

Finsler Manifolds with Nonpositive Flag Curvature and Constant S-curvature

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Abstract

The flag curvature is a natural extension of the sectional curvature in Riemannian geometry, and the S-curvature is a non-Riemannian quantity which vanishes for Riemannian metrics. There are (incomplete) non-Riemannian Finsler metrics on an open subset in \mathbb{R}^n with negative flag curvature and constant S-curvature. In this paper, we are going to show a global rigidity theorem that every Finsler metric with negative flag curvature and constant S-curvature must be Riemannian if the manifold is compact. We also study the nonpositive flag curvature case.

1 Introduction

One of important problems in Finsler geometry is to understand the geometric meanings of various quantities and their impacts on the global geometric structures. Imaging a Finsler manifold as an Easter egg and a Riemannian manifold as a white egg, Finsler manifolds are not only curved, but also very “colorful”. The flag curvature \mathbf{K} tells us how curved is the Finsler manifold at a point. There are several non-Riemannian quantities which describe the “color” and its rate of change over the manifold, such as the mean Cartan torsion \mathbf{I} , the mean Landsberg curvature \mathbf{J} and the S-curvature \mathbf{S} (see [17] or Section 2 below). These quantities interact with the flag curvature in a delicate way. The mean Landsberg curvature and the S-curvature reveal different non-Riemannian properties. For examples, there is a family of Finsler metrics on S^3 with $\mathbf{K} = 1$ and $\mathbf{S} = 0$ [5]. However, every local Finsler metric with $\mathbf{K} = 1$ and $\mathbf{J} = 0$ must be Riemannian (Theorem 9.1.1 in [17]).

An n -dimensional Finsler metric is said to have *constant S-curvature* if $\mathbf{S} = (n+1)cF$ for some constant c . It is known that every Randers metric of constant flag curvature has constant S-curvature [3], [4]. This is one of our motivations to consider Finsler metrics of constant S-curvature. In this paper, we are going to prove the following global metric rigidity theorem.

Theorem 1.1 *Let (M, F) be an n -dimensional compact boundaryless Finsler manifold with constant S-curvature, i.e., $\mathbf{S} = (n+1)cF$ for some constant c .*

- (a) If F has negative flag curvature, $\mathbf{K} < 0$, then it must be Riemannian;
- (b) If F has nonpositive flag curvature, $\mathbf{K} \leq 0$, then the mean Landsberg curvature vanishes, $\mathbf{J} = 0$, and the flag curvature $\mathbf{K}(P, y) = 0$ for the flags $P = \text{span}\{y, \mathbf{I}_y\} \subset T_x M$ whenever $\mathbf{I}_y \neq 0$.

The compactness in Theorem 1.1 (a) can not be dropped. Consider the following family of Finsler metrics on the unit ball $B^n \subset \mathbb{R}^n$,

$$F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y \rangle)^2} + \langle x, y \rangle}{1 - |x|^2} + \frac{\langle a, y \rangle}{1 + \langle a, x \rangle}, \quad (1)$$

where $y \in T_x B^n \cong \mathbb{R}^n$ and $a \in \mathbb{R}^n$ is an arbitrary constant vector with $|a| < 1$. It is proved that F has constant flag curvature $\mathbf{K} = -\frac{1}{4}$ and constant S-curvature $\mathbf{S} = \frac{1}{2}(n+1)F$ (see [17][18]). Clearly, F is not Riemannian.

The compactness in Theorem 1.1 (b) can not be dropped. Let $n \geq 2$ and

$$\mathcal{U} := \left\{ p = (s, t, \bar{p}) \in \mathbb{R}^2 \times \mathbb{R}^{n-2} \mid s^2 + t^2 < 1 \right\}.$$

Define

$$F := \frac{\sqrt{\left(-tu + sv\right)^2 + |y|^2\left(1 - s^2 - t^2\right)} - \left(-tu + sv\right)}{1 - s^2 - t^2},$$

where $y = (u, v, \bar{y}) \in T_p \mathcal{U} \cong \mathbb{R}^n$ and $p = (s, t, \bar{p}) \in \mathcal{U}$. F is an incomplete Finsler metric on Ω with $\mathbf{K} = 0$ and $\mathbf{S} = 0$, but $\mathbf{J} \neq 0$ [20].

The compactness condition in Theorem 1.1 can be replaced by a completeness condition together with certain growth condition on the mean Cartan torsion. See Theorems 4.1 and 4.2 below.

Corollary 1.2 *Let (M, F) be a compact boundaryless Berwald manifold with nonpositive flag curvature. Then the following hold,*

- (a) If F has negative flag curvature, $\mathbf{K} < 0$, then it must be Riemannian;
- (b) If F has nonpositive flag curvature, $\mathbf{K} \leq 0$, then $\mathbf{K}(P, y) = 0$ for the flag $P = \text{span}\{y, \mathbf{I}_y\}$ whenever $\mathbf{I}_y \neq 0$.

In dimension two, we have the following

Corollary 1.3 *Let (M, F) be a compact boundaryless Finsler surface. Suppose that $\mathbf{K} \leq 0$ and $\mathbf{S} = 3cF$ for some constant c , then F is either locally Minkowskian or Riemannian.*

The proof is simple. First, by Theorem 4.1 below, we know that $\mathbf{J} = 0$, then the theorem follows from Theorem 7.3.2 in [2].

