Projectively Flat Finsler Metrics with Almost Isotropic S-Curvature

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1 Introduction

One century ago, Hilbert announced his famous 23 problems. Hilbert's fourth problem in the regular case is to characterize projectively flat Finsler metrics. G. Hamel [11] first found a simple system of PDE's to characterize projectively flat Finsler metrics on a convex open subset in R^n (see (9) below). Later on, L. Berwald and P. Funk studied projectively flat Finsler metrics of constant flag curvature [4], [9], [10].

The flag curvature $\mathbf{K} = \mathbf{K}(P, y)$ is an analogue of the sectional curvature in Riemannian geometry, which is a function of a two-dimensional plane $P \subset T_xM$ and a non-zero vector $y \in P$. It is known that any locally projectively flat Finsler metric is of scalar curvature, namely, $\mathbf{K} = \mathbf{K}(x, y)$ is a scalar function of $y \in T_xM$ [4]. But there is no simple way to characterize the local structure of these metrics, except for the case when the flag curvature is constant [4], [9], [10], [18]. R. Bryant finds an elegant way to describe Finsler metrics of $\mathbf{K} = 1$ on S^n whose geodesics are great circles (hence, projectively flat) [5], [6], [7].

In [14], [15], [16], the second author introduces the notion of S-curvature. This quantity has been discussed in the recent study on Finsler metrics of scalar curvature. First, the examples constructed in [3], [19], [20], [2] are Randers metrics with constant flag curvature $\mathbf{K} = \lambda$ and constant S-curvature, $\mathbf{S} = (n+1)cF$. Then C. Robles shows that all Randers metrics of constant flag curvature have constant S-curvature. Inspired by these results, we eventually classify all Randers metrics of constant flag curvature [1].

In [8], we show that for an n-dimensional Finsler metric F of scalar curvature, if the S-curvature is almost isotropic, i.e.,

$$\mathbf{S} = (n+1)\{cF + \eta\}, \tag{1}$$

where $c = c(x)$ is a scalar function and $\eta = \eta(x, y)$ is a closed 1-form on M, then the flag curvature is given by

$$\mathbf{K} = 3\frac{c_{x^m}y^m}{F} + \sigma, \tag{2}$$

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where $\sigma = \sigma(x)$ is a scalar function on $\mathcal{U}$. There are lots of projectively flat Randers metrics with almost isotropic S-curvature [8]. Besides these Randers metrics, there is another special projectively flat Finsler metrics with almost isotropic S-curvature. Let $\Theta = \Theta(x, y)$ be a Finsler metric on an open subset $\mathcal{U} \subset \mathrm{R}^n$ satisfying

$$\Theta_{x^k} = \Theta\Theta_{y^k}. \tag{3}$$

Θ is called a *Funk metric*. Let

$$\Theta_a := \Theta(x, y) + \frac{\langle a, y\rangle}{1 + \langle a, x\rangle}, \qquad y \in T_x\mathcal{U} \cong \mathrm{R}^n, \tag{4}$$

where $a \in \mathrm{R}^n$ is a constant vector. Assume that Θ_a is still a Finsler metric on $\mathcal{U}$, then it is projectively flat with constant flag curvature $\mathbf{K} = -\frac{1}{4}$. Further, it has almost constant S-curvature

$$\mathbf{S} = (n + 1)\Big\{\frac{1}{2}\Theta_a + d\varphi\Big\},$$

where $\varphi = \varphi(x)$ is a scalar function on M [18].

It is a natural problem to to study and characterize Finsler metrics of scalar curvature with almost isotropic S-curvature.

