt{\det\Big(g_{jk}(x,y)\Big)}\Big] = \frac{1}{2}g^{jk}\frac{\partial g_{jk}}{\partial y^i} = g^{jk}C_{ijk} =: I_i. \tag{52}$$

Here σ_F does not occur in the first equality, because it is independent of y at each point x. If the distortion is isotropic at x, i.e, $\tau = \tau(x)$ is independent of the direction $y \in T_xM$, then $\tau(x) = 0$ and F_x is Euclidean (see Proposition 3.1). In this case, the infinitesimal color pattern is in the simplest form at every point.

It is natural to study the rate of change of the distortion along geodesics. For a vector $y \in T_xM \setminus \{0\}$, let $\sigma(t)$ be the geodesic with $\sigma(0) = x$ and $\dot\sigma(0) = y$.

Let

$$\mathbf{S} := \frac{d}{dt}\Big[\tau\Big(\sigma(t), \dot\sigma(t)\Big)\Big]|_{t=0}. \tag{53}$$

$\mathbf{S} = \mathbf{S}(x, y)$ is positively y-homogeneous of degree one,

$$\mathbf{S}(x, \lambda y) = \lambda \mathbf{S}(x, y), \qquad \lambda > 0.$$

$\mathbf{S}$ is called the *S-curvature.*

In a standard local coordinate system (x^i, y^i), let $G^i = G^i(x, y)$ denote the spray coefficients of F. Contracting (56) with g^{ij} yields

$$\frac{\partial G^m}{\partial y^m} = \frac{1}{2} g^{ml} \frac{\partial g_{ml}}{\partial x^i} y^i - 2 I_i G^i.$$

Then

$$\begin{aligned}
\mathbf{S} &= y^i \frac{\partial \tau}{\partial x^i} - 2 \frac{\partial \tau}{\partial y^i} G^i \\
&= \frac{1}{2} g^{ml} \frac{\partial g_{ml}}{\partial x^i} y^i - 2 I_i G^i - y^m \frac{\partial}{\partial x^m}\Big(\ln \sigma_F\Big) \\
&= \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m}\Big(\ln \sigma_F\Big).
\end{aligned} \tag{54}$$

Proposition 7.1. ([Shen 1997]) *For any Berwald metric, the S-curvature vanishes,* $\mathbf{S} = 0$.

There are many non-Berwaldian Finsler metrics with $\mathbf{S} = 0$. Namely, the class of Finsler metrics with $\mathbf{S} = 0$ strictly contains all Berwald metrics. Some comparison theorems in Riemannian geometry are still valid for Finsler metrics in this class [Shen 1997], [Shen 2001b].

By definition, the S-curvature is the covariant derivative of the distortion along geodesics. Let $\sigma(t)$ be a geodesic and

$$\tau(t) := \tau(\sigma(t), \dot\sigma(t)), \qquad \mathbf{S}(t) := \mathbf{S}(\sigma(t), \dot\sigma(t)).$$

By (53),

$$\mathbf{S}(t) = \tau'(t).$$

Thus if $\mathbf{S} = 0$, then $\tau(t) =$ *constant*. Intuitively, the distortion (twisting number) of the infinitesimal color pattern in the direction $\dot\sigma(t)$ does not change along any geodesic $\sigma = \sigma(t)$. However, the distortion might take different values along different geodesics. In the case when F is a Berwald metric, the infinitesimal color patterns do not change over the manifold (Proposition 5.2), thus the distortion of the pattern in the direction $\dot\sigma(t)$ does not change along any geodesic $\sigma = \sigma(t)$.

A Finsler metric F is said to have *isotropic S-curvature* if

$$\mathbf{S} = (n+1) cF.$$

More generally, F is said to have *almost isotropic S-curvature* if

$$\mathbf{S} = (n+1)\Big\{cF + \eta\Big\},$$

where $c = c(x)$ is a scalar function and $\eta = \eta_i(x)y^i$ is a *closed* 1-form.

Differentiating the S-curvature gives rise to another quantity. Let

$$E_{ij} := \frac{1}{2}\mathbf{S}_{y^iy^j}(x,y). \tag{55}$$

For $y \in T_xM \setminus \{0\}$, $\mathbf{E}_y = E_{ij}(x,y)dx^i \otimes dx^j$ is a symmetric bilinear form on T_xM. We call the family $\mathbf{E} := \{\mathbf{E}_y \mid y \in TM \setminus \{0\}\}$ the *mean Berwald curvature*, or simply the *E-curvature* [Shen 2001a]. Let $\mathbf{h}_y := h_{ij}(x,y)dx^i \otimes dx^j$, where $h_{ij} := FF_{y^iy^j}$. F is said to have *isotropic E-curvature* if

$$\mathbf{E} = \frac{n+1}{2} cF^{-1}\mathbf{h},$$

where $c = c(x)$ is a scalar function on M. Clearly, if the S-curvature is almost isotropic, then the E-curvature is isotropic. Conversely, if the E-curvature is isotropic, $\mathbf{E} = \frac{n+1}{2}cF^{-1}\mathbf{h}$, then there is a 1-form $\eta = \eta_i(x)dx^i$ such that $\mathbf{S} = (n+1)\{cF + \eta\}$. However, this η is not closed in general.

Finally, let us give another geometric significance of the S-curvature. Let $\rho = \rho(x)$ be a C^∞ distance function on an open subset $U \subset M$, i.e., $F(x, \nabla\rho_x) = F^*(x, d\rho_x) = 1$, $x \in U$. The gradient $\nabla\rho$ induces a Riemannian metric $\hat{F} = \hat{F}(z,v)$ on U by

$$\hat{F}(z,v) := \sqrt{\mathbf{g}_{\nabla\rho}(v,v)}, \qquad v \in T_zU.$$

Let Δ and $\hat{\Delta}$ denote the Laplacians on functions with respect to F and $\hat{F}$, respectively. Then $H = \Delta\rho|_{N_r}$ and $\hat{H} = \hat{\Delta}\rho|_{N_r}$ are the *mean curvature* of $N_r := \rho^{-1}(r)$ with respect to F and $\hat{F}$, respectively. The S-curvature can be expressed by

$$\mathbf{S}(\nabla\rho) = \hat{\Delta}\rho - \Delta\rho = \hat{H} - H.$$

By the above identities, one can estimate $\hat{\Delta}$, and obtain an estimate on $\Delta\rho$ under a Ricci curvature bound and a S-curvature bound. By these estimates, one can establish a volume comparison on the metric balls. See [Shen 2001b] for more details.

8. Landsberg Curvature

The (mean) Cartan torsion is a geometric quantity which characterizes the Euclidean norms among Minkowski norms on a vector space. On a Finsler manifold (M,F), one can view the Minkowski norm F_x on T_xM as an *infinitesimal color pattern* at x. The Cartan torsion $\mathbf{C}_y$ describes the non-Euclidean features of the pattern in the direction $y \in T_xM \setminus \{0\}$. The mean Cartan torsion $\mathbf{I}_y$ is the average value of $\mathbf{C}_y$. They reveal the non-Euclidean features which are different from that revealed by the distortion. Therefore, it is natural to study the rate of change of the (mean) Cartan torsion along geodesics.

Let (M, F) be a Finsler manifold. To differentiate the (mean) Cartan torsion along geodesics, we need linearly parallel vector fields along a geodesic. Recall that a vector field $U(t) := U^i(T)\frac{\partial}{\partial x^i}|_{\sigma(t)}$ along a geodesic $\sigma(t)$ is said to be linearly parallel along σ if $D_{\dot\sigma}U(t) = 0$, i.e.,

$$\dot U^i(t) + U^j(t)N^i_j(\sigma(t), \dot\sigma(t)) = 0. \tag{56}$$

By a direct computation using (33), one can verify that

$$y^m\frac{\partial g_{ij}}{\partial x^m} - 2G^m\frac{\partial g_{ij}}{\partial y^m} = g_{im}N^m_j + g_{mj}N^m_i. \tag{57}$$

Using (55) and (56), one can verify that for any linearly parallel vector fields $U(t), V(t)$ along σ,

$$\frac{d}{dt}\Big[\mathbf{g}_{\dot\sigma(t)}\Big(U(t), V(t)\Big)\Big] = 0.$$

In this sense, the family of inner products $\mathbf{g}_y$ does not change along any geodesic. However, for linearly parallel vector fields $U(t), V(t)$ and $W(t)$ along σ, the functions $\mathbf{C}_{\dot\sigma(t)}\Big(U(t), V(t), W(t)\Big)$ and $\mathbf{I}_{\dot\sigma(t)}\Big(U(t)\Big)$ do change, in general. Set

$$\mathbf{L}_y(u, v, w) := \frac{d}{dt}\Big[\mathbf{C}_{\dot\sigma(t)}\Big(U(t), V(t), W(t))\Big]_{|t=0}, \tag{58}$$

$$\mathbf{J}_y(u) := \frac{d}{dt}\Big[\mathbf{I}_{\dot\sigma(t)}\Big(U(t))\Big]_{|t=0}, \tag{59}$$

where $u = U(0), v = V(0), w = W(0)$ and $y = \dot\sigma(0) \in T_xM$. The family $\mathbf{L} = \{\mathbf{L}_y \mid y \in TM \setminus \{0\}\}$ is called the *Landsberg curvature* and the family $\mathbf{J} = \{\mathbf{J}_y \mid y \in TM \setminus \{0\}\}$ is called the *mean Landsberg curvature*. For the sake of simplicity, we call $\mathbf{L}$ and $\mathbf{J}$ the *L-curvature* and the *J-curvature*, respectively. A Finsler metric is called a *Landsberg metric* (resp. *weakly Landsberg metric*) if $\mathbf{L} = 0$ (resp. $\mathbf{J} = 0$).

Let (x^i, y^i) be a standard local coordinate system in TM and $C_{ijk} := \frac{1}{4}[F^2]_{y^iy^jy^k}$. From the definition, $\mathbf{L}_y = L_{ijk}dx^i \otimes dx^j \otimes dx^k$ is given by

$$L_{ijk} = y^m\frac{\partial C_{ijk}}{\partial x^m} - 2G^m\frac{\partial C_{ijk}}{\partial y^m} - C_{mjk}N^m_i - C_{imk}N^m_j - C_{ijm}N^m_k, \tag{60}$$

and $\mathbf{J} = J_idx^i$ is given by

$$J_i = y^m\frac{\partial I_i}{\partial x^m} - 2G^m\frac{\partial I_i}{\partial y^m} - I_mN^m_i. \tag{61}$$

We have

$$J_i = g^{jk}L_{ijk}.$$

It follows from (33) that

$$g_{sm}G^m = \frac{1}{4}\Big\{2\frac{\partial g_{sk}}{\partial x^m} - \frac{\partial g_{km}}{\partial x^s}\Big\}y^ky^m. \tag{62}$$

Differentiating (61) with respect to y^i, y^j and y^k, then contracting the resulting identity by $\frac{1}{2}y^s$, one obtains

$$L_{ijk} = -\frac{1}{2} y^s g_{sm} \frac{\partial^3 G^m}{\partial y^i \partial y^j \partial y^k}. \tag{63}$$

Thus if $G^m = G^m(x,y)$ are quadratic in $y \in T_xM$, then $L_{ijk} = 0$. This proves the following well-known result.

Proposition 8.1. *Every Berwald metric is a Landsberg metric.*

By definition, the (mean) Landsberg curvature is the covariant derivative of the (mean) Cartan torsion along a geodesic. Let $\sigma = \sigma(t)$ be a geodesic and $U = U(t), V = V(t), W = W(t)$ be parallel vector fields along σ. Let

$$\mathbf{L}(t) := \mathbf{L}_{\dot\sigma(t)}\Big(U(t), V(t), W(t)\Big), \qquad \mathbf{C}(t) := \mathbf{C}_{\dot\sigma(t)}\Big(U(t), V(t), W(t)\Big).$$

By (57),

$$\mathbf{L}(t) = \mathbf{C}'(t).$$

If F is Landsbergian, i.e., $\mathbf{L} = 0$, then the Cartan torsion $\mathbf{C}_{\dot\sigma}$ in the direction $\dot\sigma(t)$ is constant along σ. Intuitively, the infinitesimal color pattern in the direction $\dot\sigma(t)$ does not change along σ. But the patterns might look different at neighboring points.

It is easy to see that in dimension two, a Finsler metric is Berwaldian if and only if $\mathbf{E} = 0$ (or $\mathbf{S} = 0$) and $\mathbf{J} = 0$. It seems that $\mathbf{E}$ and $\mathbf{L}$ are complementary to each other. So we may ask the followsing question: *Is a Finsler metric Berwaldian if* $\mathbf{E} = 0$ *and* $\mathbf{L} = 0$? A more difficult problem is as follows: *Is a Finsler metric Berwaldian if* $\mathbf{L} = 0$? So far, we do not know any answer yet.

Finsler metrics with $\mathbf{L} = 0$ can be generalized as follows. Let F be a Finsler metric on an n-dimensional manifold M. F is said to have *relatively isotropic L-curvature* (resp. *relatively isotropic J-curvature*) if

$$\mathbf{L} + cF\mathbf{C} = 0, \qquad (\textit{resp.}\ \mathbf{J} + cF\mathbf{I} = 0),$$

where $c = c(x)$ is a scalar function on M.

Many interesting Finsler metrics having isotropic L-curvature or (almost) isotropic S-curvature that will be discussed in the following two sections.

