# Projectively Flat Randers Metrics with Constant Flag Curvature

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**Abstract**

We classify locally projectively flat Randers metrics with constant Ricci curvature and obtain a new family of Randers metrics of negative constant flag curvature.

## 1 Introduction

One of the fundamental problems in Finsler geometry is to study and characterize Finsler metrics of constant flag curvature. The flag curvature is an analogue of sectional curvature in Riemannian geometry. It measures the rate of changes of the Finsler metric modulo the changes of the geometry of tangent spaces.

Let $(M, F)$ be a Finsler manifold. For a non-zero vector $\mathbf{y} \in T_pM$, $F$ induces an inner product $\mathbf{g_y}$ on $T_pM$ so that $F^2(\mathbf{y}) = \mathbf{g_y}(\mathbf{y}, \mathbf{y})$. The second variation of geodesics gives rise to a family of linear maps $\mathbf{R_y} : T_pM \to T_pM$, $\mathbf{y} \in T_pM$, at any point $p \in M$. Each $\mathbf{R_y}$ is self-adjoint with respect to $\mathbf{g_y}$ and satisfies $\mathbf{R_y}(\mathbf{y}) = 0$. $\mathbf{R_y}$ is called the Riemann curvature in the direction $\mathbf{y}$. The Ricci curvature $\mathbf{Ric}(\mathbf{y})$ is the trace of $\mathbf{R_y}$. For a tangent plane $P = \mathrm{span}\{\mathbf{y}, \mathbf{u}\} \subset T_pM$, the flag curvature of $\{P, \mathbf{y}\}$ is defined by

$$\mathbf{K}(P, \mathbf{y}) = \frac{\mathbf{g_y}(\mathbf{R_y}(\mathbf{u}), \mathbf{u})}{\mathbf{g_y}(\mathbf{y}, \mathbf{y})\mathbf{g_y}(\mathbf{u}, \mathbf{u}) - \mathbf{g_y}(\mathbf{y}, \mathbf{u})\mathbf{g_y}(\mathbf{y}, \mathbf{u})}.$$

If the flag curvature is a constant, $\mathbf{K} = \lambda$, then the Ricci curvature is a constant in the sense that $\mathbf{Ric} = (n-1)\lambda F^2$. If $F$ is Riemannian, i.e., $\mathbf{g} = \mathbf{g_y}$ is independent of $\mathbf{y}$, then the flag curvature $\mathbf{K}(P, \mathbf{y}) = \mathbf{K}(P)$ is independent of $\mathbf{y} \in P$ and it is the sectional curvature of $P \subset T_pM$. Thus the flag curvature is an analogue of the sectional curvature in Riemannian geometry. It is well-known that a Riemannian metric has constant sectional curvature if and only if it is locally projectively flat, and for any number $\lambda$, there is, up to an isometry, a unique local Riemannian metric with constant sectional curvature $\mathbf{K} = \lambda$. However, this fact is no longer true for general Finsler metrics due to the complexity of the geometry of tangent spaces $(T_pM, F_p)$.

In fact, there are infinitely many non-projectively flat Finsler metrics with constant flag curvature. The first family of such examples are constructed in

[BaSh]. They are non-projectively flat Finsler metrics on $S^3$ with constant flag curvature $\mathbf{K} = 1$. Recently, the author has discovered many other non-projectively flat Finsler metrics with constant flag curvature $\mathbf{K} = -1, 0$ and 1. For example, the following Finsler metric $F$ on the cylindrical domain $\Omega := \{p = (x, y, \bar{p}) \in \mathrm{R}^n \mid x^2 + y^2 < 1\}$ has zero flag curvature $\mathbf{K} = 0$,

$$F(\mathbf{y}) := \frac{\sqrt{(-yu + xv)^2 + |\mathbf{y}|^2(1 - x^2 - y^2)} - (-yu + xv)}{1 - x^2 - y^2},$$

where $\mathbf{y} = (u, v, \bar{\mathbf{y}}) \in T_p\Omega = \mathrm{R}^n$ ([Sh3]). See [Sh4] and [BaRo] for other examples. All of these non-projectively flat examples are in the form $F = \alpha + \beta$, where $\alpha$ is a Riemannian metric and $\beta$ is a 1-form. Such Finsler metrics are called the Randers metrics [Ra].

Randers metrics are special Finsler metrics. One would like to classify Randers metrics of constant flag curvature, namely, to find an explicit formula for each isometry class. There are several equations on $\alpha$ and $\beta$ that characterize a Randers metric $F = \alpha + \beta$ of constant flag curvature [Ma] [YaSh] [BaRo]. However, it is still very difficult to find explicit solutions to these equations. From examples in [BaSh] [Sh3] [Sh4] [BaRo], we do not see any hope to find an explicit formula for each isometry class.