Theorem 1.1 *Let $F = F(x, y)$ be a projectively flat Finsler metric on an open subset $\mathcal{U} \subset \mathrm{R}^n$. Suppose that F has almost isotropic S-curvature satisfying (1). Then the flag curvature is in the form (2).*

(a) If $\mathbf{K} \neq -c^2 + \frac{c_{x^m} y^m}{F}$ on $\mathcal{U}$, then $F = \alpha + \beta$ is a projectively flat Randers metric with isotropic S-curvature $\mathbf{S} = (n + 1)cF$;

(b) If $\mathbf{K} \equiv -c^2 + \frac{c_{x^m} y^m}{F}$ on $\mathcal{U}$, then $c = constant$, and F is either locally Minkowskian ($c = 0$) or, up to a scaling, locally isometric to the metric $\Theta_a = \Theta_a(x, y)$ in (4) ($c = \frac{1}{2}$) or its reverse $\bar{\Theta}_a := \Theta_a(x, -y)$ ($c = -\frac{1}{2}$).

In Theorem 1.1 (a), the local structure of F has been completely determined in [8]. Namely, if a Randers metric $F = \alpha + \beta$ is locally projectively flat with (1), then α is locally isometric to the standard projectively metric

$$\alpha_\mu := \frac{\sqrt{(1 + \mu|x|^2)|y|^2 - \mu\langle x, y\rangle^2}}{1 + \mu|x|^2}, \qquad y \in T_x\mathrm{B}^n(r) \cong \mathrm{R}^n,$$

the scalar function $\sigma = \sigma(x)$ in (2) is given by $\sigma = \mu + 3c^2$ and β satisfies $2c_{x^k}y^k + (\mu + 4c^2)\beta = 0$. Suppose that $dc = 0$ at a point $x \in \mathcal{U}$, then at the point x, either $\beta = 0$ or $\mu + 4c^2 = 0$. In the later case, $\mathbf{K} = \mu + 3c^2 = -c^2 + \frac{c_{x^m} y^m}{F}$. This contradicts the assumption (a). We may assume that $dc \neq 0$ on $\mathcal{U}$. Then $\mu + 4c^2 \neq 0$ and β is given by

$$\beta = -\frac{c_{x^k} y^k}{\mu + 4c^2}.$$

In this case, we can completely determine the scalar function $c = c(x)$ as follows.

$$c = \begin{cases} (\lambda + \langle a, x\rangle)\sqrt{\frac{\mu}{\pm(1+\mu|x|^2)-(\lambda+\langle a,x\rangle)^2}} & \text{if } \mu \neq 0 \\ \frac{\pm 1}{2\sqrt{\lambda+2\langle a,x\rangle+|x|^2}} & \text{if } \mu = 0. \end{cases}$$

where $a \in \mathrm{R}^n$ is a constant vector and $\lambda \in \mathrm{R}$ is a constant number. See [8] for more details.

2 Projectively Flat Metrics

In this section, we are going to recall some basic facts about projecvtively flat Finsler metrics. Let $F = F(x, y)$ be a Finsler metric on an open subset $\mathcal{U} \subset \mathrm{R}^n$. The geodesics are characterized by

$$\ddot{x}^i + 2G^i(x, \dot{x}) = 0,$$

where $\dot{x}^i = \frac{dx^i}{dt}$, $\ddot{x}^i = \frac{d^2x^i}{dt^2}$, and

$$G^i = \frac{1}{4}g^{il}\Big\{[F^2]_{x^m y^l}y^m - [F^2]_{x^l}\Big\}.$$

$G := y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i}$ is a globally defined vector field on TM. We call G the *spray* and G^i the *spray coefficients* of F. The Riemann curvature is a family of linear maps $\mathbf{R}_y = R^i{}_k\frac{\partial}{\partial x^i} \otimes dx^k : T_xM \to T_xM$ defined by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - y^j\frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{5}$$

The S-curvature is a scalar function $\mathbf{S} : TM \to R$ defined by

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m} - y^m\frac{\partial(\ln \sigma_F)}{\partial y^m}, \tag{6}$$

where

$$\sigma_F = \frac{\mathrm{Vol}(\mathrm{B}^n)}{\mathrm{Vol}\{(y^i) \in \mathrm{R}^n \mid F(x, y^i\frac{\partial}{\partial x^i}|_x) < 1\}}.$$