9. Randers Metrics with Isotropic S-Curvature

In this section, we are going to discuss Randers metrics of isotropic S-curvature. Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $\beta = b_i(x)y^i$. The Randers metric is a special (α,β)-metric in the form $F = \alpha\phi(\beta/\alpha)$, where $\phi(s) = 1 + s$. By (38), the spray coefficients G^i of F and $\bar G^i$ of α are related via:

$$G^i = \bar G^i + Py^i + Q^i, \tag{64}$$

where

$$P := \frac{e_{00}}{2F} - s_0, \qquad Q^i = \alpha s^i{}_0, \tag{65}$$

and $e_{00} := e_{ij}y^iy^j$, $s_0 := s_iy^i$, $s^i{}_0 := s^i{}_jy^j$. The above formula can be found in [Antonelli et al. 1993].

Let

$$\rho := \ln\sqrt{1-\|\beta\|_x^2}.$$

The volume forms dV_F and dV_α are related by

$$dV_F = e^{(n+1)\rho(x)}dV_\alpha.$$

Since $s_{ij} = s_{ji}$, $s_{00} := s_{ij}y^iy^j = 0$ and $s^i{}_i = a^{ij}s_{ij} = 0$. Observe that

$$\begin{aligned}\frac{\partial(Py^m)}{\partial y^m} &= \frac{\partial P}{\partial y^m}y^m + nP = (n+1)P,\\ \frac{\partial Q^m}{\partial y^m} &= \alpha^{-1}s_{00} + \alpha s^m{}_m = 0.\end{aligned}$$

Since α is Riemannian, the following holds,

$$\frac{\partial\bar{G}^m}{\partial y^m} = y^m\frac{\partial}{\partial x^m}\Big(\ln\sigma_\alpha\Big).$$

By the above identities, one obtains

$$\begin{aligned}\mathbf{S} &= \frac{\partial G^m}{\partial y^m} - y^m\frac{\partial}{\partial x^m}\Big(\ln\sigma_F\Big)\\ &= \frac{\partial\bar{G}^m}{\partial y^m} + \frac{\partial(Py^m)}{\partial y^m} + \frac{\partial Q^m}{\partial y^m} - (n+1)y^m\frac{\partial\rho}{\partial x^m} - y^m\frac{\partial}{\partial x^m}\Big(\ln\sigma_\alpha\Big)\\ &= (n+1)\Big\{P-\rho_0\Big\}\\ &= (n+1)\Big\{\frac{e_{00}}{2F} - (s_0+\rho_0)\Big\}, \qquad (66)\end{aligned}$$

where $\rho_0 := \rho_{x^i}(x)y^i$. We have the following

Lemma 9.1. ([Chen–Shen 2003a]) *For a Randers metric $F = \alpha+\beta$ on an n-dimensional manifold M, the following are equivalent*

(a) *the S-curvature is isotropic,* $\mathbf{S} = (n+1)cF$*;*
(b) *the S-curvature is almost isotropic,* $\mathbf{S} = (n+1)\{cF+\eta\}$*,*
(c) *the E-curvature is isotropic,* $\mathbf{E} = \frac{1}{2}(n+1)cF^{-1}\mathbf{h}$*,*
(d) $e_{00} = 2c(\alpha^2-\beta^2)$*,*

where $c = c(x)$ is a scalar function on M and $\eta = \eta_i(x)dx^i$ is a closed 1-form on M.

Proof: The proofs for (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are obvious.

(c) $\Rightarrow$ (d). First, we have

$$\mathbf{S} = (n+1)\Big\{cF+\eta\Big\},$$

where η is a 1-form on M. By (65), (c) is equivalent to the following

$$e_{00} = 2cF^2 + 2\theta F,$$

where $\theta := s_0 + \rho_0 + \eta$. This implies that

$$e_{00} = 2c(\alpha^2+\beta^2) + 2\theta\beta, \qquad 0 = 4c\beta + 2\theta.$$

Solving for θ from the above equation on the right, $\theta = -2c\beta$, then plugging it into the one on the left, one obtains (d).

(d) $\Rightarrow$ (a). Plugging $e_{00} = 2c(\alpha^2 - \beta^2)$ into (65) yields

$$\mathbf{S} = (n+1)\Big\{c(\alpha - \beta) - (s_0 + \rho_0)\Big\}. \tag{67}$$

On the other hand, contracting $e_{ij} = 2c(a_{ij} - b_i b_j)$ with b^j gives

$$s_i + \rho_i + 2cb_i = 0.$$

Thus $s_0 + \rho_0 = -2c\beta$. Plugging it into (66) yields (a). □

Example 9.2. Let $V = (A, B, C)$ be a vector field on a domain $U \subset \mathrm{R}^3$, where $A = A(r,s,t)$, $B = B(r,s,t)$ and $C = C(r,s,t)$ are C^∞ functions on U with

$$|V(x)| = \sqrt{A(x)^2 + B(x)^2 + C(x)^2} < 1, \qquad \forall x = (r,s,t) \in U.$$

Let $\Phi := |y|$ be the standard Euclidean metric on R^3. Define $F = \alpha + \beta$ by (16) for the pair (Φ, V). α and β are given by

$$\begin{aligned} \alpha &= \frac{\sqrt{\langle V(x), y\rangle^2 + |y|^2(1 - |V(x)|^2)}}{1 - |V(x)|^2} \\ \beta &= -\frac{\langle V(x), y\rangle}{1 - |V(x)|^2}, \end{aligned}$$

where $y = (u, v, w) \in T_xU \cong \mathrm{R}^3$. One can easily verify that $\|\beta\|_x < 1$ for $x \in U$. By a direct computation, one obtains

$$\begin{aligned} e_{11} &= \frac{B^2(A_r - B_s) + C^2(A_r - C_t) - A_r + H}{1 - A^2 - B^2 - C^2} \\ e_{22} &= \frac{A^2(B_s - A_r) + C^2(B_s - C_t) - B_s + H}{1 - A^2 - B^2 - C^2} \\ e_{33} &= \frac{A^2(C_t - A_r) + B^2(C_t - B_s) - C_t + H}{1 - A^2 - B^2 - C^2} \\ e_{12} &= -\frac{A_s + B_r}{2} \\ e_{13} &= -\frac{A_t + C_r}{2} \\ e_{23} &= -\frac{B_t + C_s}{2}, \end{aligned}$$

where $H := 2ABe_{12} + 2ACe_{13} + 2BCe_{23}$. Here as usual, we denote $A_r = \frac{\partial A}{\partial r}$, etc. On the other hand,

$$a_{ij} - b_i b_j = \frac{\delta_{ij}}{1 - A^2 - B^2 - C^2}.$$

It is easy to verify that

$$e_{ij} = 2c(a_{ij} - b_i b_j)$$

holds if and only if A, B, and C satisfy the following equations:

$$A_r = B_s = C_t,$$

and

$$A_t + C_r = 0, \quad A_s + B_r = 0, \quad B_t + C_s = 0.$$

In this case,

$$c = -\frac{A_r}{2} = -\frac{B_s}{2} = -\frac{C_t}{2}.$$

By Lemma 9.1, we know that $\mathbf{S} = 4cF$.

If $F = \alpha + \beta$ on an n-dimensional manifold M is generated from the pair (Φ, V), where $\Phi = \sqrt{\phi_{ij} y^i y^j}$ is a Riemannian metric and $V = V^i \frac{\partial}{\partial x^i}$ is a vector field on M with $\phi_{ij}(x)V^i(x)v^j(x) < 1$ for any $x \in M$, then F has isotropic S-curvature, $\mathbf{S} = (n+1)c(x)F$, if and only if

$$V_{i;j} + V_{j;i} = -4c\phi_{ij},$$

where $V_i = \phi_{ij}V^j$ and $V_{i;j}$ denote the covariant derivatives of V with respect to Φ. This observation is made by Hao Xing [Xing 2003]. This fact also follows from [Bao–Robles 2003b], although it is not proved directly.

10. Randers Metrics with Relatively Isotropic L-Curvature

In this section we are going to study Randers metrics with relatively isotropic (mean) Landsberg curvature. From its definition, the mean Landsberg curvature is the mean value of the Landsberg curvature. Thus if a Finsler metric has isotropic Landsberg curvature, then it must have isotropic mean Landsberg curvature. The author does not know whether or not the converse is true as well. So far no counter-example has been founded yet. Nevertheless, for Randers metrics, "having isotropic mean Landsberg curvature" implies "having isotropic Landsberg curvature". According to Lemma 3.2, the Cartan torsion is given by (31). Differentiating (31) along a geodesic and using (59) and (60), we obtain

$$L_{ijk} = \frac{1}{n+1}\Big\{J_i h_{jk} + J_j h_{ik} + J_k h_{ij}\Big\}. \tag{68}$$

Here we have used the fact that the angular form $\mathbf{h}_y$ is constant along geodesics. By (31) and (67), one can easily show that $J_i + cFI_i = 0$ if and only if $L_{ijk} + cFC_{ijk} = 0$. This proves the above claim.

Lemma 10.1. ([Chen–Shen 2003a]) *For any non-Riemannian Randers metric $F = \alpha + \beta$ on an n-dimensional manifold M, the following are equivalent*

(a) $\mathbf{J} + c(x)F\,\mathbf{I} = 0$ *(or $\mathbf{L} + c(x)F\mathbf{C} = 0$);*
(b) $\mathbf{S} = (n+1)c(x)F$ *and β is closed;*
(c) $\mathbf{E} = \frac{1}{2}c(x)F^{-1}\mathbf{h}$ *and β is closed;*
(d) $e_{00} = 2c(x)(\alpha^2 - \beta^2)$ *and β is closed,*

where $c(x)$ is a scalar function on M.

Proof: By (67), to compute L_{ijk}, it suffices to compute J_i. First, the mean Cartan torsion is given by

$$I_i = \frac{1}{2}(n+1)F^{-1}\alpha^{-2}\Big\{\alpha^2 b_i - \beta y_i\Big\}, \tag{69}$$

where $y_i := a_{ij}y^j$. By a direct computation using (60), one obtains

$$
\begin{aligned}
J_i &= \frac{1}{4}(n+1)F^{-2}\alpha^{-2}\Big\{2\alpha\Big[(e_{i0}\alpha^2 - y_i e_{00}) \\
&\quad -2\beta(s_i\alpha^2 - y_i s_0) + s_{i0}(\alpha^2+\beta^2)\Big] \\
&\quad +\alpha^2(e_{i0}\beta - b_i e_{00}) + \beta(e_{i0}\alpha^2 - y_i e_{00}) \\
&\quad -2(s_i\alpha^2 - y_i s_0)(\alpha^2+\beta^2) + 4 s_{i0}\alpha^2\beta\Big\}.
\end{aligned} \tag{70}
$$

Using the above formulas (68) and (69), one can easily prove the lemma. □

According to Lemma 10.1, for any Randers metric $F = \alpha + \beta$, $\mathbf{J} = 0$ if and only if $e_{00} = 0$ and $d\beta = 0$. This is equivalent to $b_{i;j} = 0$, in which case, the spray coefficients of F coincide with that of α. This observation leads to the following result, which was first established by the collective efforts of the following papers: [Matsumoto 1974], [Hashiguchi–Ichijyō 1975], [Kikuchi 1979], and [Shibata et al. 1977].

Proposition 10.2. *For a Randers metric $F = \alpha + \beta$, the following are equivalent:*

(a) *F is a weakly Landsberg metric, $\mathbf{J} = 0$;*
(b) *F is a Landsberg metric, $\mathbf{L} = 0$;*
(c) *F is a Berwald metric;*
(d) *β is parallel with respect to α.*

Example 10.3. Consider the Randers metric $F = \alpha + \beta$ on R^n, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $\beta = b_i(x)y^i$ are defined by

$$
\begin{aligned}
\alpha &:= \frac{\sqrt{(1-\epsilon^2)\langle x,y\rangle^2 + \epsilon|y|^2(1+\epsilon|x|^2)}}{1+\epsilon|x|^2} \\
\beta &:= \frac{\sqrt{1-\epsilon^2}\,\langle x,y\rangle}{1+\epsilon|x|^2},
\end{aligned}
$$

where ϵ is an arbitrary constant with $0 < \epsilon \le 1$. Since β is closed, $s_{ij} = 0$ and $s_i = 0$. After computing $b_{i;j}$, one obtains

$$e_{ij} = \frac{\epsilon\sqrt{1-\epsilon^2}}{(1+\epsilon|x|^2)(\epsilon+|x|^2)}\delta_{ij}.$$

On the other hand

$$a_{ij} - b_ib_j = \frac{\epsilon}{1+\epsilon|x|^2}\delta_{ij}.$$

Thus $e_{ij} = 2c(a_{ij} - b_ib_j)$ with

$$c := \frac{\sqrt{1-\epsilon^2}}{2(\epsilon+|x|^2)}.$$

By Lemma 10.1, F satisfies

$$\mathbf{L} + cF\mathbf{C} = 0, \qquad \mathbf{S} = (n+1)cF, \qquad \mathbf{E} = \frac{1}{2}cF^{-1}\mathbf{h}.$$

See [Mo–Yang 2003] for a family of more general Randers metrics with non-constant isotropic S-curvature.

11. Riemann Curvature

The Riemann curvature is an important quantity in Finsler geometry. This quantity is introduced by Riemann for Riemannian metrics in 1854. Later on, L. Berwald extended this notion to Finsler metrics using the Berwald connection [Berwald 1926], [Berwald 1928]. Berwald's extension of the Riemann curvature is a milestone in Finsler geometry.