In this paper, we will classify locally projectively flat Randers metrics of constant flag curvature. Besides the Minkowski metrics of Randers type, the only known examples are the Funk metrics on the unit ball $\mathrm{B}^n$, which are defined by

$$F = \frac{\sqrt{|\mathbf{y}|^2 - (|\mathbf{x}|^2|\mathbf{y}|^2 - \langle \mathbf{x}, \mathbf{y}\rangle^2)}}{1 - |\mathbf{x}|^2} \pm \frac{\langle \mathbf{x}, \mathbf{y}\rangle}{1 - |\mathbf{x}|^2}, \qquad \mathbf{y} \in T_{\mathbf{x}}\mathrm{B}^n = \mathrm{R}^n. \quad (1)$$

By a direct computation, we know that the Funk metrics on $\mathrm{B}^n$ satisfy (a) $\mathbf{K} = -1/4$, (b) $\mathbf{S} = \pm\frac{1}{2}(n+1)F$ and (c) $\mathbf{E} = \pm\frac{1}{4}(n+1)F^{-1}\mathbf{h}$. Then one would like to know if there is any other Randers metrics of constant flag curvature. Before we process to discuss this problem, let us give a brief description on the quantities $\mathbf{S}$ and $\mathbf{E}$. A more precise definition will be given in Section 2 below. Both $\mathbf{S}$ and $\mathbf{E}$ measure the average rate of changes of the geometry of tangent spaces $(T_pM, F_p)$. $\mathbf{S}$ and $\mathbf{E}$ are related by $\mathbf{E}_{\mathbf{y}}(\mathbf{u}, \mathbf{v}) = \frac{1}{2}\mathbf{S}_{y^iy^j}(\mathbf{y})u^iv^j$, $\forall \mathbf{u}, \mathbf{v} \in T_pM$. Thus (b) implies (c). $\mathbf{S}(\mathbf{y})$ is called the S-curvature and $\mathbf{E}_{\mathbf{y}} : T_pM \times T_pM \to \mathrm{R}$ is called the mean Berwald curvature (or simply the E-curvature) in the direction $\mathbf{y} \in T_pM$. $\mathbf{S}$ and $\mathbf{E}$ are said to be non-Riemannian, because they all vanish when the Finsler metric is Riemannian.

In this paper, we prove the following

**Theorem 1.1** *Let $F = \alpha + \beta$ be an $n$-dimensional Randers metric of constant Ricci curvature $\mathbf{Ric} = (n-1)\lambda F^2$. Suppose that $F$ is locally projectively flat and $\beta \not\equiv 0$. Then $\lambda \le 0$. If $\lambda = 0$, $F$ is locally Minkowskian. If $\lambda = -1/4$, $F$ can be expressed in the following form*

$$F = \frac{\sqrt{|\mathbf{y}|^2 - (|\mathbf{x}|^2|\mathbf{y}|^2 - \langle \mathbf{x}, \mathbf{y}\rangle^2)}}{1 - |\mathbf{x}|^2} \pm \frac{\langle \mathbf{x}, \mathbf{y}\rangle}{1 - |\mathbf{x}|^2} \pm \frac{\langle \mathbf{a}, \mathbf{y}\rangle}{1 + \langle \mathbf{a}, \mathbf{x}\rangle}, \qquad \mathbf{y} \in T_{\mathbf{x}}\mathrm{R}^n, \quad (2)$$

*where* $\mathbf{a} \in \mathrm{R}^n$ *is a constant vector with* $|\mathbf{a}| < 1$. *The Randers metric in (2) has the following properties*

(a) $\mathbf{K} = -1/4$,

(b) $\mathbf{S} = \pm\frac{1}{2}(n+1)F$,

(c) $\mathbf{E} = \pm\frac{1}{4}(n+1)F^{-1}\mathbf{h}$,

(d) *all geodesics of* $F$ *are straight lines,*

*where* $\mathbf{h}$ *denotes the angular metric of* $F$.

Note that for the Randers metric $F = \alpha + \beta$ in (2), the Riemannian metric $\alpha$ is the well-known Klein metric of constant curvature $\mathbf{K} = -1$, and the norm $\|\beta\|_{\mathbf{x}}$ of the 1-form $\beta$ with respect to $\alpha$ at $\mathbf{x}$ satisfies

$$1 - \|\beta\|_{\mathbf{x}}^2 = \frac{(1-|\mathbf{x}|^2)(1-|\mathbf{a}|^2)}{(1+\langle \mathbf{a}, \mathbf{x}\rangle)^2}.$$

Thus $F$ is defined on the unit ball $\mathrm{B}^n$. Moreover, it is always positively complete. Note that when $\mathbf{a} = 0$, the Finsler metric in (2) is the standard Funk metric in (1) on the unit ball $\mathrm{B}^n$.

