Randers Metrics of Scalar Flag Curvature*

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Abstract

We study an important class of Finsler metrics — Randers metrics. We classify Randers metrics of scalar flag curvature whose S-curvatures are isotropic. This class of Randers metrics contains all projectively flat Randers metrics with isotropic S-curvature and Randers metrics of constant flag curvature.

1 Introduction

Curvatures are the central concept of Finsler geometry. In this paper, our main focus is on the flag curvature, the S-curvature and their interaction.

For a Finsler manifold (M, F), the flag curvature $\mathbf{K} = \mathbf{K}(P, y)$ is a function of a tangent plane $P \subset T_xM$ and a non-zero vector $y \in P$. When F is Riemannian, $\mathbf{K} = \mathbf{K}(P)$ is independent of $y \in P \setminus \{0\}$, which is called the sectional curvature. Thus the flag curvature is an analogue of sectional curvature in Riemannian geometry. A Finsler metric F is of *scalar flag curvature* if for any non-zero vector $y \in T_xM$, $\mathbf{K} = K(x, y)$ is independent of P containing $y \in T_xM$ (hence $\mathbf{K} = \sigma(x)$ when F is Riemannian). It is *of almost isotropic flag curvature* if

$$\mathbf{K} = \frac{3\theta}{F} + \sigma, \tag{1}$$

where $\sigma = \sigma(x)$ and $c = c(x)$ are scalar functions, and $\theta = c_{x^m}(x)y^m$ [4]. It is of *isotropic flag curvature* if $\mathbf{K} = \sigma(x)$ in (1). In this case, $\sigma(x) =$ *constant* if dimension $n \geq 3$ (Schur Lemma). It is one of important problems in Finsler geometry to study and characterize Finsler metrics of scalar/almost isotropic/constant flag curvature.

There is another important quantity closely related to the flag curvature. That is the so-called S-curvature $\mathbf{S} = \mathbf{S}(x, y)$. The S-curvature is introduced by the second author when he studied volume comparison in Riemann-Finsler geometry [11] [12]. The S-curvature is said to be *isotropic* if $\mathbf{S} = (n+1)cF$ where $c = c(x)$ is a scalar function on M. It is proved that, for a Finsler metric F of scalar flag curvature, if it is of isotropic S-curvature $\mathbf{S} = (n+1)cF$, then

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the flag curvature must be in the form (1). See [8] for further developments. Thus the flag curvature and the S-curvature are closely related.

Studying Randers metrics is an important step to understand general Finsler metrics. A Randers metric on a manifold M is a special Finsler metric expressed in the following form:

$$F = \alpha + \beta,$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M with $\|\beta_x\|_\alpha < 1$. Randers metrics arise naturally from the navigation problem on a Riemannian space (M, h) under an external force field W. The least time path from one point to another is a geodesic of a Randers metric $F = \alpha + \beta$ defined by

$$F = \frac{\sqrt{\lambda h^2 + W_0^2}}{\lambda} - \frac{W_0}{\lambda}, \qquad W_0 := W_i y^i, \tag{2}$$

where

$$W_i := h_{ij} W^j, \quad \lambda := 1 - W_i W^i.$$

See [14]. It is easy to see that every Randers metric can be expressed in the form (2).

For Randers metrics of scalar flag curvature with isotropic S-curvature, we prove the following

Theorem 1.1 *Let $F = \alpha + \beta$ be a Randers metric on a manifold M of dimension $n \geq 3$, which is expressed in terms of a Riemannian metric h and a vector field W by (2). Then F is of scalar flag curvature $\mathbf{K} = K(x, y)$ and of isotropic S-curvature $\mathbf{S} = (n+1)c(x)F$ if and only if at any point, there is a local coordinate system in which h, c and W are given by*

$$h = \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 + \mu|x|^2}, \tag{3}$$

