Some Constructions of Projectively Flat Finsler Metrics*

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Abstract

We construct a family of projectively flat Finsler metrics in the form $F = \alpha + \epsilon\beta + k\beta^2/\alpha$, where ϵ and k are constants with $k \neq 0$. In particular, we show that if a Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$ is projectively flat with isotropic S-curvature, then the Finsler metric $\tilde{F} := (\bar{\alpha} + \bar{\beta})^2/\bar{\alpha}$ is conformally equivalent to a projectively flat one. Further, we discover that a sub-family has zero flag curvature.

Key words and phrases: Randers metric, (α, β)-metric, Finsler metric, projectively flat metric and S-curvature

1 Introduction

It is the Hilbert Fourth Problem in the regular case to study and characterize Finsler metrics on an open domain Ω in R^n such that geodesics are straight lines. Finsler metrics on Ω with this property are said to be *projectively flat*. According to G. Hamel [4], a Finsler metric $F = F(x, y)$ on Ω is projectively flat if and only if it satisfies the following partial differential equations

$$F_{x^k y^i} y^k = F_{x^i}. \tag{1}$$

It is one of important problems in Finsler geometry to study the solutions of (1). By the Beltrami theorem, if a Riemannian metric $F = \sqrt{g_{ij}(x)y^i y^j}$ satisfies (1), then it has constant sectional curvature. Conversely, every Riemannian metric with constant sectional curvature μ is locally isometric to the following metric:

$$\bar{\alpha} = \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 + \mu|x|^2}, \tag{2}$$

Clearly, $F := \bar{\alpha}$ satisfies (1). However, there are lots of projectively flat Finsler metrics which are not of constant flag curvature (the flag curvature is an analogue of the sectional curvature in Finsler geometry). In [9], we characterize the local structure of projectively flat metrics with constant flag curvature. In [1],

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we classify projectively flat Randers metrics with isotropic S-curvature. In [2], we give a description on general projectively flat Finsler metrics with isotropic S-curvature. The main purpose of this paper is to construct some projectively flat Finsler metrics in the form $F = \alpha\phi(\beta/\alpha)$, where α is a Riemannian metric, β is a 1-form and $\phi = \phi(s)$ is a positive C^∞ function (e.g., $\phi(s) = 1 + \epsilon s + ks^2$). Such metrics are called (α, β)-*metrics*.

First, let us recall two non-trivial solutions to (1):

$$\bar{F} = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2} + \frac{\langle x, y\rangle}{1 - |x|^2}, \tag{3}$$

and

$$F = \frac{(\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle)^2}{(1 - |x|^2)^2\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}. \tag{4}$$

$\bar{F}$ is the well-known Funk metric and F is Berwald's metric defined on the unit ball $\mathrm{B}^n \subset \mathrm{R}^n$ centered at the origin [3]. Both metrics are projectively flat. Moreover, $\bar{F}$ has negative constant flag curvature $\mathbf{K} = -1/4$, and F has zero flag curvature $\mathbf{K} = 0$. $\bar{F}$ can be expressed as $\bar{F} = \bar{\alpha} + \bar{\beta}$, where

$$\bar{\alpha} := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2}, \qquad \bar{\beta} := \frac{\langle x, y\rangle}{1 - |x|^2}.$$

Let $\lambda := 1/(1 - |x|^2)$. Then F can be expressed as

$$F = \frac{(\alpha + \beta)^2}{\alpha} = \alpha + 2\beta + \frac{\beta^2}{\alpha}, \tag{5}$$

where $\alpha := \lambda\bar{\alpha}$ and $\beta = \lambda\bar{\beta} = \frac{1}{2}d\lambda$. Thus $\bar{F}$ and F are special (α, β)-metrics.

Consider a general Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$ on a manifold. It is well-known that $\bar{F}$ is locally projectively flat if and only if $\bar{\alpha}$ is locally projectively flat (equivalently, it is of constant sectional curvature μ by the Beltrami theorem) and $\bar{\beta}$ is closed. Locally, we can express $\bar{\alpha}$ by (2) and we can express $\bar{\beta}$ as a differential of some scalar function $\rho = \rho(x)$, i.e., $\bar{\beta} = \frac{1}{2}d\rho$. In this paper, we are going to find scalar functions $\lambda = \lambda(x) > 0$ and $\sigma = \sigma(x)$ such that for $\alpha := \lambda\bar{\alpha}$ and $\beta := \frac{1}{2}d\sigma$, the Finsler metric in the following form is projectively flat,

$$F := \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha},$$

where ϵ, k are constants with $k \neq 0$ satisfy certain conditions (see (12) below) so that F is a Finsler metric when $\|\beta\|_\alpha < 1$. In particular, we prove the following theorems.