By Theorem 1.1, every locally projectively flat Randers metric with $\mathbf{K} = 0$ must be locally Minkowskian. If the Randers metric is not locally projectively flat, then it is not necessarily locally Minkowskian unless it is positively complete [Sh3].

Finally, we mention several interesting locally projectively flat Finsler metrics of constant flag curvature. The Funk metrics and the Klein-Hilbert metric on a strongly convex domain in $\mathrm{R}^n$ are projectively flat with negative constant flag curvature. In 1995-96, R. Bryant constructed infinitely many locally projectively flat Finsler metrics on $\mathrm{S}^n$ with constant flag curvature $\mathbf{K} = 1$ [Br1][Br2]. Recently, the author has also discovered some interesting examples [Sh2], among them is the following Finsler metric on $\mathrm{B}^n$:

$$F(\mathbf{y}) := \frac{\Big(\sqrt{|\mathbf{y}|^2 - (|\mathbf{x}|^2|\mathbf{y}|^2 - \langle \mathbf{x}, \mathbf{y}\rangle^2)} + \langle \mathbf{x}, \mathbf{y}\rangle\Big)^2}{(1-|\mathbf{x}|^2)^2\sqrt{|\mathbf{y}|^2 - (|\mathbf{x}|^2|\mathbf{y}|^2 - \langle \mathbf{x}, \mathbf{y}\rangle^2)}}, \qquad \mathbf{y} \in T_{\mathbf{x}}\mathrm{B}^n = \mathrm{R}^n.$$

We show that the above Finsler metric $F$ is locally projectively flat with zero flag curvature $\mathbf{K} = 0$.

# 2 Preliminaries

Let $F$ be a Finsler metric on a manifold $M$. In a standard local coordinate system $(x^i, y^i)$ in $TM$, $F = F(x, y)$ is a function of $(x^i, y^i)$. Let

$$g_{ij}(x,y) := \frac{1}{2}[F^2]_{y^iy^j}(x,y)$$

and $(g^{ij}) := (g_{ij})^{-1}$. For a non-zero vector $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_p \in T_pM$, we obtain an inner product on $T_pM$,

$$\mathbf{g_y}(\mathbf{u}, \mathbf{v}) := g_{ij}(x, y)u^i v^j, \qquad \mathbf{u} = u^i \frac{\partial}{\partial x^i}|_p, \mathbf{v} = v^i \frac{\partial}{\partial x^i}|_p,$$

which satisfies that $\mathbf{g_y}(\mathbf{y}, \mathbf{y}) = F^2(\mathbf{y})$. Let

$$\mathbf{h_y}(\mathbf{u}, \mathbf{v}) := \mathbf{g_y}(\mathbf{u}, \mathbf{v}) - F^{-2}(\mathbf{y})\mathbf{g_y}(\mathbf{u}, \mathbf{y})\mathbf{g_y}(\mathbf{v}, \mathbf{y}), \qquad \mathbf{u}, \mathbf{v} \in T_pM.$$

$\mathbf{h_y}$ is called an angular form on $T_pM$ associated with $\mathbf{y}$.

The geodesics of $F$ are characterized locally by

$$\frac{d^2x^i}{dt^2} + 2G^i\Big(x, \frac{dx}{dt}\Big) = 0,$$

where

$$G^i(x, y) = \frac{1}{4} g^{ik}(x, y)\Big\{2\frac{\partial g_{pk}}{\partial x^q}(x, y) - \frac{\partial g_{pq}}{\partial x^k}(x, y)\Big\}y^p y^q.$$

The second variation of geodesics gives rise to a family of linear maps $\mathbf{R_y} = R^i{}_k(x, y)dx^k \otimes \frac{\partial}{\partial x^i}|_p : T_pM \to T_pM$, where $R^i{}_k = R^i{}_k(x, y)$ are given by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{3}$$

For any non-zero $\mathbf{y} \in T_pM$, $\mathbf{R_y}$ satisfies

$$\mathbf{R_y}(\mathbf{y}) = 0,$$

$$\mathbf{g_y}\Big(\mathbf{R_y}(\mathbf{u}),\ \mathbf{v}\Big) = \mathbf{g_y}\Big(\mathbf{u},\ \mathbf{R_y}(\mathbf{v})\Big), \qquad \mathbf{u}, \mathbf{v} \in T_pM.$$

The Ricci curvature $\mathbf{Ric}(\mathbf{y})$ is defined to be the trace of $\mathbf{R_y}$,

$$\mathbf{Ric}(\mathbf{y}) := R^m{}_m(x, y).$$

One can easily verify that for any tangent plane $P = \mathrm{span}\{\mathbf{y}, \mathbf{u}\} \subset T_pM$, the following quantity $\mathbf{K}(P, \mathbf{y})$ is well-defined,

$$\mathbf{K}(P, \mathbf{y}) := \frac{\mathbf{g_y}\Big(\mathbf{R_y}(\mathbf{u}),\ \mathbf{u}\Big)}{F^2(\mathbf{y})\mathbf{h_y}(\mathbf{u}, \mathbf{u})}.$$

$\mathbf{K}(P, \mathbf{y})$ is called the flag curvature of the "flag" $\{P, \mathbf{y}\}$.