Let (M, F) be a Finsler manifold and let $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ be the induced spray. For a vector $y \in T_xM \setminus \{0\}$, set

$$R^i{}_k := 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{71}$$

The local curvature functions $R^i{}_k$ and $R_{jk} := g_{ij} R^i{}_k$ satisfy

$$R^i{}_k y^k = 0, \qquad R_{jk} = R_{kj}. \tag{72}$$

$\mathbf{R}_y = R^i{}_k \frac{\partial}{\partial x^i} \otimes dx^k : T_xM \to T_xM$ is a well-defined linear map. We call the family $\mathbf{R} = \{\mathbf{R}_y \mid y \in TM \setminus \{0\}\}$ the *Riemann curvature*. The Riemann curvature is actually defined for sprays as shown in [Kosambi 1933], [Kosambi 1935]. When the Finsler metric is Riemannian, then

$$R^i{}_k(x, y) = R_j{}^i{}_{kl}(x) y^j y^l,$$

where $R(u, v)w = R_j{}^i{}_{kl}(x) w^j u^i v^j \frac{\partial}{\partial x^i}|_i$ denotes the Riemannian curvature tensor. Namely, $\mathbf{R}_y(u) = R(u, y)y$.

The geometric meaning of the Riemann curvature lies in the second variation of geodesics. Let $\sigma(t)$, $a \le t \le b$, be a geodesic in M. Take a geodesic variation $H(t, s)$ of $\sigma(t)$, that is, each curve $\sigma_s(t) := H(t, s)$, $a \le t \le b$, is a geodesic and $\sigma_0 = \sigma$. Let

$$J(t) := \frac{\partial H}{\partial s}(t, 0).$$

Then $J(t)$ satisfies the following so-called Jacobi equation

$$D_{\dot\sigma} D_{\dot\sigma} J(t) + \mathbf{R}_{\dot\sigma(t)}(J(t)) = 0, \tag{73}$$

where $D_{\dot\sigma}$ is defined in (39). See [Kosambi 1933], [Kosambi 1935].

There is another way to define the Riemann curvature. For a vector $y \in T_xM$, extend it to a non-zero C^∞ *geodesic field* Y in an open neighborhood U of x. Here a vector field is said to be *geodesic* if every integral curve of it is a geodesic. Define

$$\hat F(z, v) := \sqrt{\mathbf{g}_{Y_z}(v, v)}, \qquad v \in T_zU, \ z \in U.$$

$\hat F = \hat F(z, v)$ is a Riemannian metric on U. Let $\hat{\mathbf{g}} = \mathbf{g}_Y$ denote the induced inner product by $\hat F$ and $\hat{\mathbf{R}}$ denote the Riemann curvature of $\hat F$. The following properties are well-known in Riemannian geometry.

$$\hat{\mathbf{R}}_y(u) = 0, \qquad \hat{\mathbf{g}}(\hat{\mathbf{R}}_y(u), v) = \hat{\mathbf{g}}(u, \hat{\mathbf{R}}_y(v)), \tag{74}$$

where $u, v \in T_xU$. An important fact is

$$\mathbf{R}_y(u) = \hat{\mathbf{R}}_y(u), \qquad u \in T_xM. \tag{75}$$

See Proposition 6.2.2 in [Shen 2001b] for a proof of (74). Note $\hat{\mathbf{g}}_x = \mathbf{g}_y$. It follows from (73) and (74) that

$$\mathbf{R}_y(y) = 0, \qquad \mathbf{g}_y(\mathbf{R}_y(u), v) = \mathbf{g}_y(u, \mathbf{R}_y(v)), \tag{76}$$

where $u, v \in T_xM$. The equation (75) in local coordinates is just (71). See [Shen 2001b] for the application of (74) in comparison theorems in conjunction with the S-curvature.

For a two-dimensional subspace $\Pi \subset T_xM$, and a non-zero vector $y \in \Pi$, define

$$\mathbf{K}(\Pi, y) := \frac{\mathbf{g}_y(\mathbf{R}_y(u), u)}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - \mathbf{g}_y(y, u)^2}, \tag{77}$$

where $u \in \Pi$ such that $\Pi = \mathrm{span}\{y, u\}$. One can use (75) to show that $\mathbf{K}(\Pi, y)$ is independent of the choice of a particular vector u, but it is usually dependent on y. We call $\mathbf{K}(\Pi, y)$ the *flag curvature* of the "flag" (Π, y). When $F = \sqrt{g_{ij}(x)y^iy^j}$ is a Riemannian, $\mathbf{K}(\Pi, y) = \mathbf{K}(\Pi)$ is independent of $y \in \Pi$, in which case, $\mathbf{K}(\Pi)$ is usually called the *sectional curvature* of the "section" $\Pi \subset T_xM$.

A Finsler metric F on a manifold M is said to be *of scalar curvature* $\mathbf{K} = \mathbf{K}(x, y)$ if for any $y \in T_xM \setminus \{0\}$ the flag curvature $\mathbf{K}(\Pi, y) = \mathbf{K}(x, y)$ is independent of the tangent planes Π containing y. From the definition, the flag curvature is a scalar function $\mathbf{K} = \mathbf{K}(x, y)$ if and only if in a standard local coordinate system,

$$R^i{}_k = \mathbf{K}F^2 h^i_k, \tag{78}$$

where $h^i_k := g^{ij}h_{jk} = g^{ij}FF_{y^jy^k}$. F is said to be *of constant flag curvature* if the flag curvature is a constant. For a Riemannian metric, if the flag curvature $\mathbf{K}(\Pi, y) = \mathbf{K}(x, y)$ is a scalar function on TM, then $\mathbf{K}(x, y) = \mathbf{K}(x)$ is independent of $y \in T_xM$ and it is a constant when $n \geq 3$ by the Schur Lemma. In the next section, we are going to show that any locally projectively flat Finsler metric is of scalar curvature. Thus such metrics are our rich source of Finsler metrics of scalar curvature.

One of the important problems in Finsler geometry is to classify Finsler metrics of scalar curvature, in particular those of constant flag curvature. In [Shen 2003b], we characterized the local structures of projectively flat Finsler metrics of constant flag curvature. At an earlier time, R. Bryant successfully classified the global structures of projectively flat Finsler metrics of $\mathbf{K} = 1$ on S^n, he also gave some ideas for constructing non-projectively flat metrics of $\mathbf{K} = 1$ on S^n. See [Bryant 1996], [Bryant 1997], [Bryant 2002].

Very recently, some non-projectively flat metrics of constant flag curvature have been explicitly constructed [Bao–Shen 2002], [Bejancu–Farran 2002], [Shen 2003a]-[Shen 2002], [Bao–Robles 2003a], etc. These metrics are Randers metrics. Therefore it is a natural problem to classify Randers metrics of constant flag curvature. In fact, this problem was first attacked by Yasuda–Shimada [Yasuda–Shimada 1977] and Matsumoto [Matsumoto 1989]. They obtained "sufficient and necessary conditions" for a Randers metric to be of constant flag curvature. Using Yasuda–Shimada's result strictly as an inspiration, Bao–Shen constructed a family of Randers metrics on S^3 with $\mathbf{K} = 1$ [Bao–Shen 2002]. Bao–Shen's examples satisfy the "sufficient and necessary conditions" listed in [Yasuda–Shimada 1977] and [Matsumoto 1989]. It was generally believed that Yasuda–Shimada's result was completely true, until Shen found some new examples in [Shen 2003c] and [Shen 2002]. Shortly after these examples were found, Randers metrics of constant flag curvature were characterized by Bao–Robles [Bao–Robles 2003a] using a system of PDEs. The same conclusion was simultaneously obtained in [Matsumoto–Shimada 2002] by a different method. This characterization subsequently led to the corrected version of the Yasuda–Shimada theorem. Finally, using the characterization in [Bao–Robles 2003a], and motivated by some constructions in [Shen 2003c] and [Shen 2002], Bao–Robles–Shen have classified Randers metrics of constant flag curvature with the help of formula (18).

Theorem 11.1. ([Bao et al. 2003]) *Let* $\Phi = \sqrt{\phi_{ij} y^i y^j}$ *be a Riemannian metric and* $V = V^i \frac{\partial}{\partial x^i}$ *be a vector field on a manifold* M *with* $\Phi(x, V_x) < 1$ *for all* $x \in M$. *Let* F *be the Randers metric defined by (18).* F *is of constant flag curvature* $\mathbf{K} = \lambda$ *if and only if*

(a) *there is a constant* c *such that* $V = V^i \frac{\partial}{\partial x^i}$ *satisfies*

$$V_{i|j} + V_{j|i} = -4c\phi_{ij}, \tag{79}$$

where $V_i := \phi_{ij} V^j$,

(b) Φ *has constant sectional curvature* $\tilde{\mathbf{K}} = \lambda + c^2$,

where " $_|$ *" denotes the covariant derivative with respect to* Φ *and* c *is a constant.*

We should remark that the equation (78) alone is always equivalent to that $\mathbf{S} = (n+1)cF$, even $c = c(x)$ is a scalar function on M [Xing 2003].

An analogue of Theorem 11.1 still holds for Randers metrics of isotropic Ricci curvature, i.e., $\mathbf{Ric} = (n-1)\lambda F^2$, where $\lambda = \lambda(x)$ is a scalar function on M. See Bao–Robles' article [Bao–Robles 2003b] in the same volume.

We have not extended the above result to Randers metrics of scalar curvature. Usually, the isotropic S-curvature condition simplifies the classification problem. It seems possible to classify Randers metrics of scalar curvature and isotropic S-curvature. The following example is our first attempt to understand Randers metrics of scalar curvature and isotropic S-curvature.

Example 11.2. Let $F = \alpha + \beta$ be the Randers metric defined in (23). Let $\Delta := 1 - |a|^2 |x|^4$. We can write $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ and $\beta = b_i(x) y^i$, where

$$\begin{aligned} a_{ij} &= \frac{\delta_{ij}}{\Delta} + \frac{(|x|^2 a^i - 2\langle a, x\rangle x^i)(|x|^2 a^j - 2\langle a, x\rangle x^j)}{\Delta^2}, \\ b_i &= -\frac{|x|^2 a^i - 2\langle a, x\rangle x^i}{\Delta}. \end{aligned}$$

Using the notations as in Section 9, by a direct computation using a Maple program, we obtain

$$\begin{aligned} e_{00} &= \frac{2\langle a,x\rangle |y|^2}{\Delta} = 2\langle a,x\rangle(\alpha^2-\beta^2), \\ s_{j0} &= 2\frac{\langle a,y\rangle x^j - \langle x,y\rangle a^j}{\Delta^2}, \\ s_0 &= b^i s_{i0} = 2\frac{|a|^2|x|^2\langle x,y\rangle + \langle a,x\rangle\langle a,v\rangle}{\Delta}. \end{aligned}$$

By Lemma 9.1, we see that F has isotropic S-curvature,

$$\mathbf{S} = (n+1)\langle a,x\rangle F.$$

By (63), the spray coefficients $G^i = G^i(x,y)$ of F are given by

$$G^i = \bar{G}^i + Py^i + \alpha a^{ij} s_{j0},$$

where

$$P = \frac{e_{00}}{2F} - s_0 = \langle a,x\rangle(\alpha-\beta) - s_0.$$

Using the formulas for G^i and $R^i{}_k$ in (70), we can show that F is also of scalar curvature with flag curvature

$$\mathbf{K} = 3\frac{\langle a,y\rangle}{F} + 3\langle a,x\rangle^2 - 2|a|^2|x|^2.$$

12. Projectively Flat Metrics

A Finsler metric $F = F(x,y)$ on an open subset $U \subset \mathrm{R}^n$ is said to be *projectively flat* if every geodesic $\sigma(t)$ is straight in U, i.e.,

$$\sigma^i(t) = x^i + f(t)y^i,$$

where $f(t)$ is a C^∞ function with $f(0)=0, f'(0)=1$ and $x=(x^i), y=(y^i)$ are constant vectors. This is equivalent to that $G^i = Py^i$, where $P = P(x,y)$ is positively y-homogeneous of degree one. P is called the *projective factor*.

In general, it is difficult to compute the Riemann curvature. For locally projectively flat Finsler metrics, however, the formula for the Riemann curvature is relatively simple.

Consider a projectively flat Finsler metric $F = F(x,y)$ on an open subset $U \subset \mathrm{R}^n$. By definition, its spray coefficients are in the form $G^i = Py^i$. Plugging them into (70), one obtains

$$R^i{}_k = \Xi\delta^i_k + \tau_k y^i, \tag{80}$$

where

$$\Xi = P^2 - P_{x^k}y^k, \qquad \tau_k = 3(P_{x^k} - PP_{y^k}) + \Xi_{y^k}.$$

Using (75), one can show that $\tau_k = -\Xi F^{-1}F_{y^k}$ and

$$R^i{}_k = \Xi\Big\{\delta^i_k - \frac{F_{y^k}}{F}y^i\Big\}. \tag{81}$$

Thus F is of scalar curvature with flag curvature

$$\mathbf{K} = \frac{\Xi}{F^2} = \frac{P^2 - P_{x^k} y^k}{F^2}. \tag{82}$$

Using (54), one obtains

$$\mathbf{S} = (n+1)P(x,y) - y^m \frac{\partial}{\partial x^m}\Big(\ln \sigma_F(x)\Big). \tag{83}$$

By (80), one immediately obtains the following

Proposition 12.1. ([Berwald 1929a], [Berwald 1929b]) *Every locally projectively flat Finsler metric is of scalar curvature.*

See also [Szabó 1977] and [Matsumoto 1980] for related discussions. There is another way to characterize projectively flat Finsler metrics.

Theorem 12.2. ([Hamel 1903], [Rapcsák 1961]) *Let $F = F(x,y)$ be a Finsler metric on an open subset $U \subset \mathrm{R}^n$. F is projectively flat if and only if F satisfies*

$$F_{x^k y^l} y^k - F_{x^l} = 0, \tag{84}$$

in which case, the spray coefficients are given by $G^i = P y^i$ where $P = \frac{F_{x^k} y^k}{2F}$.