Theorem 1.1 *Let*

$$h := \frac{\eta|x|^2}{(1 + \sqrt{1 + \mu|x|^2})\sqrt{1 + \mu|x|^2}} + \frac{d_1 + \langle a, x\rangle}{\sqrt{1 + \mu|x|^2}},$$

where η and d_1 are constants. Let

$$\lambda : = d_3 + 2k\eta d_2 h - k\mu d_2 h^2, \tag{6}$$

$$\sigma : = \pm 2\int \sqrt{d_2\lambda}\, dh, \tag{7}$$

where d_2, d_3 are constants with $d_2 > 0$ and $d_3 + 2k\eta d_1 d_2 - k\mu d_1^2 d_2 > 0$ so that $\lambda > 0$ on an open neighborhood of the origin. Then for $\alpha := \lambda\bar{\alpha}$ and $\beta := \frac{1}{2}d\sigma$, the following Finsler metric

$$F := \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$$

is projectively flat on its domain.

When $d_1 = d_2 = 1$, $d_3 = 0$, $\eta = 0$, $\mu = -1$, $\epsilon = 2$, $k = 1$,

$$\lambda = \frac{(1 + \langle a, x\rangle)^2}{1 - |x|^2} = \sigma.$$

Then for $\alpha := \lambda\bar{\alpha}$ and $\beta := \frac{1}{2}d\sigma$, the Finsler metric $F = (\alpha + \beta)^2/\alpha$ is projectively flat. More important, this metric has zero flag curvature! Thus we obtain the following

Theorem 1.2 *Let $a \in \mathrm{R}^n$ be an arbitrary constant vector with $|a| < 1$. The following Finsler metric is projectively flat with zero flag curvature $\mathbf{K} = 0$,*

$$F := \frac{[(1 + \langle a, x\rangle)(\sqrt{|y|^2 - (|x|^2|y^2 - \langle x, y\rangle^2)} + \langle x, y\rangle) + (1 - |x|^2)\langle a, y\rangle]^2}{(1 - |x|^2)^2\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}. \tag{8}$$

The metric in (8) is a slight generalization of Berwald's famous example in (5). The curvature property follows from the classification theorem in [9]. See Section 6 below.

As we know that if a projectively flat Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$ has isotropic S-curvature, $\mathbf{S} = \frac{1}{2}(n+1)c(x)\bar{F}$, where $c = c(x)$ is a scalar function, then $\bar{\alpha}$ is given by (2) and $\bar{\beta} = \frac{1}{2}d\rho$ for some scalar function $\rho = \rho(x)$. Both $c = c(x)$ and $\rho = \rho(x)$ can be explicitly determined. See [1] or Section 5 below. By Theorem 1.1, there are scalar functions $\lambda = \lambda(x)$ and $\sigma = \sigma(x)$ in the forms (6) and (7) with $k = 1$, respectively, such that for $\alpha := \lambda\bar{\alpha}$ and $\beta := \frac{1}{2}d\sigma$, the Finsler metric $F = \alpha + \epsilon\beta + k\beta^2/\alpha$ is projectively flat. In fact, λ and σ can be chosen so that $\beta = \frac{1}{2}d\sigma = \frac{1}{2}\lambda d\rho = \lambda\bar{\beta}$. Therefore, we obtain the following

Theorem 1.3 *Let $\bar{F} = \bar{\alpha} + \bar{\beta}$ be an n-dimensional projectively flat Randers metric with isotropic S-curvature, $\mathbf{S} = \frac{1}{2}(n+1)c(x)\bar{F}$, on an open domain in R^n. There is a scalar function $\lambda = \lambda(x) > 0$ such that for $\alpha := \lambda\bar{\alpha}$ and $\beta := \lambda\bar{\beta}$, the Finsler metric in the following form is projectively flat,*

$$F := \frac{(\alpha + \beta)^2}{\alpha}. \tag{9}$$

We will give a direct proof for Theorem 1.3 in Section 5 below. The special case is when $\mu = -1$ and $c = \frac{1}{2}$. In this case, $\bar{F}$ is a generalized Funk metric [8] [9], i.e.,

$$\bar{F} = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2} + \frac{\langle x, y\rangle}{1 - |x|^2} + \frac{\langle a, y\rangle}{1 + \langle a, x\rangle}.$$

In this case, $\lambda = (1 + \langle a, x\rangle)^2/(1 - |x|^2)$. Then the Finsler metric F in (9) is given by (8). As we have claimed in Theorem 1.2, this metric has zero flag curvature.