A Finsler metric $F$ is said to be of constant flag curvature $\mathbf{K} = \lambda$, if for any $\mathbf{y} \in P \subset T_pM$,

$$\mathbf{K}(P, \mathbf{y}) = \lambda.$$

This is equivalent to the following identity,

$$R^i{}_k = \lambda\Big\{F^2\delta^i_k - FF_{y^k}y^i\Big\}. \tag{4}$$

If (4) holds, then $R^m_{\ m} = (n-1)\lambda F^2$. That is, if $\mathbf{K} = \lambda$, then $\mathbf{Ric} = (n-1)\lambda F^2$.

When $F = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, $R^i_{\ k} = R_j{}^i{}_{kl}(x)y^jy^l$, where $R_j{}^i{}_{kl}$ denote the coefficients of the usual Riemannian curvature tensor. In this case, the flag curvature $\mathbf{K}(P, \mathbf{y}) = \mathbf{K}(P)$ is independent of the direction $\mathbf{y} \in P$ and it is just the sectional curvature of $P \subset T_pM$. Thus the flag curvature is an analogue of the sectional curvature in Riemannian geometry.

There are many interesting non-Riemannian quantities in Finsler geometry such as the Cartan torsion, the Landsberg curvature and the Berwald curvature, etc. ([Sh1][Sh5]). In this paper, we will only discuss the E-curvature and the S-curvature which are closely related to the flag curvature.

Let $F$ be a Finsler metric on an $n$-dimensional manifold $M$ and $G^i = G^i(x, y)$ denote the geodesic coefficients of $F$ in a standard local coordinate system $(x^i, y^j)$ in $TM$. For any non-zero vector $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_p \in T_pM$, the E-curvature $\mathbf{E_y} : T_pM \times T_pM \to \mathrm{R}$ is a symmetric linear form defined by

$$\mathbf{E_y}(\mathbf{u}, \mathbf{v}) := \frac{1}{2}\frac{\partial^3 G^m}{\partial y^m \partial y^i \partial y^j}(x, y)u^iv^j, \tag{5}$$

where $\mathbf{u} = u^i \frac{\partial}{\partial x^i}|_p$ and $\mathbf{v} = v^i \frac{\partial}{\partial x^i}|_p \in T_pM$. $\mathbf{E_y}$ has the following properties

$$\mathbf{E}_{\lambda\mathbf{y}}(\mathbf{u}, \mathbf{v}) = \lambda^{-1}\mathbf{E_y}(\mathbf{u}, \mathbf{v}), \qquad \mathbf{E_y}(\mathbf{y}, \mathbf{v}) = 0,$$

where $\lambda > 0$.

The volume form of $F$, $dV_F = \sigma(x)dx^1 \cdots dx^n$, is defined by

$$\sigma(x) := \frac{\omega_n}{\mathrm{Vol}\{(y^i) \in \mathrm{R}^n \mid F(y^i \frac{\partial}{\partial x^i}|_x) < 1\}},$$

where $\omega_n$ denotes the volume of the unit ball in Euclidean space $\mathrm{R}^n$. For any $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_p \in T_pM$, let

$$\tau(\mathbf{y}) = \ln\Big[\frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma(x)}\Big].$$

$\tau(\mathbf{y})$ is an invariant of $F_p$ on $T_pM$. We know that at any point $p \in M$, $\tau(\mathbf{y}) = 0$, $\forall \mathbf{y} \in T_pM$ if and only if $F_p$ is Euclidean ([Sh5]). The S-curvature is defined to be the rate of changes of $\tau$ along geodesics, i.e.,

$$\mathbf{S}(\mathbf{y}) = \frac{d}{dt}\Big[\tau(\dot{c}(t))\Big]_{t=0}, \qquad \mathbf{y} \in T_pM,$$

where $c(t)$ is the geodesic with $c(0) = p$ and $\dot{c}(0) = \mathbf{y}$. In a local coordinate system,

$$\mathbf{S}(\mathbf{y}) = \frac{\partial G^m}{\partial y^m}(x, y) - \frac{y^m}{\sigma(x)}\frac{\partial \sigma}{\partial x^m}(x), \qquad \mathbf{y} = y^i\frac{\partial}{\partial x^i}|_p \in T_pM. \tag{6}$$