F is *of scalar curvature* if and only if there is a scalar function $\mathbf{K} = \mathbf{K}(x, y)$ such that

$$R^i{}_k = \mathbf{K}F^2\Big\{\delta^i_k - g_{km}\frac{y^m}{F}\Big\}. \tag{7}$$

A Finsler metric $F = F(x, y)$ is said to be *projectively flat* if its geodesics are straight line segment in $\mathcal{U}$. It is easy to see that F is projectively flat if and only if $G^i = Py^i$ where

$$P = \frac{F_{x^k}y^k}{2F}. \tag{8}$$

G. Hamel [11] finds the following sufficient and necessary condition

$$F_{x^k} = F_{x^m y^k} y^m. \tag{9}$$

By (8) and (9), one obtains

$$F_{x^k} = P_{y^k} F + P F_{y^k}. \tag{10}$$

Plugging $G^i = P y^i$ into (5), one obtains

$$R^i{}_k = \Xi \delta^i_k + \tau_k y^i,$$

where

$$\Xi = P^2 - P_{x^m} y^m, \qquad \tau_k = 3(P_{x^k} - P P_{y^k}) + \Xi_{y^k}.$$

It is well-known fact that $g_{ij} R^i{}_k = g_{ik} R^i{}_j$. Then (7) holds with

$$\mathbf{K} = \frac{\Xi}{F^2} = \frac{P^2 - P_{x^m} y^m}{\Xi}, \tag{11}$$

namely,

$$P_{x^k} - P P_{y^k} = -\frac{1}{3F} (\Xi F)_{y^k}.$$

In particular, if $\mathbf{K} = \lambda$ is a constant,

$$P_{x^k} - P P_{y^k} = -\lambda F F_{y^k}. \tag{12}$$

The above identities can be found in [4].

Funk constructed the following important example [9], [10]. Let $\phi = \phi(y)$ be a Minkowski norm on R^n and $\mathcal{U}_\phi := \{y \in \mathrm{R}^n \mid \phi(y) < 1\}$. Define $\Theta = \Theta(x, y)$ by

$$\Theta(x, y) = \phi\Big(y - \Theta(x, y) x\Big), \qquad y \in T_x \mathcal{U}_\phi \cong \mathrm{R}^n. \tag{13}$$

The key property of Θ is that it satisfies (3) (see [13]). Using this equation, one can easily show that Θ is projectively flat with $\mathbf{K} = -\frac{1}{4}$ and $\mathbf{S} = \frac{1}{2}(n+1)\Theta$ [16].

Taking the Euclidean norm $\phi = |y|$ in (13), one obtains

$$\Theta = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2} + \langle x, y\rangle}{1 - |x|^2}, \qquad y \in T_x \mathrm{B}^n \cong \mathrm{R}^n.$$

One can directly verify that Θ satisfies (3). The metric Θ_a in (4) is given by

$$\Theta_a = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2} + \langle x, y\rangle}{1 - |x|^2} + \frac{\langle a, y\rangle}{1 + \langle a, x\rangle}.$$

When $|a| < 1$, Θ_a is a projectively flat Finsler metric on B^n with $\mathbf{K} = -\frac{1}{4}$ and $\mathbf{S} = \frac{1}{2}(n+1)\Theta_a$. This fact is proved in [17].

3 Proof of Theorem 1.1

By assumption, $\mathbf{S}$ is in the form (1). Since every closed 1-form on an connected open subset in R^n is exact, we may assume that

$$\mathbf{S} = (n+1)\Big\{cF + dh\Big\},$$

where $h = h(x)$ is a scalar function on $\mathcal{U}$.