Recently, P. Senarath and G. Thornley ([7]) have given an equation in local coordinates that charaterizes projectively flat Finsler metrics in the form $F = \alpha + \beta^2/\alpha$. In particular, they show that if $\bar{F} = \bar{\alpha} + \bar{\beta}$ is projectively flat, then $\tilde{F} := \bar{\alpha} + \bar{\beta}^2/\bar{\alpha}$ can not be locally projectively flat unless it is locally Minkowskian. According to our results, there are scalar functions $\lambda = \lambda(x) > 0$ and $\delta = \delta(x)$ such that for $\alpha := \lambda\bar{\alpha}$ and $\beta := \delta\bar{\beta}$, $F = \alpha + \beta^2/\alpha$ is projectively flat (Theorem 1.1). If in addition, $\bar{F}$ has isotropic S-curvature, one can even choose $\lambda = \delta$ (Theorem 1.3).

The Finsler metric F in (9) can be expressed as $F = \lambda\tilde{F}$, where $\tilde{F} := (\bar{\alpha} + \bar{\beta})^2/\bar{\alpha}$. The Finsler metric $\tilde{F}$ is defined directly from $\bar{F} = \bar{\alpha} + \bar{\beta}$. It is then a natural problem to find a Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$ with constant flag curvature such that $\tilde{F} := (\bar{\alpha} + \bar{\beta})^2/\bar{\alpha}$ is conformal to a Finsler metric with constant flag curvature? Theorem 1.2 gives a solution to this problem. We expect to find more solutions to this problem in the near future.

2 Projectively flat (α, β)-metrics

Let $\alpha := \sqrt{a_{ij}y^iy^j}$ be a Riemannian metric, $\beta := b_iy^i$ a 1-form, and $\phi = \phi(s)$ be a positive C^∞ function defined in a neighborhood of the origin $s = 0$. Let

$$F = \alpha\phi(s), \qquad s = \frac{\beta}{\alpha}. \tag{10}$$

It is known that $F = \alpha\phi(\beta/\alpha)$ is a Finsler metric for any α and β with $\|\beta\|_\alpha < b_o$ if and only if

$$\phi(s) > 0, \quad (\phi(s) - s\phi'(s)) + (b^2 - s^2)\phi''(s) > 0, \quad (|s| \le b < b_o) \tag{11}$$

By taking $b = s$, one obtains

$$\phi(s) - s\phi'(s) > 0, \quad (|s| < b_o).$$

See [10]. A Finsler metric F is called an (α, β)-metric if it is in the above form (10) with ϕ satisfying (11) and β satisfying $\|\beta\|_\alpha < b_o$.

Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric with $\alpha = \sqrt{a_{ij}y^iy^j}$ and $\beta = b_iy^i$. Let

$$s_{ij} := \frac{1}{2}\Big(b_{i|j} - b_{j|i}\Big), \qquad r_{ij} := \frac{1}{2}\Big(b_{i|j} + b_{j|i}\Big),$$

$$s_j := b^k s_{kj}, \qquad b := \sqrt{b^i b_i},$$

where $b_{i|j}$ denote the coefficients of the covariant derivative of β with respect to α. The spray coefficients G^i of F are given by

$$G^i = \bar{G}^i + \alpha Q s^i{}_0 + \Psi\Big(-2\alpha Q s_0 + r_{00}\Big)\Big\{\chi \frac{y^i}{\alpha} + b^i\Big\},$$

where

$$\begin{aligned} Q &= \frac{\phi'}{\phi - s\phi'}, \\ \chi &= \frac{(\phi - s\phi')\phi'}{\phi\phi''} - s, \\ \Psi &= \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}. \end{aligned}$$

The above formula is given in [5], [10].

Now we consider the following special function

$$\phi = 1 + \epsilon s + k s^2,$$

where ϵ and k are constants with $k \neq 0$. In virtue of (11), we obtain the following condition on ϵ and k:

$$1 + \epsilon s + k s^2 > 0, \qquad 1 + 2kb^2 - 3ks^2 > 0, \qquad (|s| \leq b < 1) \tag{12}$$

Namely, $F = \alpha + \epsilon\beta + k\beta^2/\alpha$ is a Finsler metric for any α and β with $\|\beta\|_\alpha < 1$ if and only if ϵ and k satisfy (12).