Proof: Let $G^i = G^i(x,y)$ denote the spray coefficients of F in the standard coordinate system in $TU \cong U \times \mathrm{R}^n$. One can rewrite (33) as follows

$$G^i = P y^i + Q^i, \tag{85}$$

where

$$P = \frac{F_{x^k} y^k}{2F}, \qquad Q^i = \frac{1}{2} F g^{il} \Big\{ F_{x^k y^l} y^k - F_{x^l} \Big\}.$$

Thus F is projectively flat if and only if there is a scalar function $\tilde{P} = \tilde{P}(x,y)$ such that $G^i = \tilde{P} y^i$, i.e.,

$$P y^i + Q^i = \tilde{P} y^i. \tag{86}$$

Observe that

$$g_{ij} y^j Q^i = \frac{1}{2} F y^l \Big\{ F_{x^k y^l} y^k - F_{x^l} \Big\} = 0.$$

Assume that (85) holds. Contracting (85) with $y_i := g_{ij} y^j$ yields

$$P = \tilde{P}.$$

Then $Q^i = 0$ by (85). This implies that (83) holds. □

Since the equation (83) is linear, if F_1 and F_2 are projectively flat on an open subset $U \subset \mathrm{R}^n$, then the sum $F = F_1 + F_2$ is projectively flat on U too. If $F = F(x,y)$ is projectively flat on $U \subset \mathrm{R}^n$, then its reverse $\bar{F} := F(x,-y)$ is also projectively flat on U. Thus its symmetrization

$$\tilde{F} := \frac{1}{2}\Big\{ F(x,y) + F(x,-y) \Big\}$$

is projectively flat.

The Finsler metric $F = \alpha_\mu(x, y)$ in (5) satisfies (83), thus it is projectively flat.

Theorem 12.3. (Beltrami) *A Riemannian metric $F = F(x, y)$ on a manifold M is locally projectively flat if and only if it is locally isometric to the metric α_μ in (5).*

Using the formula (63), one can easily prove the following

Theorem 12.4. *A Randers metric $F = \alpha + \beta$ on a manifold is locally projectively flat if and only if α is locally projectively flat and β is closed.*

Besides projectively flat Randers metrics, we have the following example.

Example 12.5. Let $\phi = \phi(y)$ be a Minkowski norm on R^n and U be the strongly convex domain enclosed by the indicatrix of ϕ. Let $\Theta = \Theta(x, y)$ be the Funk metric on U. See Example 2.9. By (22),

$$\Theta_{x^k y^l} y^k = (\Theta\Theta_{y^k})_{y^l} y^k = \frac{1}{2}(\Theta^2)_{y^k y^l} y^k = \frac{1}{2}[\Theta^2]_{y^l} = \Theta_{x^l}.$$

Thus Θ is projectively flat with projective factor

$$P = \frac{\Theta_{x^k} y^k}{2\Theta} = \frac{\Theta\Theta_{y^k} y^k}{2\Theta} = \frac{1}{2}\Theta.$$

By (81), the flag curvature is given by

$$\mathbf{K} = \frac{\Theta^2 - 2\Theta_{x^k} y^k}{4\Theta^2} = \frac{\Theta^2 - 2\Theta^2}{4\Theta^2} = -\frac{1}{4}.$$

Example 12.6. ([Shen 2003b]) Let $\phi = \phi(y)$ be a Minkowski norm on R^n and U be the strongly convex domain enclosed by the indicatrix of ϕ. Let $\Theta = \Theta(x, y)$ be the Funk metric on U and define

$$F := \Theta(x, y)\Big\{1 + \Theta_{y^k}(x, y) x^k\Big\}.$$

Since $F(0, y) = \Theta(0, y) = \phi(y)$ is a Minkowski norm, by continuity, F is a Finsler metric for x nearby the origin. By (22), one can verify that

$$F_{x^k y^l} y^k = F_{x^l}, \qquad F_{x^k} y^k = 2\Theta F.$$

Thus F is projectively flat with projective factor $P = \Theta(x, y)$. By (22) and (81), we obtain

$$\mathbf{K} = \frac{\Theta^2 - \Theta_{x^k} y^k}{F^2} = \frac{\Theta^2 - \Theta\Theta_{y^k} y^k}{F^2} = 0.$$

Now let us take a look at the Finsler metric $F = F_\epsilon(x, y)$ defined in (10).

Example 12.7. Let

$$F := \frac{\sqrt{\Psi\Big[\frac{\sqrt{\Phi^2 + (1-\epsilon^2)|y|^4} + \Phi}{2}\Big] + (1-\epsilon^2)\langle x, y\rangle^2} + \sqrt{1-\epsilon^2}\langle x, y\rangle}{\Psi}, \tag{87}$$

where

$$\Phi := \epsilon|y|^2 + (|x|^2|y|^2 - \langle x, y\rangle^2), \qquad \Psi := 1 + 2\epsilon|x|^2 + |x|^4.$$

First, one can verify that $F = F_\epsilon(x, y)$ satisfies (83). Thus F is projectively flat with spray coefficients in the form $G^i = Py^i$, where $P = \frac{F_{x^k}(x,y)y^k}{2F(x,y)}$. By a direct computation using a Maple program, one obtains

$$P = \frac{\sqrt{\Psi\Big[\frac{\sqrt{\Phi^2+(1-\epsilon^2)|y|^4}-\Phi}{2}\Big] - (1-\epsilon^2)\langle x, y\rangle^2} - (\epsilon + |x|^2)\langle x, y\rangle}{\Psi}. \tag{88}$$

Further, one can verify that P satisfies the following equation,

$$P_{x^k} y^k = P^2 - F^2. \tag{89}$$

Thus

$$\mathbf{K} = \frac{P^2 - P_{x^k}y^k}{F^2} = \frac{P^2 - (P^2 - F^2)}{F^2} = 1.$$

That is, F has constant flag curvature $\mathbf{K} = 1$.

The projectively flat Finsler metrics constructed above are incomplete. They can be pulled back to S^n by (2) to form complete irreversible projectively flat Finsler metrics of constant flag curvature $\mathbf{K} = 1$. See [Bryant 1996] and [Bryant 1997].

13. Chern Connection and Some Identities

In the previous sections, we have introduced several geometric quantities, such as the Cartan torsion, the Landsberg curvature, the S-curvature and the Riemann curvature. These quantities are not completely independent of each other. To reveal the relationship among them, we use the Chern connection to describe these quantities as tensors on the slit tangent bundle, and use the exterior differentiation method to derive some important identities.

Let M be an n-dimensional manifold and TM the tangent bundle. Denote the elements in TM by (x, y) where $y \in T_xM$. The natural projection $\pi : TM \to M$ pulls back the tangent bundle TM over M to a vector bundle π^*TM over the slit tangent bundle $TM_o := TM \setminus \{0\}$. The fiber of π^*TM at each point $(x, y) \in TM_o$ is a copy of T_xM. Thus we denote the elements in π^*TM by (x, y, v) where $y \in T_xM \setminus \{0\}$ and $v \in T_xM$. Let $\partial_{i|(x,y)} := \Big(x, y, \frac{\partial}{\partial x^i}|_x\Big)$. Then $\{\partial_i\}$ is a local frame for π^*TM. Let (x^i, y^i) be a standard local coordinate system in TM_o. Then $HT^*M := \mathrm{span}\{dx^i\}$ is a well-defined subbundle of $T^*(TM_o)$. Let

$$\delta y^i := dy^i - N^i_j dx^j,$$

where $N^i_j := \frac{\partial G^i}{\partial y^j}$. Then $VT^*M := \mathrm{span}\{\delta y^i\}$ is a well-defined subbundle of $T^*(TM_o)$, so that $T^*(TM_o) = HT^*M \oplus VT^*M$. The Chern connection is a linear connection on π^*TM, locally expressed by

$$DX = \Big\{dX^i + X^j {\omega_j}^i\Big\} \otimes \partial_i, \qquad X = X^i\partial_i,$$

where the set of 1-forms $\{\omega_j^{\ i}\}$ are uniquely determined by

$$d\omega^i = \omega^j \wedge \omega_j^{\ i}, \tag{90}$$

$$dg_{ij} = g_{ik}\omega_j^{\ k} + g_{kj}\omega_i^{\ k} + 2C_{ijk}\omega^{n+k}, \tag{91}$$

where $g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}$, $C_{ijk} := \frac{1}{4}[F^2]_{y^iy^jy^k}$, $\omega^i := dx^i$, and $\omega^{n+i} := \delta y^i$. See [Bao–Chern 1993], [Bao et al. 2000], [Chern 1943], [Chern 1948], [Chern 1992]. Each 1-form $\omega_j^{\ i}$ is horizontal, i.e., $\omega_j^{\ i} = \Gamma^i_{jk}dx^k$. The coefficients $\Gamma^i_{jk} = \Gamma^i_{jk}(x,y)$ are called the *Christoffel symbols*. We have $N^i_j = y^k\Gamma^i_{jk}$. Thus

$$\omega^{n+i} = dy^i + y^j\omega_j^{\ i}. \tag{92}$$

Put

$$\Omega^i := d\omega^{n+i} - \omega^{n+j} \wedge \omega_j^{\ i}. \tag{93}$$

One can express Ω^i in the following form

$$\Omega^i = \frac{1}{2}R^i_{\ kl}\omega^k \wedge \omega^l - L^i_{\ kl}\omega^k \wedge \omega^{n+l}, \tag{94}$$

where

$$R^i_{\ kl} = \frac{\partial N^i_l}{\partial x^k} - \frac{\partial N^i_k}{\partial x^l} + N^s_l\frac{\partial N^i_k}{\partial y^s} - N^s_k\frac{\partial N^i_l}{\partial y^s}, \tag{95}$$

and

$$L^i_{\ kl} := y^j\frac{\partial \Gamma^i_{jk}}{\partial y^l} = \frac{\partial N^i_k}{\partial y^l} - \Gamma^i_{kl}. \tag{96}$$

Let $R^i_{\ k}$ be defined in (70) and L_{ijk} be defined in (59). Then

$$R^i_{\ k} = R^i_{\ kl}y^l, \qquad R^i_{\ kl} = \frac{1}{3}\Big\{\frac{\partial R^i_{\ k}}{\partial y^l} - \frac{\partial R^i_{\ l}}{\partial y^k}\Big\}. \tag{97}$$

and

$$L^i_{\ kl} = g^{ij}L_{jkl}. \tag{98}$$

Put

$$\Omega_j^{\ i} := d\omega_j^{\ i} - \omega_j^{\ k} \wedge \omega_k^{\ i}.$$

One can express $\Omega_j^{\ i}$ in the following form

$$\Omega_j^{\ i} = \frac{1}{2}R_j^{\ i}{}_{kl}\omega^k \wedge \omega^l + P_j^{\ i}{}_{kl}\omega^k \wedge \omega^{n+l}.$$

Differentiating (91) yields

$$\Omega^i = y^j\Omega_j^{\ i}.$$

Thus

$$R^i_{\ kl} = y^jR_j^{\ i}{}_{kl}, \qquad L^i_{\ kl} = -y^jP_j^{\ i}{}_{kl}.$$

There is a canonical way to define the covariant derivatives of a tensor on TM_o using the Chern connection. For the distortion τ on $TM\setminus\{0\}$, define $\tau_{|m}$ and $\tau_{.m}$ by

$$d\tau = \tau_{|i}\omega^i + \tau_{.i}\omega^{n+i}. \tag{99}$$

It follows from (52) that

$$\tau_{\cdot i} = \frac{\partial \tau}{\partial y^i} = I_i. \tag{100}$$

For the induced Riemannian tensor, $g = g_{ij}\omega^i \otimes \omega^j$, define $g_{ij|k}$ and $g_{ij\cdot k}$ by

$$dg_{ij} - g_{kj}\omega_i^{\ k} - g_{ik}\omega_j^{\ k} = g_{ij|k}\omega^k + g_{ij\cdot k}\omega^{n+k}.$$

It follows from (90) that

$$g_{ij|k} = 0, \qquad g_{ij\cdot k} = 2C_{ijk}.$$

Similarly, one can define $C_{ijk|l}$ at $I_{i|l}$. Equations (59) and (60) can be expressed as follows

$$L_{ijk} = C_{ijk|m}y^m, \qquad J_i = I_{i|m}y^m. \tag{101}$$

Differentiating (92) yields the following Bianchi identity

$$d\Omega^i = -\Omega^j \wedge \omega_j^{\ i} + \omega^{n+j} \wedge \Omega_j^{\ i}. \tag{102}$$

It follows from (101) that

$$R_j^{\ i}{}_{kl} = R^i{}_{kl\cdot j} + L^i{}_{kj|l} - L^i{}_{lj|k} + L^i{}_{lm}L^m_{kj} - L^i{}_{km}L^m_{lj}. \tag{103}$$

We are going to find other relationship between the curvature tensors and the Finsler metric. Differentiating (90) yields

$$0 = g_{ik}\Omega_j^{\ k} + g_{kj}\Omega_i^{\ k} + 2(C_{ijk|l}\omega^l + C_{ijk\cdot l}\omega^{n+l}) \wedge \omega^{n+k} + 2C_{ijk}\Omega^k.$$

It follows that

$$R_{jikl} + R_{ijkl} + 2C_{ijm}R^m_{kl} = 0, \tag{104}$$

$$P_{jikl} + P_{ijkl} + 2C_{ijl|k} - 2C_{ijm}L^m_{kl} = 0, \tag{105}$$

where $R_{jikl} := g_{im}R_j^{\ m}{}_{kl}$ and $P_{jikl} := g_{im}P_j^{\ m}{}_{kl}$. Then (102) can be expressed by

$$R_{jikl} = g_{im}R^m_{kl\cdot j} + L_{ikj|l} - L_{ilj|k} + L_{ilm}L^m_{kj} - L_{ikm}L^m_{lj}.$$