In dimension $n \geq 3$, we have some non-trivial examples satisfying the conditions and conclusions in Theorem 1.1 (b). Let (N, h) be an arbitrary closed hyperbolic Riemannian manifold. For any $\epsilon \geq 0$, let

$$F_\epsilon := \sqrt{h^2(\bar{x}, \bar{y}) + w^2 + \epsilon \sqrt{h^4(\bar{x}, \bar{y}) + w^4}},$$

where $x = (\bar{x}, s) \in M$ and $y = \bar{y} \oplus w \frac{\partial}{\partial s} \in T_x M$. This family of Finsler metrics is constructed by Z.I. Szabó in his classification of Berwald metrics [21]. It is known that each F_ϵ is a Berwald metric. Thus $\mathbf{J} = 0$ and $\mathbf{S} = 0$ [17]. Further it can be shown that F_ϵ satisfies that $\mathbf{K} \leq 0$ and $\mathbf{K}(P, y) = 0$ for $P = \text{span}\{y, \mathbf{I}_y\}$. The proof will be given in Section 5 below. A natural problem arises: Is the Finsler metric in Theorem 1.1 (b) a Berwald metric? This problem remains open.

Finally, we should point out that there are already several global rigidity results on the metric structure of Finsler manifolds with $\mathbf{K} \leq 0$. For example, H. Akbar-Zadeh proves that every closed Finsler manifold with $\mathbf{K} = -1$ must be Riemannian and every closed Finsler manifold with $\mathbf{K} = 0$ must be locally Minkowskian [1]. Mo-Shen prove that every closed Finsler manifold of scalar curvature with $\mathbf{K} < 0$ must be of Randers type in dimension ≥ 3 [15]. Here a Finsler metric F is said to be of *scalar curvature* if the flag curvature $\mathbf{K} = \mathbf{K}(x, y)$ is independent of P for any given direction $y \in T_x M$. Riemannian metrics of scalar curvature must have isotropic sectional curvature $\mathbf{K} = \mathbf{K}(x)$, hence they have constant sectional curvature in dimension $n \geq 3$ by the Schur Lemma. But there are lots of Finsler metrics of scalar curvature which have not been completely classified yet.

2 Preliminaries

In this section, we are going to give a brief description on the flag curvature and the above mentioned non-Riemannian quantities.

Let M be an n -dimensional manifold and let $\pi : TM_o := TM \setminus \{0\} \rightarrow M$ denote the slit tangent bundle. The pull-back tangent bundle is defined by $\pi^*TM := \{(x, y, v) \mid 0 \neq y, v \in T_x M\}$ and the pull-back cotangent bundle is defined by $\pi^*T^*M := \{\pi^*\theta \mid \theta \in T^*M\}$.

By definition, a Finsler metric F on a manifold M is a nonnegative function on TM which is positively y -homogeneous of degree one with positive definite fundamental tensor $\mathbf{g} := g_{ij} dx^i \otimes dx^j$ on π^*TM , where $g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}(x, y)$. A special class of Finsler metrics are Randers metrics in the form $F = \alpha + \beta$ where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form with $\|\beta\|_x := \sqrt{a^{ij}(x)b_i(x)b_j(x)} < 1$ for any $x \in M$.

For a Finsler metric F , the volume $dV = \sigma_F(x) dx^1 \cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\text{Vol}(\mathbf{B}^n(1))}{\text{Vol}\left\{(y^i) \in \mathbf{R}^n \mid F\left(x, y^i \frac{\partial}{\partial x^i} \Big|_x\right) < 1\right\}}. \quad (2)$$

When $F = \sqrt{g_{ij}(x)y^i y^j}$ is Riemannian, then $\sigma_F(x) = \sqrt{\det(g_{ij}(x))}$. In general, the following quantity is not equal to zero,

$$\tau(x, y) := \ln \left[\frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma_F(x)} \right].$$

$\tau = \tau(x, y)$ is a scalar function on TM_o , which is called the *distortion* [17]. The distortion is our primary non-Riemannian quantity. Let

$$I_i := \frac{\partial \tau}{\partial y^i}(x, y) = \frac{1}{2} g^{jk}(x, y) \frac{\partial g_{jk}}{\partial y^i}(x, y). \quad (3)$$

We have

$$I_i y^i = 0. \quad (4)$$

The tensor $\mathbf{I} := I_i dx^i$ on TM_o is called the *mean Cartan tensor*. According to Deicke's theorem [11], F is Riemannian if and only if $\mathbf{I} = 0$. Define the norm of \mathbf{I} at a point $x \in M$ by

$$\|\mathbf{I}\|_x := \sup_{0 \neq y \in T_x M} \sqrt{I_i(x, y) g^{ij}(x, y) I_j(x, y)}.$$

For a point $p \in M$, let

$$\mathcal{I}_p(r) := \sup_{\min(d(p, x), d(x, p)) < r} \|\mathbf{I}\|_x.$$

The mean Cartan tensor \mathbf{I} is said to grow sub-linearly if for any point $p \in M$,

$$\mathcal{I}_p(r) = o(r), \quad (r \rightarrow +\infty).$$

\mathbf{I} is said to grow *sub-exponentially* at rate of $k = 1$ if for any point $p \in M$,

$$\mathcal{I}_p(r) = o(e^r), \quad (r \rightarrow +\infty).$$

It is known that for a Randers metric $F = \alpha + \beta$, \mathbf{I} is bounded, i.e.,

$$\|\mathbf{I}\|_x \leq \frac{n+1}{\sqrt{2}} \sqrt{1 - \sqrt{1 - \|\beta\|_x^2}} < \frac{n+1}{\sqrt{2}}, \quad x \in M.$$

The bound in dimension two is suggested by B. Lackey. See Proposition 7.1.2 in [17] for a proof.