From now on, we always assume that ϵ and k satisfy (12). Then

$$Q = \frac{\epsilon + 2ks}{1 - ks^2}, \qquad \Psi = \frac{k}{1 + 2kb^2 - 3ks^2}, \qquad \chi = \frac{\epsilon - 3k\epsilon s^2 - 4k^2 s^3}{2k(1 + \epsilon s + ks^2)}.$$

Further, we assume that β is closed, i.e.,

$$s^i{}_0 = 0, \qquad s_0 = 0, \tag{13}$$

and it satisfies

$$r_{00} = \tau\Big\{(1/k + 2b^2)\alpha^2 - 3\beta^2\Big\}, \tag{14}$$

where $\tau = \tau(x)$ is a scalar function. Then

$$G^i = G^i_\alpha + \tau\Big\{\alpha\chi y^i + \alpha^2 b^i\Big\}. \tag{15}$$

If in addition,

$$G^i_\alpha = \tau\Big\{\theta y^i - \alpha^2 b^i\Big\}, \tag{16}$$

where $\theta = p_i y^i$ is a local 1-form, then

$$G^i = \tau\Big\{\alpha\chi + \theta\Big\} y^i.$$

In this case, F is projectively flat.

Lemma 2.1 *Let ϵ and k be constants satisfying (12). If α and β satisfies (13), (14) and (16), then $F = \alpha + \epsilon\beta + k\beta^2/\alpha$ is a solution of (1).*

M. Matsumoto ([6]) has proved that in dimension $n \geq 3$, the Finsler metric $F = \alpha + \beta^2/\alpha$ with $b \neq 0$ is a Douglas metric if and only if there is a scalar function $\tau = \tau(x)$ such that β is closed and

$$r_{00} = \tau\Big\{(1 + 2b^2)\alpha^2 - 3\beta^3\Big\}.$$

We conjecture that $F = \alpha + \epsilon\beta + k\beta^2/\alpha$ satisfy (1) if and only if (13), (14) and (16) hold.

3 Deformation of Randers metrics

First, for a scalar function $f = f(x)$, we use the following notations:

$$f_0 := \frac{\partial f}{\partial x^i}(x)y^i, \qquad f_{00} = \frac{\partial^2 f}{\partial x^i \partial x^j}(x)y^i y^j.$$

Let $\bar{\alpha}$ be given in (2) defined on a ball $\mathrm{B}^n(r)$ in R^n and $\bar{\beta} = \frac{1}{2}d\rho$, where $\rho = \rho(x)$ is a scalar function on $\mathrm{B}^n(r)$. We are going to find scalar functions $\lambda = \lambda(x) > 0$ and $\sigma = \sigma(x)$ such that for $\alpha := \lambda\bar{\alpha}$ and $\beta := \frac{1}{2}\sigma_0$, the Finsler metric $F = \alpha + \epsilon\beta + k\beta^2/\alpha$ is projectively flat.

Let $\alpha := \lambda\bar{\alpha}$ and $\beta := \frac{1}{2}\sigma_0$ for some scalar functions λ and σ. We have

$$b^2 = \frac{1}{4\lambda^2}|\bar{\nabla}\sigma|^2. \tag{17}$$

The spray coefficients of α are given by

$$\begin{aligned} G^i_\alpha &= \bar{G}^i + \frac{\lambda_0}{\lambda}y^i - \frac{1}{2}\bar{a}^{ij}\frac{\lambda_{x^j}}{\lambda}\bar{\alpha}^2 \\ &= \Big\{ - \frac{\mu\langle x, y\rangle}{1 + \mu|x|^2} + \frac{\lambda_0}{\lambda}\Big\}y^i - \frac{1}{2}a^{ij}\frac{\lambda_{x^j}}{\lambda}\alpha^2. \end{aligned}$$

Eq. (16) is equivalent to

$$\frac{\lambda_{x^i}}{\lambda} = \tau\sigma_{x^i}. \tag{18}$$

Since β is exact, it satisfies (13). Then $r_{00} = b_{0|0}$ is given by

$$\begin{aligned} r_{00} &= \frac{\partial b_i}{\partial x^j}y^i y^j - 2b_m G^m_\alpha \\ &= \frac{1}{2}\sigma_{00} - \sigma_0\Big\{ - \frac{\mu\langle x, y\rangle}{1 + \mu|x|^2} + \frac{\lambda_0}{\lambda}\Big\} + \frac{1}{2}\sigma_{x^i}\bar{a}^{ij}\frac{\lambda_{x^j}}{\lambda}\bar{\alpha}^2. \end{aligned} \tag{19}$$

By (17) and (18), we rewrite (19) as follows

$$r_{00} = \frac{1}{2}\sigma_{00} - \sigma_0\Big\{ - \frac{\mu\langle x, y\rangle}{1 + \mu|x|^2} + \tau\sigma_0\Big\} + 2\tau b^2\alpha^2. \tag{20}$$