Plugging the formulas for R_{jikl} and R_{ijkl} into (103) yields

$$L_{ijk|l} - L_{ijl|k} = -\frac{1}{2}\Big\{g_{im}R^m_{kl\cdot j} + g_{jm}R^m_{kl\cdot i}\Big\} - C_{ijm}R^m_{kl}. \tag{106}$$

$$I_{k|l} - I_{l|k} = -R^m_{kl\cdot m} - I_mR^m_{kl}. \tag{107}$$

The identity (96) can be written as

$$R^i{}_{kl} = \frac{1}{3}\Big\{R^i{}_{k\cdot l} - R^i{}_{l\cdot k}\Big\}. \tag{108}$$

Lemma 13.1. ([Mo 1999]) *L_{ijk} and $R^i{}_k$ are related by the following equation,*

$$\begin{aligned} C_{ijk|p|q}y^py^q + C_{ijm}R^m_k \quad &= \quad -\frac{1}{3}g_{im}R^m_{k\cdot j} - \frac{1}{3}g_{jm}R^m_{k\cdot i} \\ &\quad\ -\frac{1}{6}g_{im}R^m_{j\cdot k} - \frac{1}{6}g_{jm}R^m_{i\cdot k}. \end{aligned} \tag{109}$$

In particular,

$$I_{k|p|q}y^py^q + I_mR^m_{\ k} = -\frac{1}{3}\Big\{2R^m_{\ k\cdot m} + R^m_{\ m\cdot k}\Big\}. \tag{110}$$

Proof: By (100), we have

$$L_{ijk|m}y^m = C_{ijk|p|q}y^py^q, \qquad J_{k|m}y^m = I_{k|p|q}y^py^q.$$

Then contracting (105) with y^l yield (108), and contracting (108) with g^{ij} yields (109). Here we have made use of (107). □

The above equations are crucial in the study of Finsler metrics of scalar curvature. Let $F = F(x, y)$ be a Finsler metric of scalar curvature with flag curvature $\mathbf{K} = \mathbf{K}(x, y)$. Then (77) holds. Plugging (77) into (108) and (109) yields

$$C_{ijk|p|q}y^py^q + \mathbf{K}F^2C_{ijk} = -\frac{1}{3}F^2\Big\{\mathbf{K}_{\cdot i}h_{jk} + \mathbf{K}_{\cdot j}h_{ik} + \mathbf{K}_{\cdot k}h_{ij}\Big\} \tag{111}$$

$$I_{k|p|q}y^py^q + \mathbf{K}F^2I_k = -\frac{n+1}{3}F^2\mathbf{K}_{\cdot k}. \tag{112}$$

Using (110), one can show that any compact Finsler manifold of negative constant flag curvature must be Riemannian [Akbar-Zadeh 1988].

It follows from (110) and (111) that for any Finsler metric F of scalar curvature with flag curvature $\mathbf{K}$, the Matsumoto torsion satisfies

$$M_{ijk|p|q}y^py^q + \mathbf{K}F^2M_{ijk} = 0. \tag{113}$$

One can use (112) to show that for any Landsberg metric F of scalar curvature, if $\mathbf{K} \neq 0$, then it is Riemannian, provided that $n \geq 3$ [Numata 1975]. See also Corollary 17.4 below.

Using (112), one can easily prove the following

Theorem 13.2. ([Mo–Shen 2003]) *Let (M, F) be a compact Finsler manifold of dimension $n \geq 3$. If F is of scalar curvature with negative flag curvature, then F must be a Randers metric.*

Now we derive some important identities for the S-curvature. Differentiating (98) and using (92) and (99), one obtains

$$\begin{aligned} 0 = d^2\tau &= \Big\{\tau_{|k|l}\omega^l + \tau_{|k\cdot l}\omega^{n+l}\Big\} \wedge \omega^k \\ &\quad + \Big\{I_{k|l}\omega^l + I_{k\cdot l}\omega^{n+l}\Big\} \wedge \omega^k + I_m\Omega^{n+m}. \end{aligned}$$

This yields the following Ricci identities

$$\tau_{|k|l} = \tau_{|l|k} + I_pR^p_{\ kl}, \tag{114}$$

$$\tau_{|k\cdot l} = I_{l|k} - I_pL^p_{\ kl}. \tag{115}$$

From the definition (53), the S-curvature can be defined by

$$\mathbf{S} = \tau_{|m}y^m. \tag{116}$$

Contracting (114) with y^k yields

$$\begin{aligned}\mathbf{S}_{\cdot k} &= (\tau_{|m}y^m)_{\cdot k} = \tau_{|m\cdot k}y^m + \tau_{|k}\\ &= I_{k|m}y^m - I_pL^p_{mk}y^m + \tau_{|k} = J_k + \tau_{|k}.\end{aligned}$$

Here we have made use of (114) and (115). Let us restate the above equation as follows

$$\mathbf{S}_{\cdot k} = \tau_{|k} + J_k. \tag{117}$$

Lemma 13.3. ([Mo 2002], [Mo–Shen 2003]) *The S-curvature satisfies the following equation:*

$$\mathbf{S}_{\cdot k|m}y^m - \mathbf{S}_{|k} = -\frac{1}{3}\Big\{2R^m_{\ k\cdot m} + R^m_{\ m\cdot k}\Big\}. \tag{118}$$

Proof: It follows from (116) that

$$\mathbf{S}_{\cdot k|l} = \tau_{|k|l} + J_{k|l}. \tag{119}$$

By (113) and (118), one obtains

$$\begin{aligned}\mathbf{S}_{\cdot k|m}y^m - \mathbf{S}_{|k} &= \Big\{\mathbf{S}_{\cdot k|m} - \mathbf{S}_{\cdot m|k}\Big\}y^m\\ &= \{\tau_{|k|m} - \tau_{|m|k}\}y^m + \{J_{k|m} - J_{m|k}\}y^m\\ &= I_pR^{p}_{\ km}y^m + J_{k|m}y^m\\ &= I_pR^p_{\ k} - I_pR^p_{\ k} - \frac{1}{3}I_m\Big\{R^m_{\ k\cdot l} - R^m_{\ l\cdot k}\Big\}\\ &= -\frac{1}{3}I_m\Big\{R^m_{\ k\cdot l} - R^m_{\ l\cdot k}\Big\}.\end{aligned}$$

□

14. Nonpositively Curved Finsler Manifolds

In the previous section, we have derived several identities on the geometric quantities. Now we are going to use them to establish some global rigidity theorems.

First, let us consider the mean Cartan torsion. Let (M, F) be an n-dimensional Finsler manifold. The norm of the mean Cartan torsion $\mathbf{I}$ at a point $x \in M$ is defined by

$$\|\mathbf{I}\|_x := \sup_{0\neq y\in T_xM}\sqrt{I_i(x,y)g^{ij}(x,y)I_j(x,y)}.$$

It is known that if $F = \alpha + \beta$ is a Randers metric, then

$$\|\mathbf{I}\|_x \le \frac{n+1}{\sqrt{2}}\sqrt{1-\sqrt{1-\|\beta\|_x^2}} < \frac{n+1}{\sqrt{2}}.$$

The bound in dimension two is suggested by B. Lackey. See Proposition 7.1.2 in [Shen 2001b] or [Ji–Shen 2002] for a proof. Below is our first global rigidity theorem.

Theorem 14.1. ([Shen 2003d]) *Let (M, F) be an n-dimensional complete Finsler manifold with nonpositive flag curvature. Suppose that F has almost constant S-curvature $\mathbf{S} = (n+1)\{cF + \eta\}$ (c = constant and η is a closed 1-form) and bounded mean Cartan torsion $\sup_{x\in M} \|\mathbf{I}\|_x < \infty$. Then $\mathbf{J} = 0$ and $\mathbf{R} \circ \mathbf{I} = 0$. Moreover F is Riemannian at points where the flag curvature is negative.*

Proof: It follows from (109) and (117) that

$$I_{k|p|q} y^p y^q + I_m R^{\ m}_{\ k} = \mathbf{S}_{\cdot k|m} y^m - \mathbf{S}_{|k}. \tag{120}$$

Assume that the S-curvature is almost isotropic, i.e.,

$$\mathbf{S} = (n+1)\Big\{cF + \eta\Big\},$$

where $c = c(x)$ is a scalar function on M and $\eta = \eta_i dx^i$ is a closed 1-form on M. Observe that

$$\eta_{\cdot k|m} y^m - \eta_{|k} = \Big\{\eta_{k|m} - \eta_{m|k}\Big\} y^m = \Big\{\frac{\partial \eta_k}{\partial x^m} - \frac{\partial \eta_m}{\partial x^k}\Big\} y^m = 0.$$

Thus

$$\begin{aligned}\mathbf{S}_{\cdot k|m} y^m - \mathbf{S}_{|k} &= (n+1)\Big\{c_{x^m} y^m F_{\cdot k} - c_{|k} F + \eta_{\cdot k|m} y^m - \eta_{|k}\Big\} \\ &= (n+1)\Big\{c_{x^m} y^m F_{\cdot k} - c_{|k} F\Big\}.\end{aligned}$$

In this case, (117) becomes

$$2R^{\ m}_{\ k\cdot m} + R^{\ m}_{\ m\cdot k} = -3(n+1)\Big\{c_{x^m} y^m F_{\cdot k} - c_{|k} F\Big\} \tag{121}$$

and (119) becomes

$$I_{k|p|q} y^p y^q + I_m R^{\ m}_{\ k} = (n+1)\Big\{c_{x^m} y^m F_{\cdot k} - c_{|k} F\Big\}. \tag{122}$$

By assumption, $c = constant$. Thus $c_{|k} = 0$ and (121) is reduced to

$$I_{k|p|q} y^p y^q + I_m R^{\ m}_{\ k} = 0. \tag{123}$$

Let $y \in T_xM$ be an arbitrary vector and let $\sigma(t)$ be the geodesic with $\sigma(0) = x$ and $\dot\sigma(0) = y$. Since the Finsler metric is complete, one may assume that $\sigma(t)$ is defined on $(-\infty, \infty)$. The mean Cartan torsion $\mathbf{I}$ and the mean Landsberg curvature $\mathbf{J}$ restricted to $\sigma(t)$ are vector fields along $\sigma(t)$,

$$\mathbf{I}(t) := I^i(\sigma(t), \dot\sigma(t)) \frac{\partial}{\partial x^i}|_{\sigma(t)}, \qquad \mathbf{J}(t) := J^i(\sigma(t), \dot\sigma(t)) \frac{\partial}{\partial x^i}|_{\sigma(t)}.$$

It follows from (60) or (100) that

$$D_{\dot\sigma} \mathbf{I}(t) = I^i{}_{|m}(\sigma(t), \dot\sigma(t)) \dot\sigma^m(t) \frac{\partial}{\partial x^i}|_{\sigma(t)} = \mathbf{J}(t). \tag{124}$$

It follows from (122) that

$$D_{\dot\sigma} D_{\dot\sigma} \mathbf{I}(t) + \mathbf{R}_{\dot\sigma(t)}(\mathbf{I}_{\dot\sigma(t)}) = 0. \tag{125}$$

Let

$$\varphi(t) := \mathbf{g}_{\dot\sigma(t)}\Big(\mathbf{I}(t), \mathbf{I}(t)\Big).$$

We obtain

$$\varphi''(t) = 2\mathbf{g}_{\dot\sigma(t)}\Big(\mathrm{D}_{\dot\sigma}\mathrm{D}_{\dot\sigma}\mathbf{I}(t),\mathbf{I}(t)\Big)+2\mathbf{g}_{\dot\sigma(t)}\Big(\mathrm{D}_{\dot\sigma}\mathbf{I}(t),\mathrm{D}_{\dot\sigma}\mathbf{I}(t)\Big)$$
$$= -2\mathbf{g}_{\dot\sigma(t)}\Big(\mathbf{R}_{\dot\sigma(t)}(\mathbf{I}(t)),\mathbf{I}(t)\Big)+2\mathbf{g}_{\dot\sigma(t)}\Big(\mathbf{J}(t),\mathbf{J}(t)\Big). \tag{126}$$

By assumption, $\mathbf{K}\le 0$. Thus

$$\mathbf{g}_{\dot\sigma(t)}\Big(\mathbf{R}_{\dot\sigma(t)}(\mathbf{I}(t)),\mathbf{I}(t)\Big)\le 0.$$

It follows from (125) that

$$\varphi''(t)\ge 0.$$

Thus $\varphi(t)$ is convex and nonnegative. Suppose that $\varphi'(t_o)\neq 0$ for some t_o. By an elementary argument, $\lim_{t\to+\infty}\varphi(t)=\infty$ or $\lim_{t\to-\infty}\varphi(t)=\infty$. This implies that the mean Cartan torsion is unbounded, which contradicts the assumption. Therefore, $\varphi'(t)=0$ and hence $\varphi''(t)=0$. Since each term in (125) is nonnegative, one concludes that

$$\mathbf{g}_{\dot\sigma(t)}\Big(\mathbf{R}_{\dot\sigma(t)}(\mathbf{I}(t)),\mathbf{I}(t)\Big)=0,\qquad \mathbf{J}(t)=0.$$