The geodesics in a Finsler manifold are characterized by a system of second order ordinary differential equations

$$\ddot{\sigma}^i + 2G^i(\sigma, \dot{\sigma}) = 0,$$

where $G^i = G^i(x, y)$ are positively y -homogeneous functions of degree two. When F is Riemannian, $G^i = \frac{1}{2} \Gamma_{jk}^i(x) y^j y^k$ are quadratic in $y \in T_x M$. A Finsler metric with such a property called a *Berwald metric*. There are many non-Riemannian Berwald manifolds (see Section 5 below).

For a non-zero vector $y \in T_x M$, set

$$\mathbf{S}(x, y) := \frac{d}{dt} \left[\tau \left(\sigma(t), \dot{\sigma}(t) \right) \right] \Big|_{t=0},$$

where $\sigma = \sigma(t)$ is the geodesic with $\sigma(0) = x$ and $\dot{\sigma}(0) = y$. $\mathbf{S} = \mathbf{S}(x, y)$ is a scalar function on TM_o which is called the *S-curvature* [16][17]. Let $dV = \sigma_F(x) dx^1 \cdots dx^n$ be the volume form on M . The S-curvature can be expressed by

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m}(x, y) - y^m \frac{\partial}{\partial x^m} \left[\ln \sigma_F(x) \right]. \quad (5)$$

It is proved that $\mathbf{S} = 0$ for Berwald metrics [16][17]. An n -dimensional Finsler metric F is said to *have constant S-curvature* if there is a constant c such that $\mathbf{S} = (n+1)cF$. It is known that all Randers metrics of constant flag curvature must have constant S-curvature [3] (see [4] for the classification of such metrics).

There is a distinguished linear connection ∇ on π^*TM which is called the *Chern connection* [10]. Let $\{\mathbf{e}_i\}$ be a local frame for π^*TM and $\{\omega^i\}$ the dual local frame for π^*T^*M . ∇ can be expressed by

$$\nabla V = \left\{ dV^i + V^j \omega_j^i \right\} \otimes \mathbf{e}_i,$$

where $V = V^i \mathbf{e}_i \in C^\infty(\pi^*TM)$. The Chern connection can be viewed as a generalization of the Levi-Civita connection in Riemannian geometry. Let

$$\omega^{n+i} := dy^i + y^j \omega_j^i,$$

where y^i are local functions on TM_o defined by the canonical section $\mathbf{Y} = y^i \mathbf{e}_i$ of π^*TM . We obtain a local coframe $\{\omega^i, \omega^{n+i}\}$ for $T^*(TM_o)$.

Let

$$\Omega^i := d\omega^{n+i} - \omega^{n+j} \wedge \omega_j^i.$$

Ω^i can be expressed as follows,

$$\Omega^i = \frac{1}{2} R^i_{kl} \omega^k \wedge \omega^l - L^i_{kl} \omega^k \wedge \omega^{n+l},$$

where $R^i_{kl} + R^i_{lk} = 0$ and $L^i_{kl} = L^i_{lk}$. The anti-symmetric tensor $\mathbf{R} = R^i_{kl} \mathbf{e}_i \otimes \omega^k \otimes \omega^l$ is called the *Riemann tensor* and the symmetric tensor $\mathbf{L} = L^i_{kl} \mathbf{e}_i \otimes \omega^k \otimes \omega^l$ is called the Landsberg tensor.

Let

$$R^i_k := R^i_{kl} y^l, \quad R_{jk} := g_{ij} R^i_k.$$

We have

$$R^i_k y^k = 0, \quad R_{jk} = R_{kj}. \quad (6)$$

See [17] for details. The tensor $\mathbf{R} := R^i_k \mathbf{e}_i \otimes \omega^k$ is still called the *Riemann tensor*. The notion of Riemann (curvature) tensor for general Finsler metrics is introduced by L. Berwald using the Berwald connection [6][7]. Let

$$J_k := L^m_{km}.$$

The tensor $\mathbf{J} := J_i \omega^i$ is called the *mean Landsberg tensor*. For a Berwald metric, $\mathbf{J} = 0$ [17].

For a scalar function on TM_o , say τ , we define its covariant derivatives by

$$d\tau = \tau_{|k} \omega^k + \tau_{.k} \omega^{n+k}.$$

From (3), we have

$$\tau_{.i} = \frac{\partial \tau}{\partial y^i} = I_i.$$

We have

$$\mathbf{S} := \tau_{|m} y^m.$$

For a tensor, say, $\mathbf{I} = I_i \omega^i$, the covariant derivatives are defined in a canonical way by

$$dI_i - I_k \omega_i^k = I_{i|k} \omega^k + I_{i.k} \omega^{n+k}.$$

We have

$$J_i = I_{i|m} y^m \quad (7)$$

Hence

$$J_i y^i = 0. \quad (8)$$

See [17] for details.

Now we interpret the above geometric quantities in a different way.

Let F be a Finsler metric on an n -dimensional manifold M . For a non-zero tangent vector $y = y^i \frac{\partial}{\partial x^i} |_x \in T_x M$, define

$$\mathbf{g}_y(u, v) := g_{ij}(x, y) u^i v^j, \quad u = u^i \frac{\partial}{\partial x^i} |_x, v = v^j \frac{\partial}{\partial x^j} |_x \in T_x M,$$

where $g_{ij}(x, y) = \frac{1}{2}[F^2]_{y^i y^j}(x, y)$. Each \mathbf{g}_y is an inner product on the tangent space $T_x M$.