By (20), we see that (14) is equivalent to

$$\frac{1}{2}\sigma_{00}-\sigma_0\Big\{-\frac{\mu\langle x,y\rangle}{1+\mu|x|^2}+\tau\sigma_0\Big\}=\frac{\tau}{k}\alpha^2-3\tau\beta^2. \tag{21}$$

By $\beta=\frac{1}{2}\sigma_0$, we can simplify (21) to

$$\sigma_{00}+\frac{2\mu\langle x,y\rangle}{1+\mu|x|^2}\sigma_0=2\tau\Big\{\frac{\lambda^2}{k}\bar{\alpha}^2+\frac{1}{4}\sigma_0^2\Big\}. \tag{22}$$

By Lemma 2.1 and the above arguments, we obtain the following

Lemma 3.1 *Let ϵ and k be constants satisfying (12). Let $\bar{\alpha}$ be given in (2). Suppose that there are scalar functions $\lambda=\lambda(x)>0$, $\sigma=\sigma(x)$ and $\tau=\tau(x)$ such that (18) and (22) hold. Then for $\alpha:=\lambda\bar{\alpha}$ and $\beta:=\frac{1}{2}\sigma_0$, the Finsler metric $F=\alpha+\epsilon\beta+k\beta^2/\alpha$ is projectively flat.*

Let

$$\lambda=\lambda(h),\qquad \sigma=\sigma(h),$$

where $h=h(x)$ is a scalar function. Thus both λ and σ are functions of x. In this case, (18) is equivalent to

$$\tau=\frac{\lambda'}{\lambda\sigma'}, \tag{23}$$

and (22) is equivalent to

$$h_{00}+\frac{2\mu\langle x,y\rangle}{1+\mu|x|^2}h_0=\frac{(\lambda^2)'}{k(\sigma')^2}\bar{\alpha}^2+\Big(\ln\frac{\sqrt{\lambda}}{|\sigma'|}\Big)'h_0^2. \tag{24}$$

By Lemma 3.1, we obtain the following

Lemma 3.2 *Let ϵ and k be constants satisfying (12). Let $\lambda=\lambda(h)>0$ and $\sigma=\sigma(h)$ such that there is a scalar function $h=h(x)$ satisfying (24). Then for $\alpha:=\lambda\bar{\alpha}$, $\beta:=\frac{1}{2}\sigma_0$, the Finsler metric $F=\alpha+\epsilon\beta+k\beta^2/\alpha$ is projectively flat.*

4 Proof of Theorem 1.1

In this section, we are going to use Lemma 3.2 to prove Theorem 1.1. We first construct a scalar function $h=h(x)$. Then we find $\lambda=\lambda(h)$ and $\sigma=\sigma(h)$ such that (24) holds.

Let $\xi=\xi(x)$ and

$$h:=\frac{\xi(x)}{\sqrt{1+\mu|x|^2}}.$$

We have

$$h_{00}+\frac{2\mu\langle x,y\rangle}{1+\mu|x|^2}h_0=\frac{\xi_{00}}{\sqrt{1+\mu|x|^2}}-\mu h\bar{\alpha}^2, \tag{25}$$

where $\bar{\alpha}$ is given in (2). Assume that ξ satisfies the following equation

$$\xi_{00} = \eta\sqrt{1+\mu|x|^2}\,\bar{\alpha}^2, \tag{26}$$

Then it follows from (25) that

$$h_{00} + \frac{2\mu\langle x, y\rangle}{1+\mu|x|^2} h_0 + (\mu h - \eta)\bar{\alpha}^2 = 0. \tag{27}$$

Solving (26), we obtain

$$\xi := d_1 + \langle a, x\rangle + \frac{\eta|x|^2}{1+\sqrt{1+\mu|x|^2}},$$

where η and d_1 are constants and $a \in \mathrm{R}^n$ is a constant vector. Then h is given by

$$h := \frac{d_1 + \langle a, x\rangle}{\sqrt{1+\mu|x|^2}} + \frac{\eta|x|^2}{(1+\sqrt{1+\mu|x|^2})\sqrt{1+\mu|x|^2}}. \tag{28}$$

In virtue of (27), in order to find $\lambda = \lambda(h)$ and $\sigma = \sigma(h)$ satisfying (24), it suffices to solve the following equations:

$$\Big(\ln\frac{\sqrt{\lambda}}{|\sigma'|}\Big)' = 0, \tag{29}$$

$$\mu h - \eta + \frac{[\lambda^2]'}{k(\sigma')^2} = 0, \tag{30}$$

From (29), we obtain

$$\sigma' = \pm 2\sqrt{d_2\lambda}, \tag{31}$$

where d_2 is a positive constant. Plugging it to (30) yields

$$2k\mu d_2 h - 2k\eta d_2 + \lambda' = 0.$$

We obtain

$$\lambda = d_3 + 2k\eta d_2 h - k\mu d_2 h^2.$$

Plugging it into (31) we obtain the following formula for σ:

$$\sigma = \pm 2\int \sqrt{d_2\lambda}\, dh.$$

This proves the theorem. Q.E.D.