$$c = \frac{\delta + \langle a, x\rangle}{\sqrt{1 + \mu|x|^2}} \tag{4}$$

$$W = -2\Big\{\Big(\delta\sqrt{1 + \mu|x|^2} + \langle a, x\rangle\Big)x - \frac{|x|^2 a}{\sqrt{1 + \mu|x|^2} + 1}\Big\} + xQ + b + \mu\langle b, x\rangle x, \tag{5}$$

where δ, μ are constants, $Q = ({q_j}^i)$ is an anti-symmetric matrix and $a, b \in \mathrm{R}^n$ are constant vectors. In this case, the flag curvature is given by

$$\mathbf{K} = \frac{3c_{x^m} y^m}{F} + \sigma, \tag{6}$$

where $\sigma = \mu - c^2 - 2c_{x^m} W^m$.

Locally projectively flat Randers metrics are always of scalar flag curvature, hence Theorem 1.1 generalizes Theorem 1.3 in [4] which classifies the locally projectively flat Randers metrics with isotropic S-curvature. Furthermore, since every Einstein-Randers metric F must have constant S-curvature [2], the class of Randers metrics of scalar flag curvature and isotropic S-curvature contains all Randers metrics of constant flag curvature. Therefore, Theorem 1.1 also generalizes the classification theorem on Randers metrics of constant flag curvature in [3].

Let us take a look at a special example. In (3)-(5), let $\mu = 0, \delta = 0, Q = 0$ and $b = 0$. We get

$$h = |y|, \quad c = \langle a, x\rangle, \quad W = -2\langle a, x\rangle x + |x|^2 a.$$

The Randers metric $F = \alpha + \beta$ is given by

$$\begin{aligned} F &= \frac{\sqrt{(1-|a|^2|x|^4)|y|^2 + (|x|^2\langle a, y\rangle - 2\langle a, x\rangle\langle x, y\rangle)^2}}{1-|a|^2|x|^4} \\ &\quad - \frac{|x|^2\langle a, y\rangle - 2\langle a, x\rangle\langle x, y\rangle}{1-|a|^2|x|^4}. \end{aligned}$$

The above defined Randers metric F is of isotropic S-curvature and scalar flag curvature, i.e.,

$$\mathbf{S} = (n+1)\langle a, x\rangle\, F, \qquad \mathbf{K} = \frac{3\langle a, y\rangle}{F} + 3\langle a, x\rangle^2 - 2|a|^2|x|^2.$$

This example has already been constructed by the second author in [15].

The authors would like to thank D. Bao for showing us the proof of a formula for the spray coefficients (see equation (7) below) which is given in C. Robles' paper [10]. This formula is our starting point to prove Theorem 1.1.

2 Preliminaries

Consider a Randers metric $F = \alpha + \beta$ on a manifold M. We can express it in the form (2),

$$F = \frac{\sqrt{\lambda h^2 + W_0^2}}{\lambda} - \frac{W_0}{\lambda},$$

where $h = \sqrt{h_{ij}y^iy^j}$ be a Riemannian metric and $W = W^i\frac{\partial}{\partial x^i}$ be a vector field on M. Let $DW = W_{i;j}dx^i \otimes dx^j$ denote the covariant derivative of W with respect to h. Let

$$\mathcal{R}_{ij} := \frac{1}{2}(W_{i;j} + W_{j;i}), \quad \mathcal{S}_{ij} := \frac{1}{2}(W_{i;j} - W_{j;i}),$$

$$\mathcal{R}_j := W^i\mathcal{R}_{ij}, \quad \mathcal{R} := W^j\mathcal{R}_j, \quad \mathcal{S}_j := W^i\mathcal{S}_{ij}.$$