The Riemann tensor can be viewed as a family of endomorphisms on tangent spaces.

$$\mathbf{R}_y(u) := R^i_k(x, y) u^k \frac{\partial}{\partial x^i} |_x,$$

where $u = u^i \frac{\partial}{\partial x^i} |_x \in T_x M$. The coefficients $R^i_k = R^i_k(x, y)$ are given by

$$R^i_k = 2 \frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}. \quad (9)$$

It follows from (6) that

$$\mathbf{R}_y(y) = 0, \quad \mathbf{g}_y(\mathbf{R}_y(u), v) = \mathbf{g}_y(u, \mathbf{R}_y(v)), \quad (10)$$

where $u, v \in T_x M$. The family $\mathbf{R} := \{\mathbf{R}_y | y \in T_x M \setminus \{0\}\}$ is called the *Riemann curvature*.

Using the Chern connection ∇ on π^*TM , one can define the covariant derivative of a vector field $X = X^i(t)\frac{\partial}{\partial x^i}|_{c(t)}$ along a curve c by

$$D_{\dot{c}}X(t) := \left\{ \frac{dX^i}{dt}(t) + X^j(t)\Gamma_{jk}^i(c(t), \dot{c}(t))\dot{c}^k(t) \right\} \frac{\partial}{\partial x^i}|_{c(t)}.$$

If $H = H(s, t)$ is a family of geodesics, i.e., for each s , $\sigma_s(t) := H(s, t)$ is a geodesic, the variation field $V_s(t) := \frac{\partial H}{\partial s}(s, t)$ satisfies the following Jacobi field along σ_s ,

$$D_{\dot{\sigma}_s}D_{\dot{\sigma}_s}V_s(t) + \mathbf{R}_{\dot{\sigma}_s(t)}(V(t)) = 0.$$

For a tangent plane $P \subset T_xM$ and a vector $0 \neq y \in P$, let

$$\mathbf{K}(P, y) := \frac{\mathbf{g}_y(\mathbf{R}_y(u), u)}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - [\mathbf{g}_y(y, u)]^2},$$

where $P = \text{span}\{y, u\}$. By (10), one can see that $\mathbf{K}(P, y)$ is well-defined, namely, independent of the choice of a particular $u \in T_xM$.

The mean Cartan tensor and the mean Landsberg tensor can be viewed as families of vectors on the manifold, i.e.,

$$\mathbf{I}_y = I^i(x, y)\frac{\partial}{\partial x^i}|_x, \quad \mathbf{J}_y = J^i(x, y)\frac{\partial}{\partial x^i}|_x,$$

where $I^i := g^{il}I_l$ and $J^i := g^{il}J_l$. It follows from (4) and (8) that

$$\mathbf{g}_y(\mathbf{I}_y, y) = 0 = \mathbf{g}_y(\mathbf{J}_y, y).$$

Thus \mathbf{I}_y and \mathbf{J}_y are perpendicular to y with respect to \mathbf{g}_y . We call $\mathbf{I} := \{\mathbf{I}_y \mid y \in TM \setminus \{0\}\}$ and $\mathbf{J} := \{\mathbf{J}_y \mid y \in TM \setminus \{0\}\}$ the *mean Cartan torsion* and the *Landsberg curvature*, respectively.

If F is a Berwald metric, then $\mathbf{J} = 0$ and $\mathbf{S} = 0$. The converse is true too in dimension two, but it is not clear in higher dimensions (Cf. [17]).

3 Finsler metrics with constant S-curvature

The following lemma is crucial for the proof of Theorem 1.1.

Lemma 3.1 *Let (M, F) be an n -dimensional Finsler manifold. Suppose that there is a constant c and a closed 1-form γ such that*

$$\mathbf{S}(x, y) = (n+1)cF(x, y) + \gamma_x(y), \quad y \in T_xM,$$

then along any geodesic $\sigma = \sigma(t)$, the vector field $\mathbf{I}(t) := I^i(\sigma(t), \dot{\sigma}(t))\frac{\partial}{\partial x^i}|_{\sigma(t)}$ satisfies the following equation:

$$D_{\dot{\sigma}}D_{\dot{\sigma}}\mathbf{I}(t) + \mathbf{R}_{\dot{\sigma}(t)}(\mathbf{I}(t)) = 0. \quad (11)$$

Proof: It is known that the Landsberg tensor satisfies the following equation [13] [15] :

$$J_{k|m}y^m + I_m R_k^m = -\frac{1}{3}\{2R_{k\cdot m}^m + R_{m\cdot k}^m\} \quad (12)$$

and the S-curvature satisfies the following equation [8] [14]:

$$\mathbf{S}_{\cdot k|m}y^m - \mathbf{S}_{|k} = -\frac{1}{3}\{2R_{k\cdot m}^m + R_{m\cdot k}^m\}. \quad (13)$$

It follows from (12) and (13) that

$$J_{k|m}y^m + I_m R_k^m = \mathbf{S}_{\cdot k|m}y^m - \mathbf{S}_{|k}.$$

By (7), we can rewrite the above equation as follows

$$I^i_{|p|q}y^p y^q + R^i_m I^m = g^{ik}\{\mathbf{S}_{\cdot k|m}y^m - \mathbf{S}_{|k}\}. \quad (14)$$

Note that $F = \sqrt{g_{ij}y^i y^j}$ satisfies

$$F_{|m} = \frac{g_{ij|m}y^i y^j}{2F} = 0, \quad F_{\cdot k|m} = \frac{g_{ik|m}y^i}{F} = 0.$$

Since $\gamma = \gamma_i dx^i$ is closed, it satisfies

$$\gamma_{\cdot k|m}y^m - \gamma_{|k} = \left\{ \frac{\partial \gamma_k}{\partial x^m} - \frac{\partial \gamma_m}{\partial x^k} \right\} y^m = 0.$$