It follows from (5) and (6) that

$$\mathbf{E_y}(\mathbf{u},\mathbf{v}) = \frac{1}{2}\mathbf{S}_{y^iy^j}(\mathbf{y})u^iv^j. \tag{7}$$

$F$ is said to have constant E-curvature $c$, denoted by $\mathbf{E} = (n+1)c\,F^{-1}\mathbf{h}$, if for any $p \in M$ and any non-zero vector $\mathbf{y} \in T_pM$,

$$\mathbf{E_y}(\mathbf{u},\mathbf{v}) = (n+1)cF^{-1}(\mathbf{y})\mathbf{h_y}(\mathbf{u},\mathbf{v}), \qquad \mathbf{u},\mathbf{v} \in T_pM.$$

$F$ is said to have constant S-curvature $c$, denoted by $\mathbf{S} = (n+1)c\,F$, if for any $p \in M$ and any non-zero vector $\mathbf{y} \in T_pM$,

$$\mathbf{S}(\mathbf{y}) = (n+1)cF(\mathbf{y}).$$

From (7), we see that $\mathbf{S} = (n+1)cF$, then $\mathbf{E} = \frac{1}{2}(n+1)cF^{-1}\mathbf{h}$. The converse might not be true, although no example has been found yet. See [Sh1][Sh5] for more details on the E-curvature and the S-curvature.

It is worth mentioning the recent result by X. Mo [Mo] that if an $n$-dimensional Finsler metric is isotropic, i.e., the flag curvature $\mathbf{K}(P,\mathbf{y}) = \mathbf{K}(\mathbf{y})$ is independent of $P$ containing $\mathbf{y}$, and it has constant S-curvature, $\mathbf{S} = (n+1)cF$, then the flag curvature $\mathbf{K}(\mathbf{y}) = \mathbf{K}(p)$ is independent of $\mathbf{y} \in T_pM$ for any point $p \in M$ and $\mathbf{K}(p) = \mathit{constant}$ when $n \geq 3$.

Randers metrics are among the simplest non-Riemannian Finsler metrics, because that many well-known geometric quantities are computable for Randers metrics.

Let $F = \alpha + \beta$ be a Randers metric on a manifold $M$, where

$$\alpha(y) = \sqrt{a_{ij}(x)y^iy^j}, \qquad \beta(y) = b_i(x)y^i$$

with $\|\beta\|_p := \sup_{y\in T_pM}\beta(y)/\alpha(y) < 1$. Define $b_{i|j}$ by

$$b_{i|j}\theta^j := db_i - b_j\theta_i^{\ j},$$

where $\theta^i := dx^i$ and $\theta_i^{\ j} := \bar{\Gamma}^j_{ik}dx^k$ denote the Levi-Civita connection forms of $\alpha$. Let

$$r_{ij} := \frac{1}{2}\Big(b_{i|j} + b_{j|i}\Big), \qquad s_{ij} := \frac{1}{2}\Big(b_{i|j} - b_{j|i}\Big),$$

$$s^i_{\ j} := a^{ih}s_{hj}, \qquad s_j := b_is^i_{\ j}, \qquad e_{ij} := r_{ij} + b_is_j + b_js_i.$$

Then $G^i$ are given by

$$G^i = \bar{G}^i + \frac{e_{00}}{2F}y^i - s_0y^i + \alpha s^i_{\ 0}, \tag{8}$$

where $e_{00} := e_{ij}y^iy^j$, $s_0 := s_iy^i$ and $s^i_{\ 0} := s^i_{\ j}y^j$. See [AIM]. Let

$$P := \frac{e_{00}}{2F} - s_0.$$

The S-curvature is given by

$$\mathbf{S} = (n+1)\Big\{P - d\rho(\mathbf{y})\Big\} = (n+1)\Big\{\frac{e_{00}}{2F} - s_0 - d\rho(\mathbf{y})\Big\}, \tag{9}$$

where $\rho := \ln\sqrt{1 - \|\beta\|^2}$. See (5.34) in [Sh1].

## 3 Projectively Flat Randers Metrics

In this section, we are going to study locally projectively flat Randers metrics with constant Ricci curvature.