The spray coefficients G^i of F are defined by (cf. [13])

$$G^i = \frac{1}{4}g^{il}\Big\{[F^2]_{x^jy^l}y^j - [F^2]_{x^l}\Big\}.$$

Let $\bar{G}^i$ denote the spray coefficients of h. We have the following

$$G^i = \bar{G}^i - \frac{1}{2}F^2(\mathcal{S}^i + \mathcal{R}^i) - F\mathcal{S}^i{}_0 + \frac{1}{2}(y^i/F - W^i)(2\mathcal{R}_0 F - \mathcal{R}_{00} - \mathcal{R}F^2), \quad (7)$$

where $\mathcal{S}^i := h^{ij}\mathcal{S}_j$, $\mathcal{R}^i := h^{ij}\mathcal{R}_j$, $\mathcal{S}^i{}_j := h^{il}\mathcal{S}_{lj}$, $\mathcal{S}^i{}_0 := \mathcal{S}^i{}_j y^j$, $\mathcal{R}_0 := \mathcal{R}_i y^i$ and $\mathcal{R}_{00} := \mathcal{R}_{ij}y^i y^j$. Formula (7) is given in [10].

Let $dV_F := \sigma_F(x)dx^1 \cdots dx^n$ denote the volume form of a Finsler metric F, where

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathrm{B}^n(1))}{\mathrm{Vol}\{(y^i)|F(x,y) < 1\}}.$$

Here Vol denotes the Euclidean volume and $\mathrm{Vol}(\mathrm{B}^n(1))$ denotes the Euclidean volume of the unit ball in R^n. Then the S-curvature $\mathbf{S}$ of F is given by (cf. [12][13])

$$\mathbf{S}(x,y) = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m}\Big(\ln \sigma_F\Big). \quad (8)$$

The S-curvature $\mathbf{S}(x,y)$ measures the average rate of changes of (T_xM, F_x) in the direction $y \in T_xM$. An important property is that $\mathbf{S} = 0$ for Berwald spaces [11][12].

Lemma 2.1 ([17]) *Let F be a Randers metric defined by (2) and $c = c(x)$ be a scalar function on an n-dimensional manifold. Then $\mathbf{S} = (n+1)cF$ if and only if*

$$\mathcal{R}_{00} = -2ch^2. \quad (9)$$

Assume that F has isotropic S-curvature, $\mathbf{S} = (n+1)cF$. By Lemma 2.1, W satisfies (9). Then the spray coefficients G^i are reduced to the following expression:

$$G^i = \bar{G}^i - F\mathcal{S}^i{}_0 - \frac{1}{2}F^2\mathcal{S}^i + cFy^i. \quad (10)$$

It is known that a Randers metric $F = \alpha + \beta$ expressed in the form (2) is of constant flag curvature $\mathbf{K} = k$ if and only if h has constant sectional curvature $\bar{\mathbf{K}} = \mu$ and W is homothetic, i.e., it satisfies (9). In this case, $k = \mu - c^2$. Moreover, $c = 0$ if $\mu \neq 0$. This leads to the classification of Randers metrics of constant flag curvature [3]. See also [1], [2] and [7] for some early work on Randers metrics of constant flag curvature.

According to [5] and [17], for any scalar function $c = c(x)$, the equation (**??**) is equivalent to the following condition on the S-curvature:

$$\mathbf{S} = (n+1)cF. \quad (11)$$

This fact also follows from the arguments in [2]. Thus a Randers metric of constant flag curvature must have constant S-curvature.

3 The Riemann Curvature

Let F be a Finsler metric on a manifold M and G^i be the spray coefficients of F. The Riemann curvature $R = R^i{}_k \frac{\partial}{\partial x^i} \otimes dx^k$ is defined by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^m \partial y^k} y^m + 2G^m \frac{\partial^2 G^i}{\partial y^m \partial y^k} - \frac{\partial G^i}{\partial y^m}\frac{\partial G^m}{\partial y^k}. \tag{12}$$

It is known that (cf. ([13]) F is of scalar flag curvature $\mathbf{K} = K(x,y)$ if and only if in a standard local coordinate system,

$$R^i{}_k = K(x,y)\big\{F^2 \delta^i{}_k - F F_{y^k} y^i\big\}. \tag{13}$$

From now on, we always assume that F is a Randers metric given by (2) with isotropic S-curvature, $\mathbf{S} = (n+1)cF$. We are going to use (10) to express the Riemann curvature in terms of h and W.