We have

$$\mathbf{S}_{\cdot k|m}y^m - \mathbf{S}_{|k} = (n+1)c\{F_{\cdot k|m}y^m - F_{|k}\} + \gamma_{\cdot k|m}y^m - \gamma_{|k} = 0.$$

Then (14) is reduced to

$$I^i_{|p|q}y^p y^q + R^i_m I^m = 0. \quad (15)$$

Since σ is a geodesic, we have

$$D_{\dot{\sigma}} D_{\dot{\sigma}} \mathbf{I}(t) = I^i_{|p|q}(\sigma(t), \dot{\sigma}(t)) \dot{\sigma}^p(t) \dot{\sigma}^q(t) \frac{\partial}{\partial x^i} |_{\sigma(t)}.$$

Then (15) restricted to $\sigma(t)$ gives rise to (11).

Q.E.D.

4 Proof of Theorem 1.1

In this section, we are going to prove a slightly more general version of Theorem 1.1.

Theorem 4.1 *Let (M, F) be an n -dimensional complete Finsler manifold with nonpositive flag curvature $\mathbf{K} \leq 0$ and almost constant S -curvature $\mathbf{S} = (n + 1)cF + \gamma$ ($c = \text{constant}$ and γ is a closed 1-form). Suppose that the mean Cartan torsion grows sub-linearly. Then $\mathbf{J} = 0$ and $\mathbf{K}(P, y) = 0$ for the flag $P = \text{span}\{\mathbf{I}_y, y\}$ whenever $\mathbf{I}_y \neq 0$. Moreover F is Riemannian at points where $\mathbf{K} < 0$.*

Proof. Let $y \in T_x M$ be an arbitrary non-zero vector and let $\sigma = \sigma(t)$ be the geodesic with $\sigma(0) = x$ and $\dot{\sigma}(0) = y$. Since the Finsler metric is complete, one may assume that σ is defined on $(-\infty, \infty)$. \mathbf{I} and \mathbf{J} restricted to σ are vector fields along σ ,

$$\mathbf{I}(t) := I^i(\sigma(t), \dot{\sigma}(t)) \frac{\partial}{\partial x^i} |_{\sigma(t)}, \quad \mathbf{J}(t) := J^i(\sigma(t), \dot{\sigma}(t)) \frac{\partial}{\partial x^i} |_{\sigma(t)}.$$

It follows from (7) that

$$D_{\dot{\sigma}} \mathbf{I}(t) = I^i |_{\dot{m}}(\sigma(t), \dot{\sigma}(t)) \dot{\sigma}^m(t) \frac{\partial}{\partial x^i} |_{\sigma(t)} = \mathbf{J}(t). \quad (16)$$

If $\mathbf{I}(t) \equiv 0$, then by (16), $\mathbf{J}_y = D_{\dot{\sigma}} \mathbf{I}(0) = 0$. From now on, we assume that $\mathbf{I}(t) \not\equiv 0$. Let

$$\varphi(t) := \sqrt{\mathbf{g}_{\dot{\sigma}(t)}(\mathbf{I}(t), \mathbf{I}(t))}. \quad (17)$$

Let $I = (a, b) \neq \emptyset$ be a maximal interval on which $\varphi(t) > 0$. We have

$$\varphi \varphi' = \mathbf{g}_{\dot{\sigma}}(\mathbf{I}, D_{\dot{\sigma}} \mathbf{I}) \leq \sqrt{\mathbf{g}_{\dot{\sigma}}(\mathbf{I}, \mathbf{I})} \sqrt{\mathbf{g}_{\dot{\sigma}}(D_{\dot{\sigma}} \mathbf{I}, D_{\dot{\sigma}} \mathbf{I})} = \varphi \sqrt{\mathbf{g}_{\dot{\sigma}}(D_{\dot{\sigma}} \mathbf{I}, D_{\dot{\sigma}} \mathbf{I})}.$$

This is,

$$\varphi' \leq \sqrt{\mathbf{g}_{\dot{\sigma}}(D_{\dot{\sigma}} \mathbf{I}, D_{\dot{\sigma}} \mathbf{I})}. \quad (18)$$

By assumption $\mathbf{K} \leq 0$ and (18), we have

$$\begin{aligned} \frac{1}{2}[\varphi^2]'' &= \mathbf{g}_{\dot{\sigma}}(D_{\dot{\sigma}} D_{\dot{\sigma}} \mathbf{I}, \mathbf{I}) + \mathbf{g}_{\dot{\sigma}}(D_{\dot{\sigma}} \mathbf{I}, D_{\dot{\sigma}} \mathbf{I}) \\ &= -\mathbf{g}_{\dot{\sigma}}(\mathbf{R}_{\dot{\sigma}}(\mathbf{I}), \mathbf{I}) + \mathbf{g}_{\dot{\sigma}}(D_{\dot{\sigma}} \mathbf{I}, D_{\dot{\sigma}} \mathbf{I}) \\ &\geq \mathbf{g}_{\dot{\sigma}}(D_{\dot{\sigma}} \mathbf{I}, D_{\dot{\sigma}} \mathbf{I}) \geq \varphi'^2. \end{aligned} \quad (19)$$

We obtain that $\varphi''(t) \geq 0$.