5 Proof of Theorem 1.3

In this section, we are going to prove Theorem 1.3. Let $\bar{F} = \bar{\alpha} + \bar{\beta}$ be a locally projectively flat metric on an n-dimensional manifold. We may assume that $\bar{\alpha}$ is given by (2) on an open neighborhood of the origin and $\bar{\beta} = \frac{1}{2}\rho_0$ for some scalar function $\rho = \rho(x)$. Now we assume that $\bar{F}$ has isotropic S-curvature, $\mathbf{S} = \frac{1}{2}(n+1)c\bar{F}$, where $c = c(x)$ is a scalar function. According to [1],

$$\rho = \begin{cases} \ln \frac{(1+\langle a,x\rangle)^2}{1-|x|^2} & \text{if } \mu + 4c^2 = 0,\ \mu = -1 \\ 2(1+\langle a,x\rangle) & \text{if } \mu + 4c^2 = 0,\ \mu = 0 \\ -\int \frac{4}{\mu+4c^2} dc & \text{if } \mu + 4c^2 \neq 0 \end{cases}.$$

When $\mu + 4c^2 \neq 0$, the scalar function $c = c(x)$ is determined by

$$c_{00} + \frac{2\mu\langle x,y\rangle}{1+\mu|x|^2} c_0 = -c(\mu + 4c^2)\bar{\alpha}^2 + \frac{12cc_0^2}{\mu + 4c^2}. \tag{32}$$

The general solution of (32) will be given below. See [1].

Let

$$\lambda := \begin{cases} (1+\langle a,x\rangle)^2/(1-|x|^2) & \text{if } \mu + 4c^2 = 0,\ \mu = -1 \\ 1 & \text{if } \mu + 4c^2 = 0,\ \mu = 0 \\ 1/|16c^2 \pm 4| & \text{if } \mu + 4c^2 \neq 0,\ \mu = \pm 1 \\ 4/c^2 & \text{if } \mu + 4c^2 \neq 0,\ \mu = 0 \end{cases}$$

We are going to show that for $\alpha := \lambda\bar{\alpha}$, $\beta = \lambda\bar{\beta}$, the Finsler metric in the following form is projectively flat,

$$F = \frac{(\alpha+\beta)^2}{\alpha} \tag{33}$$

(a) Assume that $\mu + 4c^2 = 0$ and $\mu = 0$. In this case, we let $h = h(x)$ be the function in (28) with $k = 1$ and $\mu = 0$, i.e.,

$$h = d_1 + \langle a,x\rangle + \frac{\eta}{2}|x|^2.$$

Let

$$\lambda = d_3 + 2\eta d_2 h, \qquad \sigma = \pm 2\int \sqrt{d_2\lambda}\, dh.$$

Then $\lambda = \lambda(h), \sigma = \sigma(h)$ and $h = h(x)$ satisfy (24) with $k = 1$ and $\mu = 0$. By Lemma 3.2, for $\alpha := \lambda\bar{\alpha}$ and $\beta := \frac{1}{2}\sigma_0$, the Finsler metric $F = (\alpha+\beta)^2/\alpha$ is projectively flat.

If $d_1 = 1, d_2 = 1, d_3 = 1$ and $\eta = 0$,

$$h = 1 + \langle a,x\rangle, \qquad \rho = 2h = \sigma, \quad \lambda = 1,$$

then

$$\beta = \frac{1}{2}\sigma_0 = h_0 = \frac{1}{2}\rho_0 = \bar{\beta}.$$

Then

$$F = \frac{(\bar{\alpha} + \bar{\beta})^2}{\bar{\alpha}}$$

is a Minkowski metric.

Assume that $\mu + 4c^2 = 0$, $\mu = -1$. By reversing the metric, if necessary, We can also assume that $c = \frac{1}{2}$. Let $h = h(x)$ be the function in (28) with $k = 1$.

$$h = \frac{d_1 + \langle a, x\rangle}{\sqrt{1 - |x|^2}} + \frac{\eta |x|^2}{(1 + \sqrt{1 - |x|^2})\sqrt{1 - |x|^2}}.$$

Let

$$\lambda = d_3 + 2\eta d_2 h + d_2 h^2, \qquad \sigma = \pm 2 \int \sqrt{d_2 \lambda}\, dh.$$

Then $\lambda = \lambda(h), \sigma = \sigma(h)$ and $h = h(x)$ satisfy (24) with $k = 1$ and $\mu = -1$. By Lemma 3.2, for $\alpha := \lambda\bar{\alpha}$ and $\beta := \frac{1}{2}\sigma_0$, the Finsler metric $F = (\alpha + \beta)^2/\alpha$ is projectively flat.