Rewrite (10) as follows

$$G^i = \bar{G}^i + Q^i,$$

where

$$Q^i := -F\mathcal{S}^i{}_0 - \frac{1}{2}F^2\mathcal{S}^i + cFy^i.$$

Then

$$R^i{}_k = \bar{R}^i{}_k + 2Q^i{}_{;k} - [Q^i{}_{;m}]_{y^k} y^m + 2Q^m [Q^i]_{y^m y^k} - [Q^i]_{y^m}[Q^m]_{y^k}, \tag{14}$$

where $\bar{R} = \bar{R}^i{}_k \frac{\partial}{\partial x^i} \otimes dx^k$ denote the Riemann curvature of h, ";" denotes the horizontal covariant differentiation with respect to h (cf. [13]). We first compute the horizontal and vertical derivatives of Q^i and express them in terms of h, W and the covariant derivatives of W with respect to h. Since W satisfies (9), we have

$$W_{i;j;k} = 2\Big(c_{x^i} h_{jk} - c_{x^j} h_{ik} - c_{x^k} h_{ij}\Big) - \bar{R}_{kpij} W^p, \tag{15}$$

where $\bar{R}_{kpij}$ denote the Riemann curvature tensor of h.

Equation (15) is proved in Lemma 3.1 in [16]. By (15) and the properties of the Riemann curvature tensor of h, we obtain

$$\begin{aligned}
\mathcal{S}^i{}_{k;0} &= 2(h^{im} c_{x^m} y_k - c_{x^k} y^i) - \bar{R}_k{}^i{}_{mq} W^m y^q,\\
\mathcal{S}^i{}_{0;k} &= 2(h^{im} c_{x^m} y_k - c_{x^m} y^m \delta^i_k) + \bar{R}_p{}^i{}_{kq} y^p W^q,\\
\mathcal{S}^i{}_{;k} &= 2c\mathcal{S}^i{}_k - \mathcal{S}^i{}_m \mathcal{S}^m_k + 2(c_{x^m} W^m \delta^i_k - h^{im} c_{x^m} W_k) - \bar{R}_p{}^i{}_{kq} W^p W^q,\\
\mathcal{S}^i{}_{;0} &= 2cS^i{}_0 - \mathcal{S}^i{}_m \mathcal{S}^m_0 + 2(c_{x^m} W^m y^i - h^{im} c_{x^m} W_0) - \bar{R}_p{}^i{}_{mq} W^p W^q y^m,\\
\mathcal{S}^i{}_{0;0} &= 2(h^{im} c_{;m} h^2 - c_{;0} y^i) - \bar{R}_p{}^i{}_{mq} y^p y^q W^m.
\end{aligned}$$

Let $A := \sqrt{\lambda h^2 + W_0^2}$. From (2), we have $A = \lambda F + W_0$. It is easy to verify that

$$h^2 - 2FW_0 = \lambda F^2. \tag{16}$$

By (16), we have the following identity

$$h^2 - FW_0 - AF = 0. \tag{17}$$

Further, by (17) and (9) we obtain the following formulas

$$\begin{aligned}
F_{;k} &= \frac{2cF(y_k - FW_k) + F(F\mathcal{S}_k + \mathcal{S}_{k0})}{A}, \\
F_{;0} &= 2cF^2 + \frac{F^2}{A}\mathcal{S}_0, \\
(F_{y^k})_{;0} &= \Big(\frac{h^2}{A^3}\mathcal{S}_0 + 2c\frac{F}{A}\Big)\{y_k - FW_k\} - \frac{F^2}{A^2}\mathcal{S}_0 W_k - \frac{F}{A}\mathcal{S}_{k0}.
\end{aligned}$$