We claim that $\varphi'(t) \equiv 0$. Suppose that $\varphi'(t_o) \neq 0$ for some $t_o \in I$. If $\varphi'(t_o) > 0$, then

$$\varphi(t) \geq \varphi'(t_o)(t - t_o) + \varphi(t_o), \quad t > t_o.$$

Thus $b = +\infty$. If $\varphi'(t_o) < 0$, then

$$\varphi(t) \geq \varphi'(t_o)(t - t_o) + \varphi(t_o) > \varphi(t_o) > 0, \quad t < t_o.$$

Thus $a = -\infty$. In either case, $\varphi(t)$ grows at least linearly. Note that for $p = \sigma(t_o)$, $\mathcal{I}_p(|t-t_o|) \geq \varphi(t)$. We see that \mathbf{I} grows at least linearly. This is impossible. Thus $\varphi'(t) \equiv 0$ and $\varphi(t) = \text{constant} > 0$. In this case, $I = (-\infty, \infty)$.

It follows from (19) that

$$\mathbf{g}_{\dot{\sigma}}(\mathbf{R}_{\dot{\sigma}}(\mathbf{I}), \mathbf{I}) = 0, \quad D_{\dot{\sigma}}\mathbf{I} = 0.$$

By (16), we get $\mathbf{J}_y = D_{\dot{\sigma}}\mathbf{I}(0) = 0$. Since \mathbf{I}_y is orthogonal to y with respect to \mathbf{g}_y , $\mathbf{K}(P, y) = 0$ for $P = \text{span}\{\mathbf{I}_y, y\}$ whenever $\mathbf{I}_y \neq 0$.

Assume that $\mathbf{K} < 0$ at a point $x \in M$. It follows from $\mathbf{g}_y(\mathbf{R}_y(\mathbf{I}_y), \mathbf{I}_y) = 0$ that $\mathbf{I}_y = 0$ for all $y \in T_x M \setminus \{0\}$. By Deicke's theorem [11], F is Riemannian. Q.E.D.

Two natural problems arise:

- (a) Is there any *complete* non-Landsberg metric on \mathbb{R}^n with $\mathbf{K} \leq 0$, $\mathbf{S} = (n+1)cF$ and $\mathcal{I}_p(r) \sim Cr$ (as $r \rightarrow +\infty$)?
- (b) What is the metric structure of a complete Finsler metric on \mathbb{R}^n ($n \geq 3$) satisfying $\mathbf{K} = 0$, $\mathbf{J} = 0$ and $\mathbf{S} = 0$?

If the flag curvature is strictly negative, we have the following

Theorem 4.2 *Let (M, F) be an n -dimensional complete Finsler manifold with $\mathbf{K} \leq -1$ and almost constant S -curvature. Suppose that the mean Cartan torsion \mathbf{I} grows sub-exponentially a rate of $k = 1$. Then F is Riemannian.*

Proof: The proof is similar. Assume that $\mathbf{I}_y \neq 0$ for some non-zero vector $y \in T_x M$. Let $y \in T_x M$ be an arbitrary vector and $\sigma = \sigma(t)$ be the geodesic with $\sigma(0) = x$ and $\dot{\sigma}(0) = y$. Let $\varphi(t)$ be defined by (17). Let $I = (a, b) \neq \emptyset$ be the maximal interval on which $\varphi(t) > 0$ and $0 \in I$. By assumption $\mathbf{K} \leq -1$ and (18), we obtain

$$\begin{aligned} \frac{1}{2}[\varphi^2]'' &= \mathbf{g}_{\dot{\sigma}}(D_{\dot{\sigma}}D_{\dot{\sigma}}\mathbf{I}, \mathbf{I}) + \mathbf{g}_{\dot{\sigma}}(D_{\dot{\sigma}}\mathbf{I}, D_{\dot{\sigma}}\mathbf{I}) \\ &= -\mathbf{g}_{\dot{\sigma}}(\mathbf{R}_{\dot{\sigma}}(\mathbf{I}), \mathbf{I}) + \mathbf{g}_{\dot{\sigma}}(D_{\dot{\sigma}}\mathbf{I}, D_{\dot{\sigma}}\mathbf{I}) \\ &\geq \varphi^2 + \varphi'^2. \end{aligned}$$

This gives rise to the following inequality

$$\varphi'' - \varphi \geq 0. \tag{20}$$

We claim that $\varphi'(t) \equiv 0$. Suppose that $\varphi'(t_o) \neq 0$ for some $t_o \in I$. Let

$$\varphi_o(t) := \varphi(t_o) \cosh(t - t_o) + \varphi'(t_o) \sinh(t - t_o).$$

Let $h(t) := \varphi'(t)/\varphi(t)$ and $h_o(t) := \varphi'_o(t)/\varphi_o(t)$.

$$\chi(t) := e^{\int_{t_o}^t (h(\tau) + h_o(\tau)) d\tau} [h(t) - h_o(t)].$$

It is easy to verify that $\chi'(t) \geq 0$ and $\chi(t_o) = 0$. Thus $\chi(t) \geq 0$ for $t > t_o$ and $\chi(t) \leq 0$ for $t < t_o$. This implies that $h(t) \geq h_o(t)$ for $t > t_o$ and $h(t) \leq h_o(t)$ for $t < t_o$. Note that

$$h(t) - h_o(t) = \frac{\varphi'(t)}{\varphi(t)} - \frac{\varphi'_o(t)}{\varphi_o(t)} = \frac{d}{dt} \left[\ln \frac{\varphi(t)}{\varphi_o(t)} \right].$$

Thus $[\varphi/\varphi_o]'(t) \geq 0$ for $t > t_o$ and $[\varphi/\varphi_o]'(t) \leq 0$ for $t < t_o$. We conclude that

$$\varphi(t) \geq \varphi_o(t), \quad a < t < b.$$

If $\varphi'(t_o) > 0$, then

$$\varphi(t) \geq \varphi_o(t) > 0, \quad t > t_o.$$

Thus $b = +\infty$ and

$$\liminf_{t \rightarrow +\infty} \frac{\varphi(t)}{e^{t-t_o}} \geq \frac{\varphi(t_o) + \varphi'(t_o)}{2} > 0.$$