Setting $t=0$ yields

$$\mathbf{g}_y(\mathbf{R}_y(\mathbf{I}_y),\mathbf{I}_y)=0 \tag{127}$$

and $\mathbf{J}_y=0$. By (75), $\mathbf{R}_y(y)=0$ and $\mathbf{R}_y$ is self-adjoint with respect to $\mathbf{g}_y$, i.e., $\mathbf{g}_y(\mathbf{R}_y(u),v)=\mathbf{g}_y(u,\mathbf{R}_y(v))$, $u,v\in T_xM$. Thus there is an orthonormal basis $\{\mathbf{e}_i\}_{i=1}^n$ with $\mathbf{e}_n=y$ such that

$$\mathbf{R}_y(\mathbf{e}_i)=\lambda_i\mathbf{e}_i,\qquad i=1,\cdots,n,$$

with $\lambda_n=0$. By assumption, the flag curvature is nonpositive. Then

$$\mathbf{g}_y(\mathbf{R}_y(\mathbf{e}_i),\mathbf{e}_i)=\lambda_i\le 0,\qquad i=1,\cdots,n-1.$$

Since $\mathbf{I}_y$ is perpendicular to y with respect to $\mathbf{g}_y$, one can express it by $\mathbf{I}_y=\mu_1\mathbf{e}_1+\cdots+\mu_{n-1}\mathbf{e}_{n-1}$. By (126), one obtains

$$0=\mathbf{g}_y(\mathbf{R}_y(\mathbf{I}_y),\mathbf{I}_y)=\sum_{i=1}^{n-1}\mu_i^2\lambda_i.$$

Since each term $\mu_i^2\lambda_i\le 0$, one concludes that $\mu_i\lambda_i=0$, namely,

$$\mathbf{R}_y(\mathbf{I}_y)=\sum_{i=1}^{n-1}\mu_i\lambda_i=0. \tag{128}$$

Now suppose that F has negative flag curvature at a point $x\in M$. Then $\lambda_i<0$ for $i=1,\cdots,n-1$. By (127), one concludes that $\mu_i=0$, $i=1,\cdots,n-1$, namely, $\mathbf{I}_y=0$. By Deicke's theorem [Deicke 1953], F is Riemannian. □

Corollary 14.2. *Every complete Berwald manifold with negative flag curvature is Riemannian.*

Proof: This is because for a Berwald metric F on a manifold M, the Minkowski spaces (T_xM, F_x) are all linearly isometric (Proposition 5.2). Thus the Cartan torsion is bounded from above. Meanwhile, the S-curvature vanishes (Proposition 7.1). Thus F must be Riemannian. □

Example 14.3. Let (M_i, α_i), $i = 1, 2$, be Riemannian manifolds and $F = F(x, y)$ be the product metric on $M = M_1 \times M_2$, defined in Example 5.1.

$$F(x,y) := \sqrt{f\Big([\alpha_1(x_1,y_1)]^2,\ [\alpha_2(x_2,y_2)]^2\Big)}.$$

We have computed the spray coefficients of F in Example 5.1. Using (70), one obtains the Riemann tensor of F,

$$R^i{}_j = \bar{R}^i{}_j, \qquad R^a{}_\beta = 0 = R^\alpha{}_b, \qquad R^\alpha{}_\beta = \bar{R}^\alpha{}_\beta,$$

where $\bar{R}^a{}_b$ and $\bar{R}^\alpha{}_\beta$ are the coefficients of the Riemann tensor of α_1 and α_2 respectively. Let $R_{ij} := g_{ik}R^k{}_j$, $\bar{R}_{ab} := \bar{g}_{ac}\bar{R}^c{}_b$ and $\bar{R}_{\alpha\beta} := \bar{g}_{\alpha\gamma}\bar{R}^\gamma{}_\beta$. Using (43), one obtains

$$R_{ab} = f_s\bar{R}_{ab}, \quad R_{a\beta} = 0 = R_{\alpha b}, \quad R_{\alpha\beta} = f_t\bar{R}_{\alpha\beta}.$$

For any vector $v = v^i\frac{\partial}{\partial x^i}|_x \in T_xM$,

$$\mathbf{g}_y\Big(\mathbf{R}_y(v), v\Big) = f_s\bar{R}_{ab}v^av^b + f_t\bar{R}_{\alpha\beta}v^\alpha v^\beta.$$

Thus if α_1 and α_2 both have nonpositive sectional curvature, then F has nonpositive flag curvature.

Using (44), one can compute the mean Cartan torsion. First, observe that

$$I_i = \frac{\partial}{\partial y^i}\Big[\ln\sqrt{\det(g_{jk})}\Big] = \frac{\partial}{\partial y^i}\Big[\ln\sqrt{h\Big([\alpha_1]^2,[\alpha_2]^2\Big)}\Big],$$

where $h = h(s,t)$ is defined in (45). One obtains

$$I_a = \frac{h_s}{h}\bar{y}_a \qquad I_\alpha = \frac{h_t}{h}\bar{y}_\alpha,$$

where $\bar{y}_a := \bar{g}_{ab}y^b$ and $\bar{y}_\alpha := \bar{g}_{\alpha\beta}y^\beta$. Since $\bar{y}_a\bar{R}^a{}_b = 0$ and $\bar{y}_\alpha\bar{R}^\alpha{}_\beta = 0$, one obtains

$$\mathbf{g}_y\Big(\mathbf{R}_y(\mathbf{I}_y), \mathbf{I}_y\Big) = I_iR^i{}_jI^j = \frac{h_s}{h}\bar{y}_a\bar{R}^a{}_bI^b + \frac{h_t}{h}\bar{y}_\alpha\bar{R}^\alpha{}_\beta I^\beta = 0.$$

Since $\mathbf{R}_y$ is self-adjoint and nonpositive definite with respect to $\mathbf{g}_y$, $\mathbf{R}_y(\mathbf{I}_y) = 0$. Therefore F satisfies the conditions and conclusions in Theorem 14.1.

The following example shows that the completeness in Theorem 14.1 can not be replaced by positive completeness.

Example 14.4. Let $\phi(y)$ be a Minkowski norm on R^n. Let $\Theta = \Theta(x, y)$ be the Funk metric on $U := \{y \in \mathrm{R}^n \mid \phi(y) < 1\}$ defined in (19). Let $a \in \mathrm{R}^n$ be an arbitrary constant vector. Let

$$F := \Theta(x,y) + \frac{\langle a, y\rangle}{1 + \langle a, x\rangle}, \qquad y \in TU \cong U \times \mathrm{R}^n.$$

Clearly, F is a Finsler metric nearby the origin. By (22), one sees that the spray coefficients of F are given by $G^i = Py^i$, where

$$P := \frac{1}{2}\Big\{\Theta(x,y) - \frac{\langle a, y\rangle}{1+\langle a, x\rangle}\Big\}.$$

Using the above formula for G^i and (81), one obtains

$$\mathbf{K} = \frac{\frac{1}{4}\Big[\Theta - \frac{\langle a,y\rangle}{1+\langle a,x\rangle}\Big]^2 - \frac{1}{2}\Big[\Theta^2 + \Big(\frac{\langle a,y\rangle}{1+\langle a,x\rangle}\Big)^2\Big]}{\Big[\Theta(x,y) + \frac{\langle a,y\rangle}{1+\langle a,x\rangle}\Big]^2} = -\frac{1}{4}.$$

That is, F has constant flag curvature $\mathbf{K} = -\frac{1}{4}$ (see also Example 5.3 in [Shen 2003b]). Now let us compute the S-curvature of F. A direct computation gives

$$\frac{\partial G^m}{\partial y^m} = (n+1)P.$$

Let $dV = \sigma_F(x)dx^1 \cdots dx^n$ be the Finsler volume form on M. By (82), we obtain

$$\begin{aligned}\mathbf{S} &= \frac{n+1}{2}F(x,y) - (n+1)\frac{\langle a,y\rangle}{1+\langle a,x\rangle} - y^m\frac{\partial}{\partial x^m}\Big(\ln\sigma_F(x)\Big)\\ &= (n+1)\Big\{\frac{1}{2}F(x,y) + d\varphi_x(y)\Big\},\end{aligned}$$

where $\varphi(x) := -\ln\Big[(1+\langle a,x\rangle)\sigma_F(x)^{\frac{1}{n+1}}\Big]$. Thus

$$\mathbf{E} = \frac{n+1}{4}F^{-1}\mathbf{h},$$

where $\mathbf{h}_y = h_{ij}(x,y)dx^i \otimes dx^j$ is given by $h_{ij} = F(x,y)F_{y^iy^j}(x,y)$.

When $\phi(y) = |y|$ is the standard Euclidean norm, $U = \mathrm{B}^n$ is the standard unit ball in R^n and

$$\Theta = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1-|x|^2}.$$

Thus

$$F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1-|x|^2} + \frac{\langle a,y\rangle}{1+\langle a,x\rangle}.$$

Assume that $|a| < 1$. It is easy to verify that F is a Randers metric defined on the whole B^n with constant S-curvature, i.e., $\mathbf{S} = \frac{1}{2}(n+1)F(x,y)$. Further, one can show that F is positively complete on B^n, namely, every geodesic defined on an interval (λ, μ) can be extended to a geodesic defined on $(\lambda, +\infty)$.

15. Flag Curvature and Isotropic S-Curvature

It is a difficult task to classify Finsler metrics of scalar curvature. All known Randers metrics of scalar curvature have isotropic S-curvature. Thus it is a natural idea to investigate Finsler metrics of scalar curvature which also have isotropic S-curvature.

Proposition 15.1. ([Chen et al. 2003]) *Let (M, F) be an n-dimensional Finsler manifold of scalar curvature with flag curvature $\mathbf{K} = \mathbf{K}(x, y)$. Suppose that the S-curvature is almost isotropic,*

$$\mathbf{S} = (n+1)\Big\{cF + \eta\Big\}, \tag{129}$$

where $c = c(x)$ is a scalar function on M and $\eta = \eta_i(x)y^i$ is a closed 1-form. Then there is a scalar function $\sigma = \sigma(x)$ on M such that the flag curvature is in the following form,

$$\mathbf{K} = 3\frac{c_{x^m}y^m}{F} + \sigma. \tag{130}$$

Proof: By assumption, the flag curvature $\mathbf{K} = \mathbf{K}(x, y)$ is a scalar function on TM_o. Thus (77) holds. Plugging (77) into (120) yields

$$c_{x^m}y^m F_{\cdot k} - c_{x^k}F = -\frac{1}{3}\mathbf{K}_{y^k}F^2. \tag{131}$$

Rewriting (130) as follows

$$\Big[\frac{1}{3}\mathbf{K} - \frac{c_{x^m}y^m}{F}\Big]_{y^k} = 0,$$

one concludes that the following quantity

$$\sigma := \mathbf{K} - \frac{3c_{x^m}y^m}{F}$$

is a scalar function on M. This proves the proposition. □

Corollary 15.2. ([Mo 2002]) *Let F be an n-dimensional Finsler metric of scalar curvature with flag curvature $\mathbf{K} = \mathbf{K}(x, y)$. If F has almost constant S-curvature, $\mathbf{S} = (n+1)\{cF + \eta\}$, where $c = constant$ and η is closed, then $\mathbf{K} = \mathbf{K}(x)$ is a scalar function on M.*

From the definition of flag curvature, one can see that every two-dimensional Finsler metric is of scalar curvature. One immediately obtains the following

Corollary 15.3. *Let F be a two-dimensional Finsler metric with almost isotropic S-curvature. Then the flag curvature is in the form (129).*

Let $F = F(x, y)$ be a two-dimensional Berwald metric on a surface M. It follows from Corollaries 15.3 and 15.2 that the Gauss curvature $\mathbf{K} = \mathbf{K}(x)$ is a scalar function of $x \in M$. Since F is a Berwald metric, $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$ are quadratic in $y = y^i\frac{\partial}{\partial x^i}|_x \in T_xM$. By (70), the Riemann curvature, $R^i{}_k = R^i{}_k(x, y)$, are quadratic in y. This implies that the Ricci scalar $\mathbf{Ric} = R^m{}_m(x, y)$ is quadratic in y. Suppose that $\mathbf{K}(x_o) \neq 0$ at some point $x_o \in M$. Then

$$F(x_o, y)^2 = \frac{\mathbf{Ric}(x_o, y)}{\mathbf{K}(x_o)}$$

is quadratic in $y \in T_{x_o}M$. Namely, $F_{x_o} = F|_{T_{x_o}M}$ is Euclidean at x_o. By Proposition 5.2, all tangent spaces (T_xM, F_x) are linearly isometric to each other. One concludes that F_x is Euclidean for any $x \in M$ and F is Riemannian. Now we suppose that $\mathbf{K} \equiv 0$. Since F is Berwaldian, F must be locally Minkowskian. See [Szabó 1981] for a different argument.

16. Projectively Flat Metrics with Isotropic S-Curvature

Recall that a Finsler metric F on a manifold M is locally projectively flat if at any point $x \in M$, there is a local coordinate system (x^i) in M such that every geodesic $\sigma(t)$ is straight, i.e., $\sigma^i(t) = f(t)a^i + b^i$. This is equivalent to saying that in the standard local coordinate system (x^i, y^i), the spray coefficients G^i are in the form $G^i = Py^i$ with $P = \frac{1}{2F} F_{x^k} y^k$. It is well-known that any locally projectively flat Finsler metric F is of scalar curvature and the flag curvature is given by $\mathbf{K} = \frac{1}{F^2}\{P^2 - P_{x^k} y^k\}$ (see Proposition 12.1). Our goal is to characterize those with almost isotropic S-curvature.

First, by Beltrami's theorem and the Cartan classification theorem, a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. Every Riemannian metric of constant sectional curvature μ is locally isometric to the metric α_μ on a ball in R^n, which is defined in (5). A Randers metric $F = \alpha + \beta$ is locally projectively flat if and only if α is locally projectively flat (hence, it is of constant sectional curvature) and β is closed. This is a direct consequence of a result in [Bácsó–Matsumoto 1997] and the Beltrami theorem on projectively flat Riemannian metrics. If in addition, the S-curvature is almost isotropic, then β can be determined explicitly.