If $d_1 = 1$, $d_2 = 1$, $d_3 = 0$ and $\eta = 0$, then

$$h = \frac{1 + \langle a, x\rangle}{\sqrt{1 - |x|^2}}, \qquad \rho = 2\ln h, \qquad \lambda = h^2 = \sigma.$$

In this case,

$$\beta = \frac{1}{2}\sigma_0 = h h_0 = \lambda \frac{h_0}{h} = \frac{1}{2}\lambda\rho_0 = \lambda\bar{\beta}.$$

Then

$$F = \frac{(\alpha + \beta)^2}{\alpha} = \lambda \frac{(\bar{\alpha} + \bar{\beta})^2}{\bar{\alpha}}. \tag{34}$$

(b) Assume that $\mu + 4c^2 \neq 0$. There is a constant η and a function $f = f(c)$ satisfying

$$\frac{\mu f - \eta}{f'} = c(\mu + 4c^2), \qquad \frac{f''}{f'} + \frac{12c}{\mu + 4c^2} = 0. \tag{35}$$

The function f is given by

$$f(c) = \begin{cases} \frac{c}{\sqrt{|\mu + 4c^2|}} & \text{if } \mu \neq 0 \text{ (taking } \eta = 0) \\ -\frac{1}{2c^2} & \text{if } \mu = 0 \text{ (taking } \eta = -4) \end{cases} \tag{36}$$

We use η and f to define a scalar function $c = c(x)$.

$$f(c) := h(x) = \begin{cases} \frac{d_1 + \langle a, x\rangle}{\sqrt{1 + \mu|x|^2}} & \text{if } \mu \neq 0 \\ d_1 + \langle a, x\rangle - 2|x|^2 & \text{if } \mu = 0 \end{cases}.$$

By (27) and (35), one can easily verify that $c = c(x)$ satisfies (32). Thus we obtain a general solution of (32) by solving $f(c) = h$ for c. Compare [1].

Let

$$\lambda: \;=\; d_3 + 2\eta d_2 h - \mu d_2 h^2,$$
$$\sigma: \;=\; \pm 2\int \sqrt{d_2\lambda}\, dh,$$

where $\eta = 0$ if $\mu \neq 0$ and $\eta = -4$ if $\mu = 0$. Then $\lambda = \lambda(h), \sigma = \sigma(h)$ and $h = h(x)$ satisfy (24) with $k = 1$. By Lemma 3.2, for $\alpha := \lambda\bar{\alpha}$ and $\beta := \frac{1}{2}\sigma_0$, the Finsler metric $F = (\alpha+\beta)^2/\alpha$ is projectively flat.

We can express β as

$$\beta = \frac{1}{2}\sigma_0 = \pm\sqrt{d_2\lambda}\, h_0 = \pm\sqrt{d_2\lambda}\, f'(c) c_0 = \delta\bar{\beta},$$

where

$$\delta := \mp\frac{1}{2}(\mu + 4c^2)\sqrt{d_2\lambda}\, f'(c).$$

We can choose d_3 and the sign of σ such that $\delta = \lambda$.

Observe that

$$\lambda = \begin{cases} \frac{d_3|\mu+4c^2| - d_2\mu c^2}{|\mu+4c^2|} & \text{if } \mu \neq 0 \\ \frac{d_3 c^2 + 4d_2}{c^2} & \text{if } \mu = 0 \end{cases}$$

$$\delta = \mp\begin{cases} \frac{\mu\sqrt{d_2[d_3|\mu+4c^2| - d_2\mu c^2]}}{2|\mu+4c^2|} & \text{if } \mu \neq 0 \\ \frac{2\sqrt{d_2(d_3c^2+4d_2)}}{c|c|} & \text{if } \mu = 0 \end{cases}$$

Take

$$d_3 = \begin{cases} \frac{d_2\mu}{4}\text{sign}(\mu+4c^2) & \text{if } \mu \neq 0 \\ 0 & \text{if } \mu = 0 \end{cases}$$

Then

$$\lambda = \begin{cases} \frac{d_2\mu^2}{4|\mu+4c^2|} & \text{if } \mu \neq 0 \\ \frac{4d_2}{c^2} & \text{if } \mu = 0 \end{cases}$$

$$\delta = \mp\begin{cases} \frac{d_2\mu|\mu|}{4|\mu+4c^2|} & \text{if } \mu \neq 0 \\ \frac{4d_2}{c|c|} & \text{if } \mu = 0 \end{cases}$$

Clearly, we can chose the sign of σ such that $\delta = \lambda$. This proves Theorem 1.3.