By (14) and the above identities and in virtue of Maple program, we first obtain the following very simple formula:

$$\begin{aligned}
R^i{}_k &= \bar{R}_p{}^i{}_{kq} y^p y^q - F\bar{R}_p{}^i{}_{kq} W^p y^q - F\bar{R}_p{}^i{}_{kq} y^p W^q \\
&\quad + F^2 \bar{R}_p{}^i{}_{kq} W^p W^q - F_{y^k}\bar{R}_p{}^i{}_{mq} y^p y^q W^m + FF_{y^k}\bar{R}_p{}^i{}_{mq} y^q W^p W^m \\
&\quad + \Big(\frac{3c_{x^m}y^m}{F} - c^2 - 2c_{x^m}W^m\Big)\Big\{F^2\delta^i_k - FF_{y^k}y^i\Big\}.
\end{aligned} \tag{18}$$

It is surprised that all the terms with $\mathcal{S}^i$ or $\mathcal{S}^i{}_k$ do not occur in (18).

Observe that

$$\begin{aligned}
&\bar{R}_p{}^i{}_{kq}(y^p - FW^p)(y^q - FW^q) \\
&= \bar{R}_p{}^i{}_{kq} y^p y^q - F\bar{R}_p{}^i{}_{kq} y^p W^q - F\bar{R}_p{}^i{}_{kq} W^p y^q + F^2\bar{R}_p{}^i{}_{kq} W^p W^q
\end{aligned}$$

and

$$\begin{aligned}
&\bar{R}_p{}^i{}_{mq}(y^p - FW^p)(y^q - FW^q)W^m \\
&\quad = \bar{R}_p{}^i{}_{mq} y^p y^q W^m - F\bar{R}_p{}^i{}_{mq} W^p y^q W^m \\
&\qquad - F\bar{R}_p{}^i{}_{mq} y^p W^q W^m + F^2\bar{R}_p{}^i{}_{mq} W^p W^q W^m \\
&\quad = \bar{R}_p{}^i{}_{mq} y^p y^q W^m - F\bar{R}_p{}^i{}_{mq} W^p y^q W^m.
\end{aligned}$$

Substituting them into (18), we obtain

$$\begin{aligned}
R^i{}_k &= \bar{R}_p{}^i{}_{kq}(y^p - FW^p)(y^q - FW^q) \\
&\quad - F_{y^k}\bar{R}_p{}^i{}_{mq}(y^p - FW^p)(y^q - FW^q)W^m \\
&\quad + \Big(\frac{3c_{x^m}y^m}{F} - c^2 - 2c_{x^m}W^m\Big)\Big\{F^2\delta^i_k - FF_{y^k}y^i\Big\}.
\end{aligned} \tag{19}$$

Let

$$\xi^i := y^i - F(x,y)W^i, \qquad \xi_k := h_{ik}\xi^i$$

and

$$\tilde{h} := h(x,\xi) = \sqrt{h_{pq}\xi^p\xi^q} = \sqrt{\xi_k\xi^k}, \qquad \tilde{W}_0 := W_i\xi^i.$$

We have

$$\tilde{h}^2 = h_{pq}(y^p - FW^p)(y^q - FW^q) = h^2 - 2FW_0 + F^2h(x,W)^2 = F^2.$$

Thus

$$y^i = \xi^i + \tilde{h}W^i.$$

Observe that

$$\begin{aligned} \lambda\tilde{h} &= \lambda F = A - W_0 \\ &= A - W_i(\xi^i + \tilde{h}W^i) \\ &= A - \tilde{W}_0 - \tilde{h}(1-\lambda). \end{aligned}$$