If $\varphi'(t_o) < 0$, then

$$\varphi(t) \geq \varphi_o(t) > 0, \quad t < t_o.$$

Thus $a = -\infty$ and

$$\liminf_{t \rightarrow -\infty} \frac{\varphi(t)}{e^{t-t_o}} \geq \frac{\varphi(t_o) - \varphi'(t_o)}{2} > 0.$$

Note that $\mathcal{I}_p(|t - t_o|) \geq \varphi(t)$ for $p = \sigma(t_o)$. Thus \mathbf{I} grows at least exponentially at rate of $k = 1$. This contradicts the assumption. Therefore, $\varphi'(t) \equiv 0$.

Since $\varphi'(t) \equiv 0$, we conclude that $\varphi(t) \equiv 0$ by (20). In particular, $\mathbf{I}_y = \varphi(0) = 0$. This contradicts the assumption at the beginning of the argument.

Therefore $\mathbf{I} \equiv 0$ and F is Riemannian by Deicke's theorem [11]. Q.E.D.

A natural problem arises: Is there any non-Riemannian *complete* Finsler metric on \mathbb{R}^n satisfying $\mathbf{K} \leq -1$, $\mathbf{S} = (n+1)cF$, but $\mathcal{I}_p(r) \sim Ce^r$ (as $r \rightarrow +\infty$)? This problems remains open.

Example 4.3 Let $\phi = \phi(y)$ be a Minkowski norm on \mathbb{R}^n and $\mathcal{U} := \{y \in \mathbb{R}^n \mid \phi(y) < 1\}$. Let $\Theta = \Theta(x, y)$ be a function on $T\mathcal{U} \cong \mathcal{U} \times \mathbb{R}^n$ defined by

$$\Theta(x, y) = \phi(y - \Theta(x, y)x).$$

Θ is a Finsler metric on \mathcal{U} which is called the *Funk metric* [12]. The Funk metric satisfies the following important equation

$$\Theta_{x^k}(x, y) = \Theta(x, y)\Theta_{y^k}(x, y). \quad (21)$$

Let $a \in \mathbb{R}^n$ be an arbitrary constant vector $a \in \mathbb{R}^n$ with $|a| < 1$. Let

$$F := \Theta(x, y) + \frac{\langle a, y \rangle}{1 + \langle a, x \rangle}, \quad y \in T\mathcal{U} \cong \mathcal{U} \times \mathbb{R}^n.$$

Clearly, F is a Finsler metric near the origin. By (21), one sees that the spray coefficients of F are given by $G^i = Py^i$, where

$$P := \frac{1}{2} \left\{ \Theta(x, y) - \frac{\langle a, y \rangle}{1 + \langle a, x \rangle} \right\}.$$

Then using the above formula for G^i and (9), one can easily show that F has constant flag curvature $\mathbf{K} = -\frac{1}{4}$ (see Example 5.3 in [19]). Now let us compute the S-curvature of F . A direct computation gives

$$\frac{\partial G^m}{\partial y^m} = (n+1)P.$$

Let $dV = \sigma_F(x) dx^1 \cdots dx^n$ be the Finsler volume form on M . Using (5), we obtain

$$\begin{aligned} \mathbf{S} &= (n+1)P(x, y) - y^m \frac{\partial}{\partial x^m} \left(\ln \sigma_F(x) \right) \\ &= \frac{n+1}{2} F(x, y) - (n+1) \frac{\langle a, y \rangle}{1 + \langle a, x \rangle} - y^m \frac{\partial}{\partial x^m} \left(\ln \sigma_F(x) \right) \\ &= \frac{1}{2} (n+1) F(x, y) + d\varphi_x(y), \end{aligned}$$

where

$$\varphi := -\ln \left[(1 + \langle a, x \rangle)^{n+1} \sigma_F(x)^{\frac{1}{n+1}} \right]. \quad (22)$$

Thus F has almost constant S-curvature. Note that F is not Riemannian in general.

Example 4.3 shows that the completeness in Theorem 4.1 can not be replaced by the positive completeness.

5 An Example

The local/global structures of Berwald metrics have been completely determined by Z.I. Szabo [21], but their curvature properties have not been discussed thoroughly. Here we are going to compute the Riemann curvature and the mean Cartan torsion for a special class of Berwald manifolds constructed from a pair of Riemannian manifolds. Then we show that these metrics satisfy the conditions and the conclusions in Theorem 1.1 (b).

Let (M_i, α_i) , $i = 1, 2$, be arbitrary Riemannian manifolds and $M = M_1 \times M_2$. Let $f : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ be an arbitrary C^∞ function satisfying

$$f(\lambda s, \lambda t) = \lambda f(s, t), \quad (\lambda > 0) \quad \text{and} \quad f(s, t) \neq 0 \text{ if } (s, t) \neq 0.$$

Define

$$F := \sqrt{f\left([\alpha_1(x_1, y_1)]^2, [\alpha_2(x_2, y_2)]^2\right)}, \quad (23)$$

where $x = (x_1, x_2) \in M$ and $y = y_1 \oplus y_2 \in T_{(x_1, x_2)}(M_1 \times M_2) \cong T_{x_1}M_1 \oplus T_{x_2}M_2$. Clearly, F has the following properties:

- (a) $F(x, y) \geq 0$ with equality holds if and only if $y = 0$;
- (b) $F(x, \lambda y) = \lambda F(x, y)$, $\lambda > 0$;
- (c) $F(x, y)$ is C^∞ on $TM \setminus \{0\}$.