6 Zero flag curvature

In this section, we are going to show that the Finsler metric F in (34) has zero flag curvature. This fact follows from Theorem 1.3 in [9].

According to Theorem 1.3 in [9], for any Minkwski norm $\psi = \psi(y)$ on R^n and any positively homogeneous function of degree one, $\varphi = \varphi(y)$, on R^n, the following function F is a projectively flat Finsler metric with zero flag curvature on an open neigborhood of the origin in R^n,

$$F := \psi(y + Px)\Big\{1 + P_{y^m}x^m\Big\}, \tag{37}$$

where $P = P(x, y)$ is defined by

$$P = \varphi(y + Px). \tag{38}$$

The proof is based on the following important equation satisfied by P,

$$P_{x^k} = P P_{y^k}. \tag{39}$$

When φ is a Minkowski norm on R^n, the function P is a Finsler metric on the open domain

$$\Omega_\varphi := \Big\{ y \in \mathrm{R}^n \mid \varphi(y) < 1 \Big\}.$$

This metric is called the *Funk metric* of φ. Let

$$\psi := \varphi(y)\phi\Big(\frac{\langle a, y\rangle}{\varphi(y)}\Big),$$

where $\phi = \phi(s)$ is a positve C^∞ function of s and $a \in \mathrm{R}^n$ is a constant vector such that ψ is a Minkowsi norm on R^n. By (38), we have

$$\frac{\langle a, y + P\ x\rangle}{\varphi(y + P\ x)} = \frac{\langle a, y\rangle + P\langle a, x\rangle}{P} = \langle a, x\rangle + \frac{\langle a, y\rangle}{P}.$$

Thus

$$\begin{aligned} \psi(y + P\ x) &= \varphi(y + P\ x)\phi\Big(\frac{\langle a, y + P\ x\rangle}{\varphi(y + P\ x)}\Big) \\ &= P\phi\Big(\langle a, x\rangle + \frac{\langle a, y\rangle}{P}\Big). \end{aligned}$$

Then the Finsler metric in (37) can be expressed by

$$F = P\phi\Big(\langle a, x\rangle + \frac{\langle a, y\rangle}{P}\Big)\Big\{1 + P_{y^m}x^m\Big\}.$$

By (39), we have

$$P\Big\{1 + P_{y^m}x^m\Big\} = P + P_{x^m}x^m.$$

Then we obtain the following version of Theorem 1.3 in [9].

Theorem 6.1 *Let $\varphi = \varphi(y)$ be a Minkowski norm on R^n and $P = P(x, y)$ be the Funk metric of φ on a strongly convex domain Ω_φ. Let $\phi = \phi(s)$ be an arbitrary positive C^∞ function and $a \in \mathrm{R}^n$ is a constant vector such that $\psi := \varphi(y)\phi\Big(\langle a, y\rangle/\varphi(y)\Big)$ is a Minkowski norm on R^n. Let*

$$F := \phi\Big(\langle a, x\rangle + \frac{\langle a, y\rangle}{P(x, y)}\Big)\Big\{P(x, y) + P_{x^m}(x, y)x^m\Big\}.$$

Then F is projectively flat with projective factor P and flag curvature $\mathbf{K} = 0$.

When $\varphi = |y|$, we have

$$P = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)} + \langle x,y\rangle}{1-|x|^2}.$$

Then

$$P + P_{x^m}x^m = \frac{(\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)} + \langle x,y\rangle)^2}{(1-|x|^2)^2\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}.$$

In this case,

$$\begin{aligned} F &= \phi\Big(\langle a,x\rangle + \frac{(1-|x|^2)\langle a,y\rangle}{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)} + \langle x,y\rangle}\Big) \times \\ &\quad \times \frac{(\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)} + \langle x,y\rangle)^2}{(1-|x|^2)^2\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}. \end{aligned}$$

Further, if $\phi(s) = (1+s)^2$, then

$$F := \frac{[(1+\langle a,x\rangle)(\sqrt{|y|^2 - (|x|^2|y^2 - \langle x,y\rangle^2)} + \langle x,y\rangle) + (1-|x|^2)\langle a,y\rangle]^2}{(1-|x|^2)^2\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}. \tag{40}$$

By Theorem 6.1, F is projectively flat with $\mathbf{K} = 0$. Note that the Finsler metric in (40) is the same metric we obtained in (34).

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