On the other hand, F is projectively flat, hence the spray coefficients are in the form $G^i = Py^i$. By (6), one obtains

$$\mathbf{S} = (n+1)P - y^m \frac{\partial(\ln \sigma_F)}{\partial y^m}.$$

Thus

$$P = cF + d\varphi, \tag{14}$$

where $\varphi = [\sigma_F(x)]^{\frac{1}{n+1}}h(x)$. It follows from (8) and (14) that

$$F_{x^i}y^i = 2FP = 2F\Big\{cF + \varphi_{x^i}y^i\Big\}. \tag{15}$$

Plugging (14) into (11) and using (15), one obtains

$$\begin{aligned}\mathbf{K} &= \frac{\Big\{cF + \varphi_{x^i}y^i\Big\}^2 - \Big\{c_{x^i}y^iF + cF_{x^i}y^i + \varphi_{x^ix^j}y^iy^j\Big\}}{F^2} \\ &= \frac{-c^2F^2 - c_{x^m}y^mF + [\varphi_{x^i}\varphi_{x^j} - \varphi_{x^ix^j}]y^iy^j}{F^2}.\end{aligned} \tag{16}$$

Comparing (16) with (2) yields

$$[\sigma + c^2]F^2 + 4c_{x^m}y^mF + [\varphi_{x^ix^j} - \varphi_{x^i}\varphi_{x^j}]y^iy^j = 0. \tag{17}$$

Assume that $\mathbf{K} \neq -c^2 + \frac{c_{x^m}y^m}{F}$. By (2), this is equivalent to the following inequality:

$$\sigma + c^2 + \frac{2c_{x^m}y^m}{F} \neq 0. \tag{18}$$

From (17) and (18), we can see that $\sigma + c^2 \neq 0$. In this case, one can solve the quadratic equation (17) for F,

$$F = \frac{\sqrt{[\sigma + c^2][\varphi_{x^ix^j} - \varphi_{x^i}\varphi_{x^j}]y^iy^j + 4[c_{x^m}y^m]^2} - 2c_{x^m}y^m}{\sigma + c^2}.$$

That is, $F = \alpha + \beta$ is a Randers metric. We have classified projectively flat Randers metrics with almost isotropic S-curvature [8].

We now assume that $\mathbf{K} \equiv -c^2 + \frac{c_{x^m}y^m}{F}$. It follows from (2) that

$$\sigma + c^2 + \frac{2c_{x^m}y^m}{F} \equiv 0.$$

This implies that $c = constant$, hence $\sigma = -c^2$ is a constant too. In this case, the flag curvature is given by $\mathbf{K} = -c^2$. The equation (17) is reduced to

$$\varphi_{x^i x^j} - \varphi_{x^i}\varphi_{x^j} = 0.$$

It is easy to solve this equation,

$$\varphi = -\ln\Big(1 + \langle a, x\rangle\Big) + C,$$

where $a \in \mathrm{R}^n$ is a constant vector and C is a constant.

When $c = 0$, $\mathbf{K} = -c^2 = 0$. It follows from (14) that the projective factor $P = d\varphi$ is a 1-form, hence the spray coefficients $G^i = Py^i$ are quadratic in $y \in T_x\mathcal{U}$. By definition, F is a Berwald metric. It is known that every Berwald metric with vanishing flag curvature is locally Minkowskian.

When $c \neq 0$, we may assume that $c = \pm\frac{1}{2}$ after a suitable scaling. Let

$$\Psi := P + cF.$$

Since F is projectively flat and P is the projective factor, it follows from By (10) and (12), one can easily verify that

$$\Psi_{x^i} = \Psi\Psi_{y^i}.$$

Let

$$\Theta := \begin{cases} \Psi(x,y) & \text{if } c = \frac{1}{2} \\ -\Psi(x,-y) & \text{if } c = -\frac{1}{2} \end{cases}.$$