This gives

$$A = \tilde{h} + \tilde{W}_0.$$

By the above identities, we obtain

$$F_{y^k} = \frac{1}{A}(y_k - FW_k) = \frac{\xi_k}{\tilde{h} + \tilde{W}_0},$$

$$F^2\delta^i_k - FF_{y^k}y^i = \tilde{h}^2\delta^i_k - \xi_k\xi^i - \frac{1}{\tilde{h} + \tilde{W}_0}\xi_k(\tilde{h}^2\delta^i_p - \xi_p\xi^i)W^p,$$

where $y_k := h_{ik}y^i$. Let

$$\tilde{R}^i{}_k := \bar{R}_p{}^i{}_{kq}\xi^p\xi^q.$$

By (19), we obtain the following

Lemma 3.1 *Let $F = \alpha + \beta$ be a Randers metric expressed by (2). Suppose that it has isotropic S-curvature, $\mathbf{S} = (n+1)cF$. Then for any scalar function $\mu = \mu(x)$ on M,*

$$\begin{aligned} & R^i{}_k - \Big(\frac{3c_{x^m}y^m}{F} + \mu - c^2 - 2c_{x^m}W^m\Big)\Big\{F^2\delta^i_k - FF_{y^k}y^i\Big\} \\ &= \tilde{R}^i{}_k - \mu\Big(\tilde{h}^2\delta^i_k - \xi_k\xi^i\Big) - \frac{\xi_k}{\tilde{h} + \tilde{W}_0}\Big\{\tilde{R}^i{}_p - \mu\Big(\tilde{h}^2\delta^i_p - \xi_p\xi^i\Big)\Big\}W^p \end{aligned} \quad (20)$$

4 The Ricci Curvature

In this section, we shall study the Ricci curvature of a Randers metric with isotropic S-curvature. Let Ric and $\overline{\mathrm{Ric}}$ denote the Ricci curvature of F and h respectively. They are defined by

$$\mathrm{Ric} := R^m{}_m, \qquad \overline{\mathrm{Ric}} := \bar{R}^m{}_m.$$

Let

$$\widetilde{\mathrm{Ric}} := \tilde{R}^m{}_m = \bar{R}_p{}^m{}_{mq}\xi^p\xi^q.$$

Clearly, $\overline{\mathrm{Ric}} = (n-1)\mu h^2$ if and only if $\widetilde{\mathrm{Ric}} = (n-1)\mu\tilde{h}^2$.

First we have the following

Lemma 4.1 *Let $F = \alpha + \beta$ be a Randers metric expressed by (2). Suppose that it has isotropic S-curvature, $\mathbf{S} = (n+1)cF$. Then for any scalar function $\mu = \mu(x)$ on M,*

$$\mathrm{Ric} - (n-1)\Big(\frac{3c_{x^m}y^m}{F} + \mu - c^2 - 2c_{x^m}W^m\Big)F^2 = \widetilde{\mathrm{Ric}} - (n-1)\mu\tilde{h}^2. \quad (21)$$

Proof: Observe that

$$\xi_m \tilde{R}^m_{\ p} = \xi_m \bar{R}_i{}^m{}_{pj}\xi^i\xi^j = \xi^m \bar{R}_{impj}\xi^i\xi^j = 0$$

and

$$\xi_m\Big(\tilde{h}^2\delta^m_p - \xi_p\xi^m\Big) = \tilde{h}^2\xi_p - \xi_p\tilde{h}^2 = 0.$$

Then (21) follows from (20). Q.E.D.

From Lemma 4.1 we immediately obtain the following

Theorem 4.2 *Let F be a Randers metric on n-dimensional manifold M defined by (2) and let $c = c(x)$ and $\mu = \mu(x)$ be scalar functions on M. Suppose $\mathbf{S} = (n+1)cF$. Then $\overline{\mathrm{Ric}} = (n-1)\mu h^2$ if and only if*

$$\mathrm{Ric} = (n-1)\Big\{\frac{3c_{x^m}y^m}{F} + \mu - c^2 - 2c_{x^m}W^m\Big\}F^2. \quad (22)$$

Corollary 4.3 *Let F be a Randers metric defined by (2). If W is an infinitesimal homothety of h (or equivalently, $\mathbf{S} = (n+1)cF$ for some constant c), then for any scalar function $\mu = \mu(x)$, $\overline{\mathrm{Ric}} = (n-1)\mu h^2$ if and only if $\mathrm{Ric} = (n-1)\big(\mu - c^2\big)F^2$.*

Corollary 4.3 is already proved in Theorem 9 in [2] (see also [9]). In fact, Bao-Robles prove that for a Randers metric F defined by (2), F is Einstein $\mathrm{Ric} = (n-1)K(x)F^2$, if and only if $\mathbf{S} = (n+1)cF$ for some constant c and $\overline{\mathrm{Ric}} = (n-1)\big(K(x) + c^2\big)h^2$.