Let $F = \alpha + \beta$ be a Randers metric on an $n$-dimensional manifold $M$. First we assume that $F$ is locally projectively flat. Let

$$\Phi := b_{i|j} y^i y^j, \qquad \Psi := b_{i|j|k} y^i y^j y^k.$$

According to [HaIc], $\beta$ must be closed , that is,

$$s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}) = 0.$$

Hence $s_i = 0$ and

$$P = \frac{e_{00}}{2F} - s_0 = \frac{r_{00} + 2\beta s_0}{2F} - s_0 = \frac{r_{00}}{2F} = \frac{\Phi}{2F}.$$

It follows from (8) that

$$G^i = \bar{G}^i + P y^i.$$

This implies that the Riemann curvature $\mathbf{R_y} = R^i{}_k dx^k \otimes \frac{\partial}{\partial x^i}$ of $F$ is related to the Riemann curvature $\bar{\mathbf{R}}_\mathbf{y} = \bar{R}^i{}_k dx^k \otimes \frac{\partial}{\partial x^i}$ of $\alpha$ by

$$R^i{}_k = \bar{R}^i{}_k + \Big(3\Big(\frac{\Phi}{2F}\Big)^2 - \frac{\Psi}{2F}\Big)\Big\{\delta^i_k - \frac{F_{y^k}}{F} y^i\Big\} + \tau_k\, y^i, \tag{10}$$

where

$$\tau_k = \frac{1}{F}(b_{i|j|k} - b_{i|k|j}) y^i y^j = -\frac{1}{F} b_j \bar{R}^j{}_k.$$

Note that the last term $\tau_k y^i$ in [Sh1, (8.54)] is missing.

From (10), we obtain a relationship between the Ricci curvature $\mathbf{Ric} = (n-1)R$ of $F$ and the Ricci curvature $\overline{\mathbf{Ric}} = (n-1)\bar{R}$ of $\alpha$.

$$R = \bar{R} + 3\Big(\frac{\Phi}{2F}\Big)^2 - \frac{\Psi}{2F}. \tag{11}$$

Since $\beta$ is closed , $F$ must be pointwise projectively equivalent to $\alpha$. Thus $\alpha$ must be locally projectively flat. This implies that $\alpha$ is of constant curvature $\bar{\mathbf{K}} = \mu$,

$$\bar{R}^i{}_k = \mu\alpha^2 \Big\{\delta^i_k - \frac{\alpha_{y^k}}{\alpha} y^i\Big\}. \tag{12}$$

In particular, $\bar{R} = \mu\alpha^2$. Therefore $F$ has constant Ricci curvature $\mathbf{Ric} = (n-1)\lambda F^2$ if and only if

$$\mu\alpha^2 + 3\Big(\frac{\Phi}{2F}\Big)^2 - \frac{\Psi}{2F} = \lambda F^2. \tag{13}$$

**Remark 3.1** It follows from (10) and (12) that

$$R^i{}_k = \lambda F^2 \delta^i_k + \tilde{\tau}_k y^i,$$

where $\tilde{\tau}_k$ are some local functions on $TM$ with $\tilde{\tau}_k y^k = -\lambda F^2$. Thus $F$ must have constant flag curvature $\mathbf{K} = \lambda$.

**Lemma 3.2** *Let $F = \alpha + \beta$ be a locally projectively flat Randers metric. Suppose that $F$ has constant Ricci curvature, $\mathbf{Ric} = (n-1)\lambda F^2$. Then*

(a) *$F$ has constant flag curvature $\mathbf{K} = \lambda \leq 0$,*

(b) *$\alpha$ has constant curvature, $\bar{\mathbf{K}} = \mu = 4\lambda$, and*

(c)
$$e_{00} = \pm 2\sqrt{-\lambda}(\alpha^2 - \beta^2). \tag{14}$$

*where $e_{00} := e_{ij}y^i y^j$.*

*Proof*: It follows from (13) that

$$\mu\alpha^2(\alpha+\beta)^2 + \frac{3}{4}\Phi^2 - \frac{1}{2}\Psi(\alpha+\beta) = \lambda(\alpha+\beta)^4. \tag{15}$$

(15) is equivalent to the following two equations

$$\frac{3}{4}\Phi^2 = \frac{1}{2}\beta\Psi + (\lambda-\mu)\alpha^4 + (6\lambda-\mu)\alpha^2\beta^2 + \lambda\beta^4 \tag{16}$$

$$\frac{1}{2}\Psi = (2\mu - 4\lambda)\alpha^2\beta - 4\lambda\beta^3. \tag{17}$$

Plugging (17) into (16) yields

$$\frac{3}{4}\Phi^2 = (\lambda-\mu)\alpha^4 + (2\lambda+\mu)\alpha^2\beta^2 - 3\lambda\beta^4. \tag{18}$$

Differentiating (18), we obtain

$$\frac{3}{2}\Phi\; b_{i|j|k}y^i y^j = 2(2\lambda+\mu)\alpha^2\beta\; b_{i|k}y^i - 12\lambda\beta^3\; b_{i|k}y^i. \tag{19}$$