Proposition 16.1. ([Chen et al. 2003]) *Let $F = \alpha + \beta$ be a locally projectively flat Randers metric on an n-dimensional manifold M. Suppose that F has almost isotropic S-curvature, $\mathbf{S} = (n+1)\{cF + \eta\}$, where $c = c(x)$ is a scalar function on M and $\eta = \eta_i(x)y^i$ is a closed 1-form on M. Then α is locally isometric to α_μ and β is a closed 1-form which satisfies*

$$(\mu + 4c^2)\beta = -c_{x^k} y^k.$$

In this case, the flag curvature is given by

$$\mathbf{K} = \frac{3c_{x^k} y^k}{\alpha + \beta} + 3c^2 + \mu = \frac{3}{4}\Big\{\mu + 4c^2\Big\}\frac{\alpha - \beta}{\alpha + \beta} + \frac{\mu}{4}. \tag{132}$$

Furthermore,

(A) *if $\mu + 4c^2 \equiv 0$, then c is a constant and the flag curvature $\mathbf{K} = -c^2$. In this case, $F = \alpha + \beta$ is either locally Minkowskian ($c = 0$) or, up to a scaling ($c = \pm 1/2$), locally isometric to the generalized Funk metric $\Theta_a = \Theta_a(x, y)$ in (7) or its reverse $\bar{\Theta}_a = \Theta_a(x, -y)$;*

(B) *if $\mu + 4c^2 \neq 0$, then $F = \alpha + \beta$ must be locally given by*

$$\alpha = \alpha_\mu(x, y), \qquad \beta = -\frac{2c_{x^k}(x) y^k}{\mu + 4c^2} \tag{133}$$

where $c := c_\mu(x)$ is given by

$$c_\mu = (\lambda + \langle a, x\rangle)\sqrt{\frac{\mu}{\pm(1+\mu|x|^2) - (\lambda + \langle a, x\rangle)^2}}, \qquad (\mu \neq 0), \tag{134}$$

$$c_\mu = \frac{\pm 1}{2\sqrt{\lambda + 2\langle a, x\rangle + |x|^2}}, \qquad (\mu = 0)$$

where $a \in \mathrm{R}^n$ is a constant vector and $\lambda \in \mathrm{R}$ is a constant number.

Proof: Let $\alpha_\mu = \sqrt{a_{ij}(x)y^iy^j}$ and $\beta = b_i(x)y^i$. We may assume that $\alpha = \alpha_\mu$ in a local coordinate system

$$a_{ij} = \frac{\delta_{ij}}{1+\mu|x|^2} - \frac{\mu x^i x^j}{(1+\mu|x|^2)^2}.$$

The Christoffel symbols of α are given by

$$\bar{\Gamma}^i_{jk} = -\mu\frac{x^j\delta^i_k + x^k\delta^i_j}{1+\mu|x|^2}.$$

Thus

$$\bar{G}^i = -\frac{\mu\langle x, y\rangle}{1+\mu|x|^2}\; y^i.$$

The spray coefficients of F are given by

$$G^i = \bar{G}^i + Py^i + Q^i,$$

where $P = \frac{1}{2F}e_{00}$ and $Q^i = \alpha s^i{}_0$ are given by (63) and (64) respectively. Since β is closed, $s_{ij} := \frac{1}{2}\{b_{i;j} + b_{j;i}\} = 0$ and $s_i := b_j s^j{}_i = 0$. Thus $Q^i = 0$. By assumption, $\mathbf{S} = (n+1)\{cF + \eta\}$ and Lemma 9.1,

$$e_{00} = \beta_{|k}y^k = 2c(\alpha^2 - \beta^2). \tag{135}$$

Thus

$$P = \frac{e_{00}}{2F} - s_0 = c(\alpha - \beta)$$

and

$$\beta_{x^k}y^k = \beta_{|k}y^k + 2\bar{G}^k\beta_{y^k} = 2c(\alpha^2 - \beta^2) - \frac{2\mu\langle x, y\rangle\beta}{1+\mu|x|^2}.$$

Then $G^i = \tilde{P}y^i$, where

$$\tilde{P} = -\frac{\mu\langle x, y\rangle}{1+\mu|x|^2} + c(\alpha - \beta).$$

By (81), we obtain

$$\begin{aligned}\mathbf{K}F^2 &= \tilde{P}^2 - \tilde{P}_{x^k}y^k \\ &= \mu\alpha^2 + c^2(3\alpha+\beta)(\alpha-\beta) - c_{x^k}y^k(\alpha-\beta).\end{aligned} \tag{136}$$

On the other hand, by Theorem 15.1, the flag curvature is in the following form

$$\mathbf{K} = \frac{3c_{x^k}y^k}{\alpha+\beta} + \sigma, \tag{137}$$

where $\sigma = \sigma(x)$ is a scalar function on M. It follows from (135) and (136) that

$$2\Big\{2c_{x^k}y^k + (\sigma + c^2)\beta\Big\}\alpha + \Big\{2c_{x^k}y^k + (\sigma + c^2)\beta\Big\}\beta + \Big\{\sigma - 3c^2 - \mu\Big\}\alpha^2 = 0.$$

This gives

$$2c_{x^k}y^k + (\sigma + c^2)\beta = 0, \tag{138}$$

$$\sigma - 3c^2 - \mu = 0. \tag{139}$$

From (138), one obtains that $\sigma = 3c^2 + \mu$. Plugging it into (137) yields

$$(\mu + 4c^2)\beta = -2c_{x^k}y^k. \tag{140}$$

Case A: Suppose that $\mu + 4c^2 \equiv 0$. Then $c = constant$. It follows from (131) that

$$\mathbf{K} = 3c^2 + \mu = -c^2.$$

In this case, the local structure of F can be easily determined [Shen 2003a].

Case B: Suppose that $\mu + 4c^2 \neq 0$ on an open subset $U \subset M$. Then by (139), β is given by

$$\beta = -\frac{2c_{x^k}y^k}{\mu + 4c^2}. \tag{141}$$

Note that β is exact. It follows from (134) and (140) that

$$\begin{aligned} & c_{x^ix^j} + \frac{\mu(x^i c_{x^j} + x^j c_{x^i})}{1 + \mu|x|^2} \\ = \quad & -c(\mu + 4c^2)\Big\{\frac{\delta_{ij}}{1 + \mu|x|^2} - \frac{\mu x^i x^j}{(1 + \mu|x|^2)^2}\Big\} + \frac{12cc_{x^i}c_{x^j}}{\mu + 4c^2}. \end{aligned} \tag{142}$$

We are going to solve (141) for c. Let

$$\begin{aligned} f \quad &:= \quad \frac{2c\sqrt{1 + \mu|x|^2}}{\sqrt{\pm(\mu + 4c^2)}}, \qquad (\mu \neq 0) \\ f \quad &:= \quad \frac{1}{c^2}, \qquad (\mu = 0) \end{aligned} \tag{143}$$

where the sign $\pm$ depends on the value of c such that $\pm(\mu + 4c^2) > 0$. Then (141) is reduced to the following equation:

$$f_{x^ix^j} = 0 \ \ (\text{if } \mu \neq 0), \qquad f_{x^ix^j} = 8\delta_{ij} \ \ (\text{if } \mu = 0).$$

We obtain

$$f = \lambda + \langle a, x\rangle \ \ (\mu \neq 0),$$

$$f = 4(\lambda + 2\langle a, x\rangle + |x|^2) \ \ (\mu = 0),$$

where $a \in \mathrm{R}^n$ is a constant vector and λ is a constant. This gives (133). □

By Proposition 16.1, one immediately obtains the following

Corollary 16.2. *Let $F = \alpha + \beta$ be a locally projectively flat Randers metric on an n-dimensional manifold M. Suppose that F has almost constant S-curvature $\mathbf{S} = (n+1)\{cF + \eta\}$, where c is a constant. Then F is locally Minkowskian, or Riemannian with constant curvature, or up to a scaling, locally isometric to the generalized Funk metric in (7).*

Proof: Let μ be the constant sectional curvature of α. First assume that $\mu + 4c^2 = 0$. Then by Proposition 16.1 (A), $F = \alpha + \beta$ is either locally Minkowskian or, up to a scaling, locally isometric to the generalized Funk metric in (7). Suppose that $\mu + 4c^2 \neq 0$. Then by Proposition 16.1 (B), $F = \alpha + \beta$ is given by (132). Since $c_{x^k} = 0$, $\beta = 0$ and $F = \alpha$ is a Riemannian metric. □

In Proposition 16.1, we have completely classified projectively flat Randers metrics of almost isotropic S-curvature. If a Randers metric $F = \alpha + \beta$ has almost isotropic S-curvature, then the E-curvature is isotropic. By Lemma 9.1, the S-curvature is isotropic. Thus a Randers metric is of almost isotropic S-curvature if and only if it is of isotropic S-curvature. We emphasize that this is not true for general Finsler metrics. Consider the following Finsler metric,

$$F = \Theta(x, y) + \frac{\langle a, y\rangle}{1 + \langle a, x\rangle}, \qquad y \in T_xU \cong \mathrm{R}^n,$$

where $\Theta(x, y)$ is the Funk metric on a strongly convex domain $U \subset \mathrm{R}^n$. According to Example 14.4, F is projectively flat with almost isotropic S-curvature. Thus it has isotropic E-curvature. However, this F is of isotropic S-curvature only when U is a special domain (such as the standard unit ball B^n).

A natural problem is whether or not there are other types of projectively flat Finsler metrics of almost isotropic S-curvature. We answer the question in the following

Proposition 16.3. ([Chen–Shen 2003b]) *Let $F = F(x, y)$ be a projectively flat Finsler metric on a simply connected open subset $U \subset \mathrm{R}^n$. Suppose that F has almost isotropic S-curvature, i.e.,*

$$\mathbf{S} = (n+1)c\Big\{F + \eta\Big\}, \tag{144}$$

where $c = c(x)$ is a scalar function on M and $\eta = \eta_i(x)dx^i$ is a closed 1-form on U. Then F is determined as follows.

(a) *If $\mathbf{K} \neq -c^2 + \frac{c_{x^m}y^m}{F}$ at every point $x \in U$, then $F = \alpha + \beta$ is a Randers metric on U. Further, α is of constant sectional curvature $\bar{\mathbf{K}} = \mu$ with $\mu + 4c^2 \neq 0$ and α and β are given by (132) and (133);*

(b) *If $\mathbf{K} \equiv -c^2 + \frac{c_{x^m}y^m}{F}$ on U, then c is a constant, and either F is locally Minkowskian ($c = 0$) or up to a scaling, F can be expressed as*

$$\begin{aligned} F &= \Theta(x, y) + \frac{\langle a, y\rangle}{1 + \langle a, x\rangle} \quad \text{if } c = \frac{1}{2} \\ F &= \Theta(x, -y) - \frac{\langle a, y\rangle}{1 + \langle a, x\rangle} \quad \text{if } c = -\frac{1}{2}, \end{aligned} \tag{145}$$

where $a \in \mathrm{R}^n$ is a constant vector and $\Theta = \Theta(x,y)$ is a Funk metric defined by (22).

Proof: Since F is projectively flat, the spray coefficients are given by $G^i = Py^i$, where

$$P := \frac{F_{x^k} y^k}{2F}. \tag{146}$$

Thus the S-curvature is given by (82) and the flag curvature of F is given by (81).

By assumption, $\mathbf{S}$ is in the form (143). Since $\eta = \eta(x,y)$ is closed on U, it can be expressed in the form $\eta(x,y) = dh_x(y)$ where $h = h(x)$ is a scalar function on U. Thus

$$P = cF + d\varphi_x, \tag{147}$$

where $\varphi(x) := h(x) + \frac{1}{n+1}\ln[\sigma_F(x)]$. It follows from (145) and (146) that

$$F_{x^i} y^i = 2FP = 2F\Big\{cF + \varphi_{x^i} y^i\Big\}. \tag{148}$$

Plugging (146) into (81) and using (147), one obtains

$$\begin{aligned} \mathbf{K} &= \frac{\Big\{cF + \varphi_{x^i} y^i\Big\}^2 - \Big\{c_{x^i} y^i F + cF_{x^i} y^i + \varphi_{x^i x^j} y^i y^j\Big\}}{F^2} \\ &= \frac{-c^2F^2 - c_{x^m} y^m F + \Big\{\varphi_{x^i}\varphi_{x^j} - \varphi_{x^i x^j}\Big\} y^i y^j}{F^2}. \end{aligned} \tag{149}$$

On the other hand, since F is of scalar curvature, by Proposition 15.1, the flag curvature of F is given by (129), i.e.,

$$\mathbf{K} = 3\frac{c_{x^m} y^m}{F} + \sigma, \tag{150}$$

where $\sigma = \sigma(x)$ is a scalar function on U. Comparing (148) with (149) yields

$$\Big\{\sigma + c^2\Big\}F^2 + 4c_{x^m} y^m F + \Big\{\varphi_{x^i x^j} - \varphi_{x^i}\varphi_{x^j}\Big\} y^i y^j = 0. \tag{151}$$

Assume that $\mathbf{K} \neq -c^2 + \frac{c_{x^m} y^m}{F}$ at every point $x \in U$. Then, by (149), for any $x \in U$, there is a non-zero vector $y \in T_xU$ such that

$$\sigma + c^2 + \frac{2c_{x^m} y^m}{F} \neq 0.$$

We claim that $\sigma + c^2 \neq 0$ on U. If not, there is a point $x_o \in U$ such that $\sigma(x_o) + c(x_o)^2 = 0$. The above inequality implies that $dc \neq 0$ at x_o. Then (150) at x_o is reduced to

$$4c_{x^m}(x_o) y^m F(x_o, y) + \Big\{\varphi_{x^i x^j}(x_o) - \varphi_{x^i}(x_o)\varphi_{x^j}(x_o)\Big\} y^i y^j = 0. \tag{152}$$

Differentiating (151) with respect to y^i, then restricting it to the hyperplane $V := \{y \mid c_{x^m}(x_o) y^m = 0\}$, one obtains

$$4c_{x^i}(x_o) F(x_o, y) + \Big\{\varphi_{x^i x^j}(x_o) - \varphi_{x^i}(x_o)\varphi_{x^j}(x_o)\Big\} y^j = 0.$$

Namely, $F(x_o, y)$ is a homogeneous linear function of $y \in V$. This is impossible, because $F(x_o, y)$ is always positive for $y \in V \setminus \{0\}$.