Then $\Theta = \Theta(x,y)$ satisfies (3). Thus by definition it is a Funk metric. By (14), $\Psi = 2cF + d\varphi$. Thus

$$F = \frac{1}{2c}\Big\{\Psi(x,y) - d\varphi_x\Big\}.$$

When $c = \frac{1}{2}$, $\Psi(x,y) = \Theta(x,y)$. Thus

$$F = \Theta(x,y) + \frac{\langle a, y\rangle}{1 + \langle a, x\rangle} =: \Theta_a(x,y).$$

When $c < 0$, $\Psi(x,y) = -\Theta(x,-y)$. Thus

$$F = \Theta(x,-y) - \frac{\langle a, y\rangle}{1 + \langle a, x\rangle} =: \bar{\Theta}_a(x,y),$$

where $\bar{\Theta}_a(x,y) := \Theta_a(x,-y)$. Q.E.D.

References

[1] D. Bao, C. Robles and Z. Shen, *Zermelo Navigation on Riemannian manifolds*, preprint, 2003.

[2] D. Bao and C. Robles, *On Randers metrics of constant curvature*, Rep. on Math. Phys. **51** (2003), 9-42.

[3] D. Bao and Z. Shen, *Finsler metrics of constant curvature on the Lie group S^3*, J. London Math. Soc. **66**(2002), 453-467.

[4] L. Berwald, *Über die n-dimensionalen Geometrien konstanter Krümmung, in denen die Geraden die kürzesten sind*, Math. Z. **30**(1929), 449-469.

[5] R. Bryant, *Finsler structures on the 2-sphere satisfying $K = 1$*, Finsler Geometry, Contemporary Mathematics **196**, Amer. Math. Soc., Providence, RI, 1996, 27-42.

[6] R. Bryant, *Projectively flat Finsler* 2*-spheres of constant curvature*, Selecta Math., N.S. **3**(1997), 161-204.

[7] R. Bryant, *Some remarks on Finsler manifolds with constant flag curvature*, Houston J. Math. **28**(2) (2002), 221-262.

[8] X. Chen, X. Mo and Z. Shen, *On the flag curvature of Finsler metrics of scalar curvature*, J. of London Math. Soc. (to appear).

[9] P. Funk, *Über Geometrien bei denen die Geraden die Kürzesten sind*, Math. Ann. **101**(1929), 226-237.

[10] P. Funk, *Über zweidimensionale Finslersche Räume, insbesondere über solche mit geradlinigen Extremalen und positiver konstanter Krümmung*, Math. Z. **40**(1936), 86-93.

[11] G. Hamel, *Über die Geometrien in denen die Geraden die Kürzesten sind*, Math. Ann. **57**(1903), 231-264.

[12] D. Hilbert, *Mathematical Problems*, Bull. of Amer. Math. Soc. **37**(2001), 407-436. Reprinted from Bull. Amer. Math. Soc. **8** (July 1902), 437-479.

[13] T. Okada, *On models of projectively flat Finsler spaces of constant negative curvature*, Tensor, N. S. **40**(1983), 117-123.

[14] Z. Shen, *Volume comparison and its applications in Riemann-Finsler geometry*, Advances in Math. **128**(1997), 306-328.

[15] Z. Shen, *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, 2001.

[16] Z. Shen, *Lectures on Finsler Geometry*, World Scientific, Singapore, 2001.

[17] Z. Shen, *Projectively flat Randers metrics of constant flag curvature*, Math. Ann. **325**(2003), 19-30.

[18] Z. Shen, *Projectively flat Finsler metrics of constant flag curvature*, Trans. of Amer. Math. Soc. **355**(4) (2003), 1713-1728.

[19] Z. Shen, *Finsler metrics with K=0 and S=0*, preprint, 2001, Canadian J. of Math. **55**(2003), no. 1, 112-132.

[20] Z. Shen, *Two-dimensional Finsler metrics of constant flag curvature*, Manuscripta Mathematica, **109**(3) (2002), 349-366.

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