Contracting (19) with $y^k$ yields

$$\frac{3}{2}\Phi\Psi = 2(2\lambda+\mu)\alpha^2\beta\Phi - 12\lambda\beta^3\Phi. \tag{20}$$

Plugging (17) into (20) yields

$$4(\mu - 4\lambda)\Phi\alpha^2\beta^2 = 0.$$

First, we assume that $\Phi\beta^2 \not\equiv 0$. Then

$$\mu = 4\lambda.$$

Plugging it into (18) yields

$$\Phi^2 = -4\lambda(\alpha^2 - \beta^2)^2. \tag{21}$$

We see that $\lambda \leq 0$ and

$$\Phi = 2\epsilon\sqrt{-\lambda}(\alpha^2 - \beta^2). \tag{22}$$

where $\epsilon = \pm 1$.

Now we assume that $\Phi\beta^2 \equiv 0$. We are going to show that (22) still holds with $\lambda = 0$. Let $\mathcal{U} := \{p \in M \mid \beta_p \not\equiv 0\}$. By assumption $\beta \not\equiv 0$, we know that $\mathcal{U}$ is a non-empty open subset of $M$. For a point $p \in \mathcal{U}$, for any $\mathbf{y} \in T_pM$ with $\beta_p(\mathbf{y}) \neq 0$, we have

$$\Psi_p(\mathbf{y}) = 0.$$

Since the hyperplane $\beta_p = 0$ has zero measure in $T_pM$, we conclude that $\Phi_p = 0$. Then $\beta$ is parallel and $\Psi = 0$ on $\mathcal{U}$. It follows from (15) that the following holds on $\mathcal{U}$,

$$\mu\alpha^2 = \lambda(\alpha + \beta)^2.$$

We conclude that $\mu = 4\lambda = 0$. Now at any point in $M$, (16) and (17) simplify to

$$\frac{3}{4}\Phi^2 = \frac{1}{2}\beta\Psi, \qquad \frac{1}{2}\Psi = 0.$$

We conclude that $\Phi \equiv 0$ on $M$ and (22) holds everywhere on $M$ with $\lambda = 0$. Since $\beta$ is closed, $\Phi = e_{00}$. Thus (22) is equivalent to (14). Q.E.D.

**Remark 3.3** During the preparation of this paper, the author was informed that D. Bao and C. Robles has shown that for any Randers metric with constant Ricci curvature, there is a scalar function $c = c(x)$ such that $e_{00} = 2c(\alpha^2 - \beta^2)$. See [BaRo]. But it is not clear if $c$ is a constant.

# 4 Proof of Theorem 1.1

First, we assume that $\lambda = 0$. Then by Lemma 3.2, $\alpha$ is flat and $\Phi = 0$. This implies that $b_{i|j} = 0$. Thus $\beta$ is parallel with respect to $\alpha$. In this case, $F$ is flat with $G^i(x,y) = \bar{G}^i(x,y)$ quadratic in $y$. We conclude that $F$ is locally Minkowskian.

Now we assume that $\lambda < 0$. Then $\alpha$ has negative constant curvature $\mu = 4\lambda$. By scaling, we may assume that $\alpha$ has constant curvature $\mu = -1$, hence $\lambda = -1/4$. It follows from (22) that

$$b_{i|j} = \epsilon(a_{ij} - b_ib_j). \tag{23}$$

We express $\alpha = \sqrt{a_{ij}y^i y^j}$ in the following Klein form,

$$\alpha = \frac{\sqrt{|\mathbf{y}|^2 - (|\mathbf{x}|^2|\mathbf{y}|^2 - \langle \mathbf{x}, \mathbf{y}\rangle^2)}}{1-|\mathbf{x}|^2}, \qquad \mathbf{y} \in T_{\mathbf{x}}\mathrm{B}^n. \tag{24}$$

We can also express $\beta = b_i y^i$ in the gradient form,

$$\beta = \epsilon \, \frac{\langle \mathbf{x}, \mathbf{y}\rangle}{1-|\mathbf{x}|^2} + \epsilon \, d\varphi(\mathbf{y}),$$

where $\varphi$ is a scalar function on $\mathrm{B}^n$ and $\epsilon = \pm 1$ is same as in (22) and (23). The Christoffel symbols of $\alpha$ are given by

$$\bar{\Gamma}^i_{jk} = \frac{x^k \delta^i_j + x^j \delta^i_k}{1-|\mathbf{x}|^2}.$$