Now we may assume that $\sigma + c^2 \neq 0$ on U. One can solve the quadratic equation (150) for F,

$$F = \frac{\sqrt{[\sigma + c^2][\varphi_{x^i x^j} - \varphi_{x^i}\varphi_{x^j}]y^i y^j + 4[c_{x^m} y^m]^2} - 2c_{x^m} y^m}{\sigma + c^2}.$$

That is, F is expressed in the form $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij} y^i y^j}$ and $\beta = b_i y^i$ are given by

$$a_{ij} = \frac{[\sigma + c^2][\varphi_{x^i x^j} - \varphi_{x^i}\varphi_{x^j}] + 4c_{x^i} c_{x^j}}{(\sigma + c^2)^2}, \qquad b_i = -\frac{2c_{x^i}}{\sigma + c^2}.$$

Since F is a Randers metric, by Lemma 9.1, one concludes that $\mathbf{S}$ is isotropic, i.e., $\eta = 0$ and

$$\mathbf{S} = (n + 1)cF.$$

Since F is projectively flat, α is of constant sectional curvature $\bar{\mathbf{K}} = \mu$ and β is closed. Moreover, by Proposition 16.1, the flag curvature is given by (149) with $\sigma = 3c^2 + \mu$. See (131). Note that the inequality $\sigma + c^2 \neq 0$ is equivalent to the following inequality

$$\mu + 4c^2 \neq 0.$$

By Proposition 16.1 (B), F is given by (132).

We now assume that $\mathbf{K} \equiv -c^2 + \frac{c_{x^i} y^i}{F}$. It follows from (149) that

$$\sigma + c^2 + \frac{2c_{x^m} y^m}{F} \equiv 0.$$

Suppose that $c_{x^m}(x_o) y^m \neq 0$ at some point x_o. Then from the above identity, one can see that $\sigma(x_o) + c(x_o)^2 \neq 0$. Thus

$$F(x_o, y) = -\frac{2c_{x^m}(x_o) y^m}{\sigma(x_o) + c(x_o)^2}$$

is a linear function. This is impossible. One concludes that $c_{x^m} y^m = 0$ on U, and hence c is a constant and $\sigma(x) = -c^2$ is a constant too. In this case, the flag curvature is given by $\mathbf{K} = -c^2$. The equation (150) is reduced to

$$\varphi_{x^i x^j} - \varphi_{x^i}\varphi_{x^j} = 0.$$

It is easy to solve the above equation,

$$\varphi = -\ln\Big(1 + \langle a, x\rangle\Big) + C,$$

where $a \in \mathrm{R}^n$ is a constant vector and C is a constant.

Assume that $c = 0$. Then $\mathbf{K} = -c^2 = 0$. It follows from (146) that the projective factor $P = d\varphi_x$ is a 1-form, hence the spray coefficients $G^i = P y^i$ are quadratic in $y \in T_x U$. By definition, F is a Berwald metric. Every Berwald metric with $\mathbf{K} = 0$ is locally Minkowskian. See [Bao et al. 2000] for a proof.

Assume that $c \neq 0$. By (146), $P = cF + d\varphi$. Let

$$\Psi := P + cF = 2cF + d\varphi_x.$$

Then

$$F = \frac{1}{2c}\Big\{\Psi(x,y) - d\varphi_x\Big\} = \frac{1}{2c}\Big\{\Psi(x,y) + \frac{\langle a, y\rangle}{1+\langle a, x\rangle}\Big\}.$$

Since F is projectively flat and P is the projective factor,

$$F_{x^k} = (PF)_{y^k}, \qquad P_{x^k} = PP_{y^k} + c^2 F F_{y^k}.$$

These equations imply that $\Psi_{x^i} = \Psi\Psi_{y^i}$. Let $\Theta := \Psi(x,y)$ if $c > 0$ and $\Theta := -\Psi(x,-y)$ if $c < 0$. Then Θ satisfies (22) and F can be expressed in the form (144). □

17. Flag Curvature and Relatively Isotropic L-Curvature

Although the relatively isotropic J-curvature condition is stronger than the isotropic S-curvature condition for Randers metrics (Lemma 10.1), it seems that there is no direct relationship between these two conditions. Nevertheless, for Finsler metrics of scalar curvature, the relatively isotropic J-curvature condition also implies that the flag curvature takes a special form in certain cases.

Proposition 17.1. ([Chen et al. 2003]) *Let F be an n-dimensional Finsler manifold of scalar curvature. Suppose it has relatively constant J-curvature, i.e.,*

$$\mathbf{J} + cF\mathbf{I} = 0, \tag{153}$$

where $c = constant$. Then

$$\mathbf{K} = -c^2 + \sigma e^{-\frac{3\tau}{n+1}},$$

where $\tau = \tau(x,y)$ is the distortion and $\sigma = \sigma(x)$ is a scalar function on M;

Proof: By assumption $J_k = -cFI_k$. Using (99) and (100), one obtains

$$I_{k|p|q}y^p y^q = J_{k|m}y^m = -cFI_{k|m}y^m = c^2F^2\tau_{\cdot k}.$$

Plugging it into (111) yields

$$\frac{n+1}{3}\mathbf{K}_{\cdot k} + \Big(\mathbf{K} + c^2\Big)\tau_{\cdot k} = 0.$$

This implies that

$$\Big[\Big(\mathbf{K} + c^2\Big)^{\frac{n+1}{3}} e^{\tau}\Big]_{\cdot k} = \Big(\mathbf{K} + c^2\Big)^{\frac{n-2}{3}} e^{\tau}\Big\{\frac{n+1}{3}\mathbf{K}_{\cdot k} + \mathbf{K}\tau_{\cdot k}\Big\} = 0.$$

Thus the function $(\mathbf{K} + c^2)^{\frac{n+1}{3}}e^{\tau}$ is independent of $y \in T_xM$. □

Proposition 17.1 in the case when $c = 0$ is essentially proved in Matsumoto's book. See Proposition 26.2 in [Matsumoto 1972a]. Matsumoto assumes that F is a Landsberg metric, but what he actually needs in his proof is that $\mathbf{J} = 0$. Since the notion of distortion has not been introduced in [Matsumoto 1972a], his result is stated in a local coordinate system.

Corollary 17.2. *Let F be a Finsler metric on a manifold M. Suppose that the following hold,*

(i) *F has isotropic flag curvature which is not equal to $-c^2$, i.e., $\mathbf{K} = \mathbf{K}(x) \neq -c^2$ is a scalar function on M,*

(ii) *F has relatively constant J-curvature, i.e.,* $\mathbf{J} + cF\mathbf{I} = 0$.
Then F is Riemannian.

Proof: By Proposition 17.1,

$$\mathbf{K}(x) = -c^2 + \sigma(x)e^{-\frac{3\tau}{n+1}}.$$

Since $\mathbf{K}(x) \neq -c^2$, one concludes that $\sigma(x) \neq 0$ and hence $\tau = \tau(x)$ is independent of $y \in T_xM$. It follows from (52) that $I_i = \tau_{y^i} = 0$. Thus F is Riemannian by Deicke's theorem [Deicke 1953]. □

Proposition 17.3. *Let F be a Finsler metric of scalar curvature on an n-dimensional manifold. Suppose that F has relatively isotropic L-curvature, i.e.,*

$$\mathbf{L} + cF\mathbf{C} = 0, \tag{154}$$

where $c = c(x)$ *is a scalar function on* M.

(a) *If* $c = constant$, *then*

$$\mathbf{K} = -c^2 + \sigma e^{-\frac{3\tau}{n+1}},$$

where $\sigma = \sigma(x)$ *is a scalar function on* M.

(b) *If* $n \geq 3$ *and* $\mathbf{K} \neq -c^2 + \frac{c_{x^m}y^m}{F}$ *for almost all* $y \in T_xM \setminus \{0\}$ *at any point x in an open domain U of M, then* $F = \alpha + \beta$ *is a Randers metric in U.*

Proof: Note that if (153) holds, then (152) holds by taking the average of (153) on both sides. Proposition 17.3 (a) follows from Proposition 17.1.

Now we assume that $\mathbf{K} \neq -c^2 + c_{x^m}(x)y^m/F$ for almost all $y \in T_xM \setminus \{0\}$ at any point x in an open domain $U \subset M$. By assumption, $L_{ijk} = -cFC_{ijk}$, one obtains

$$C_{ijk|p|q}y^py^q = -c_{x^m}y^mFC_{ijk} - cFL_{ijk} = \Big(c^2 - \frac{c_{x^m}y^m}{F}\Big)F^2C_{ijk}.$$

Since $J_k = -cFI_k$ by (152), we have

$$I_{k|p|q}y^py^q = -c_{x^m}y^mFI_k - cFJ_k = \Big(c^2 - \frac{c_{x^m}y^m}{F}\Big)F^2I_k.$$

By the formula for M_{ijk} in (25), one obtains

$$M_{ijk|p|q}y^py^q = \Big(c^2 - \frac{c_{x^m}y^m}{F}\Big)F^2M_{ijk}.$$

Since F is of scalar curvature, equation (112) holds. One obtains

$$\Big(\mathbf{K} + c^2 - \frac{c_{x^m}y^m}{F}\Big)F^2M_{ijk} = 0. \tag{155}$$

It follows from (154) that the Matsumoto torsion vanishes, $M_{ijk} = 0$. By Proposition 3.3, $F = \alpha + \beta$ is a Randers metric on U. □

Proposition 17.3 is proved by H. Izumi. See [Izumi 1976], [Izumi 1977], and [Izumi 1982]. Proposition 17.3 in the case when $c = 0$ is proved by S. Numata [Numata 1975].

Corollary 17.4. ([Numata 1975]) *Let F be a Finsler metric of scalar curvature on an n-dimensional manifold ($n \geq 3$). Suppose that $\mathbf{L} = 0$ and $\mathbf{K} \neq 0$. Then F is Riemannian.*

Proof: By Proposition 17.3, $F = \alpha + \beta$ is a Randers metric with $\mathbf{L} = 0$. By Lemma 10.1, $\mathbf{S} = 0$ and β is closed. By Proposition 15.1, one concludes that $\mathbf{K} = \sigma(x)$ is a scalar function on M. It follows from (109) that

$$0 = -F^2\sigma(x)I_k.$$

By assumption, $\mathbf{K} = \sigma(x) \neq 0$. Thus $I_k = 0$ and F is Riemannian by Deicke's theorem. □

We may ask the following question again: is there any non-Berwaldian Finsler metric satisfying the following conditions

$$\mathbf{K} = 0, \qquad \mathbf{L} = 0 \ \ (\text{or } \mathbf{J} = 0)?$$

If such a metric exists, it can not be locally projectively flat and it can not be a Randers metric. Why?

Example 17.5. Let $F = \alpha + \beta$ be a Randers metric on R^n defined by

$$F := |y| + \frac{\langle x, y\rangle}{\sqrt{1+|x|^2}}, \qquad y \in T_x\mathrm{R}^n \cong \mathrm{R}^n.$$

Note that

$$\|\beta\|^2 = \frac{|x|^2}{1+|x|^2} < 1.$$

F is indeed a Randers metric on the whole R^n. One can verify that F satisfies (83). Thus it is a projectively flat Randers metric on R^n. Further, the spray coefficients $G^i = Py^i$ are given by

$$P = c\Big\{|y| - \frac{\langle x, y\rangle}{\sqrt{1+|x|^2}}\Big\},$$

where $c = \frac{1}{2\sqrt{1+|x|^2}}$. Let

$$\rho := \ln\sqrt{1-\|\beta\|^2} = -\ln\sqrt{1+|x|^2}.$$

By (65), one obtains

$$\mathbf{S} = (n+1)\Big\{P - \rho_0\Big\} = (n+1)cF.$$

Since β is closed, by Proposition 10.1, the above identity is equivalent to the following identity:

$$\mathbf{L} + cF\mathbf{C} = 0.$$

Since F is projectively flat, it is of scalar curvature. By further computation, one can find the flag curvature

$$\mathbf{K} = \frac{P^2 - P_{x^k}y^k}{F^2} = \frac{3}{4(1+|x|^2)} \cdot \frac{|y|\sqrt{1+|x|^2} - \langle x, y\rangle}{|y|\sqrt{1+|x|^2} + \langle x, y\rangle}.$$

Note that $\mathbf{K} \neq -c^2 + \frac{c_{x^k}(x)y^k}{F(x,y)}$ and F is a Randers metric. This matches the conclusion in Proposition 17.3 (b).

The Randers metric in Example 17.5 is locally projectively flat. There are non-projectively flat Randers metrics of scalar curvature and isotropic S-curvature. See Example 11.2. This example is a Randers metric generated by a special vector field on the Euclidean space by (16). In fact, we can determine all vector fields V on a Riemannian space form (M, α_μ) of constant curvature μ such that the generated Randers metric $F = \alpha + \beta$ by (α_μ, V) is of scalar curvature and isotropic S-curvature. This work will appear somewhere else.

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