Now we are going to find additional condition on $f = f(s, t)$ under which the matrix $g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}$ is positive definite. Take standard local coordinate systems (x^a, y^a) in TM_1 and (x^α, y^α) in TM_2 . Then $(x^i, y^j) := (x^a, x^\alpha, y^a, y^\alpha)$ is a standard local coordinate system in TM . Express

$$\alpha_1(x_1, y_1) = \sqrt{\bar{g}_{ab}(x_1)y^a y^b}, \quad \alpha_2(x_2, y_2) = \sqrt{\bar{g}_{\alpha\beta}(x_2)y^\alpha y^\beta},$$

where $y_1 = y^a \frac{\partial}{\partial x^a}$ and $y_2 = y^\alpha \frac{\partial}{\partial x^\alpha}$. We obtain

$$(g_{ij}) = \begin{pmatrix} 2f_{ss}\bar{y}_a\bar{y}_b + f_s\bar{g}_{ab} & 2f_{st}\bar{y}_a\bar{y}_\beta \\ 2f_{st}\bar{y}_\alpha\bar{y}_\beta + f_t\bar{g}_{\alpha\beta} & \end{pmatrix}, \quad (24)$$

where $\bar{y}_a := \bar{g}_{ab}y^b$ and $\bar{y}_\alpha := \bar{g}_{\alpha\beta}y^\beta$. By an elementary argument, one can show that (g_{ij}) is positive definite if and only if f satisfies the following conditions:

$$f_s > 0, \quad f_t > 0, \quad f_s + 2sf_{ss} > 0, \quad f_t + 2tf_{tt} > 0,$$

and

$$f_s f_t - 2f f_{st} > 0.$$

In this case,

$$\det(g_{ij}) = h([\alpha_1]^2, [\alpha_2]^2) \det(\bar{g}_{ab}) \det(\bar{g}_{\alpha\beta}), \quad (25)$$

where

$$h := (f_s)^{n_1-1} (f_t)^{n_2-1} \{f_s f_t - 2f f_{st}\},$$

where $n_1 := \dim M_1$ and $n_2 := \dim M_2$.

By a direct computation, one knows that the spray coefficients of F are splitted as the direct sum of the spray coefficients of α_1 and α_2 , that is,

$$G^a(x, y) = \bar{G}^a(x_1, y_1), \quad G^\alpha(x, y) = \bar{G}^\alpha(x_2, y_2), \quad (26)$$

where \bar{G}^a and \bar{G}^α are the spray coefficients of α_1 and α_2 respectively. From (26), one can see that the spray of F is independent of the choice of a particular function f . In particular, G^i are quadratic in $y \in T_x M$. Thus F is a Berwald

metric. This fact is claimed in [21]. Since F is a Berwald metric, $\mathbf{J} = 0$ and $\mathbf{S} = 0$ [17].

The Riemann tensor of F is given by

$$\left(R^i_j\right) = \left(\bar{R}^i_j\right) = \begin{pmatrix} \bar{R}^a_b & 0 \\ 0 & \bar{R}^{\alpha\beta} \end{pmatrix},$$

where \bar{R}^a_b and $\bar{R}^{\alpha\beta}$ are the coefficients of the Riemann tensor of α_1 and α_2 respectively. Let $\bar{R}_{ij} := g_{ik}\bar{R}^k_j$, $\bar{R}_{ab} := \bar{g}_{ac}\bar{R}^c_b$ and $\bar{R}_{\alpha\beta} := \bar{g}_{\alpha\gamma}\bar{R}^{\gamma\beta}$. Using (24), one obtains

$$\left(R_{ij}\right) = \begin{pmatrix} f_s\bar{R}_{ab} & 0 \\ 0 & f_t\bar{R}_{\alpha\beta} \end{pmatrix}.$$

For any vector $v = v^i \frac{\partial}{\partial x^i}|_x \in T_x M$,

$$\mathbf{g}_y\left(\mathbf{R}_y(v), v\right) = f_s\bar{R}_{ab}v^av^b + f_t\bar{R}_{\alpha\beta}v^\alpha v^\beta. \quad (27)$$

It follows from (27) that if α_1 and α_2 both have nonpositive sectional curvature, then F has nonpositive flag curvature.

Using (25), one can compute the mean Cartan torsion. First, observe that

$$I_i = \frac{\partial}{\partial y^i} \left[\ln \sqrt{\det(g_{jk})} \right] = \frac{\partial}{\partial y^i} \left[\ln \sqrt{h([\alpha_1]^2, [\alpha_2]^2)} \right].$$

One obtains

$$I_a = \frac{h_s}{h}\bar{y}_a \quad I_\alpha = \frac{h_t}{h}\bar{y}_\alpha,$$

where $\bar{y}_a := \bar{g}_{ab}y^b$ and $\bar{y}_\alpha := \bar{g}_{\alpha\beta}y^\beta$. Since $\bar{y}_a\bar{R}^a_b = 0$ and $\bar{y}_\alpha\bar{R}^{\alpha\beta} = 0$, one obtains

$$\mathbf{g}_y\left(\mathbf{R}_y(\mathbf{I}_y), \mathbf{I}_y\right) = I_i R^i_j I^j = \frac{h_s}{h}\bar{y}_a\bar{R}^a_b I^b + \frac{h_t}{h}\bar{y}_\alpha\bar{R}^{\alpha\beta} I^\beta = 0.$$

Therefore F satisfies the conditions and conclusions in Theorem 4.1.

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