The covariant derivatives of $\beta$ with respect to $\alpha$ are given by

$$b_{i|j} = \epsilon \Big\{ \frac{\partial^2 \varphi}{\partial x^i \partial x^j} + \frac{1}{1-|\mathbf{x}|^2}\Big( \delta_{ij} - x^i \frac{\partial \varphi}{\partial x^j} - x^j \frac{\partial \varphi}{\partial x^i}\Big)\Big\},$$

and

$$a_{ij} - b_i b_j = \frac{1}{1-|\mathbf{x}|^2}\Big( \delta_{ij} - x^i \frac{\partial \varphi}{\partial x^j} - x^j \frac{\partial \varphi}{x^i}\Big) - \frac{\partial \varphi}{\partial x^i}\frac{\partial \varphi}{\partial x^j}.$$

It follows from (23) that

$$\frac{\partial^2 \varphi}{\partial x^i \partial x^j} + \frac{\partial \varphi}{\partial x^i}\frac{\partial \varphi}{\partial x^j} = 0. \tag{25}$$

Let $f = \exp(\varphi)$. Then (25) simplifies to

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = 0. \tag{26}$$

Thus $f$ is a linear function

$$f = c(1 + \langle \mathbf{a}, \mathbf{x}\rangle), \qquad c > 0.$$

We obtain that

$$\varphi = \ln c + \ln(1 + \langle \mathbf{a}, \mathbf{x}\rangle).$$

Finally, we find the most general solution for $\beta$,

$$\beta = \epsilon \frac{\langle \mathbf{x}, \mathbf{y}\rangle}{1-|\mathbf{x}|^2} + \epsilon \frac{\langle \mathbf{a}, \mathbf{y}\rangle}{1+\langle \mathbf{a}, \mathbf{x}\rangle}, \qquad \mathbf{y} \in T_{\mathbf{x}}\mathrm{B}^n. \tag{27}$$

Let $F = \alpha + \beta$ be a Randers metric, where $\alpha$ and $\beta$ are given by (24) and (27), respectively. By the above argument, (23) holds. Thus

$$\Phi := b_{i|j} y^i y^j = \epsilon(\alpha^2 - \beta^2). \tag{28}$$

Since $\beta$ is closed and $\alpha$ has constant curvature $\mu = -1$, $F$ is locally projectively flat. It follows from (23) that

$$b_{i|j|k} = -\epsilon(b_{i|k}b_j + b_i b_{j|k}) = -a_{ik}b_j - a_{jk}b_i + 2b_i b_j b_k.$$

This gives

$$\Psi := b_{i|j|k}y^i y^j y^k = -2\beta(\alpha^2 - \beta^2). \tag{29}$$

Using (28) and (29), we compute the terms in (10) as follows.

$$3\Big(\frac{\Phi}{2F}\Big)^2 - \frac{\Psi}{2F} = \alpha^2 - \frac{1}{4}F^2 \tag{30}$$

$$\tau_k = \frac{\alpha}{F}(\alpha b_k - \alpha_{y^k}\beta). \tag{31}$$

Plugging (30) and (31) into (10) yields

$$R^i{}_k = -\frac{1}{4}F^2\Big\{\delta^i_k - \frac{F_{y^k}}{F}y^i\Big\}.$$

Thus $F$ has constant flag curvature $\mathbf{K} = -1/4$.

Now we compute the S-curvature. By a direct computation, we obtain

$$1 - \|\beta\|^2_{\mathbf{x}} = \frac{(1-|\mathbf{x}|^2)(1-|\mathbf{a}|^2)}{\Big(1 + \langle \mathbf{a}, \mathbf{x}\rangle\Big)^2}. \tag{32}$$

Since $F$ is a Randers metric defined nearby the origin, we conclude that

$$|\mathbf{a}| < 1.$$

By (32), $\rho := \ln\sqrt{1 - \|\beta\|^2_\alpha}$ is given by

$$\rho = \ln\sqrt{1-|\mathbf{x}|^2} - \ln(1 + \langle \mathbf{a}, \mathbf{x}\rangle) + \ln\sqrt{1-|\mathbf{a}|^2}.$$

We obtain

$$d\rho(\mathbf{y}) = -\frac{\langle \mathbf{x}, \mathbf{y}\rangle}{1-|\mathbf{x}|^2} - \frac{\langle \mathbf{a}, \mathbf{y}\rangle}{1+\langle \mathbf{a}, \mathbf{x}\rangle} = -\epsilon\beta.$$

On the other hand,

$$P = \frac{\Phi}{2F} = \frac{\epsilon}{2}(\alpha - \beta).$$

Plugging them into (9), we obtain

$$\mathbf{S} = \frac{\epsilon}{2}(n+1)F.$$

Differentiating $\mathbf{S}$ and using (7), we obtain

$$\mathbf{E} = \frac{\epsilon}{4}(n+1)F^{-1}h.$$

This completes the proof of Theorem 1.1. Q.E.D.

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