5 Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. First we prove the following

Theorem 5.1 *Let F be a Randers metric on n-dimensional manifold M defined by (2). Suppose that $\mathbf{S} = (n+1)cF$ where $c = c(x)$ is a scalar function. Then F is of scalar flag curvature if and only if h is of sectional curvature $\bar{\mathbf{K}} = \mu$, where $\mu = \mu(x)$ is a scalar function (=constant if $n \geq 3$). In this case, the flag curvature of F is given by*

$$\mathbf{K} = \frac{3c_{x^m}y^m}{F} + \sigma, \quad (23)$$

where $\sigma := \mu - c^2 - 2c_{x^m}W^m$.

Proof: Assume that F is of scalar curvature, then by Theorem 1.1 in [4], the flag curvature is given by

$$\mathbf{K} = \frac{3c_{x^m}y^m}{F} + \sigma,$$

where $\sigma = \sigma(x)$ is a scalar function on M. That is, we have

$$R^i{}_k = \big(\frac{3c_{x^m}y^m}{F} + \sigma\big)\big\{F^2\delta^i_k - FF_{y^k}y^i\big\}.$$

Let

$$\mu := \sigma + c^2 + 2c_{x^m}W^m.$$

It suffices to show that h has sectional curvature $\bar{\mathbf{K}} = \mu$. It follows from (20) that

$$\tilde{R}^i{}_k - \mu\Big(\tilde{h}^2\delta^i_k - \xi_k\xi^i\Big) - \frac{1}{\tilde{h} + \tilde{W}_0}\xi_k\Big\{\tilde{R}^i{}_p - \mu\Big(\tilde{h}^2\delta^i_p - \xi_p\xi^i\Big)\Big\}W^p = 0.$$

Clearly, we have

$$\tilde{R}^i{}_k = \mu\Big(\tilde{h}^2\delta^i_k - \xi_k\xi^i\Big). \tag{24}$$

Thus h has sectional curvature $\bar{K} = \mu$. By the Schur lemma, $\mu = \mathit{constant}$ in dimension $n \geq 3$.

Conversely, if h has sectional curvature $\bar{K} = \mu$, then (24) holds. By (20) again, we get

$$R^i{}_k = \Big(\frac{3c_{x^m}y^m}{F} + \sigma\Big)\Big\{F^2\delta^i_k - FF_{y^k}y^i\Big\}, \tag{25}$$

where $\sigma = \mu - c^2 - 2c_{x^m}W^m$. Thus F is of scalar curvature. Q.E.D.

Proof of Theorem 1.1: By assumption, the dimension of M is not less than 3. First we assume that $F = \alpha + \beta$ is of isotropic S-curvature and of scalar flag curvature. By Theorem 5.1, the flag curvature of F is given by (23) and h has constant sectional curvature $\bar{\mathbf{K}} = \mu$. At any point, there is a local coordinate system in which h is given by (3). By the Theorem 1.2 in [16], if $\mathbf{S} = (n+1)cF$, then c and W are given by (4) and (5) respectively in the same local coordinate system.

Conversely, assume that there is a local coordinate system in which h, c and W are given by (3), (4) and (5) respectively, then by Theorem 1.2 in [16], $\mathbf{S} = (n+1)cF$. Since h has constant sectional curvature $\bar{\mathbf{K}} = \mu$, by Theorem 5.1, F is of scalar curvature with flag curvature given by (23). Q.E.D.

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