CURVATURE, DIAMETER AND BOUNDED BETTI NUMBERS

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Abstract. In this paper, we introduces the notion of bounded Betti numbers, and show that the bounded Betti numbers of a closed Riemannian n-manifold (M, g) with $\mathrm{Ric}(M) \geq -(n-1)$ and $\mathrm{Diam}(M) \leq D$ are bounded by a number depending on D and n. We also show that there are only finitely many isometric isomorphism types of bounded cohomology groups $(\hat{H}^*(M), \|\cdot\|_\infty)$ among closed Riemannian manifolds (M, g) with $K(M) \geq -1$ and $\mathrm{Diam}(M) \leq D$.

Key words: Diameter, Ricci curvature, sectional curvature, bounded cohomology and bounded Betti number

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1. Introduction

One of the fundamental problems in Riemannian geometry is to understand the relation between the topology and geometry of a Riemannian manifold.

Every closed manifold M admits a Riemannian metric g with the following curvature bound

$$K(M) \geq -1. \tag{1.1}$$

Thus the curvature bound in (1.1) alone does not have any implication for the topological structure of the manifold. With this normalized metric, the topology depends on the "size" of the manifold. The diameter is one of the global geometric quantities to measure the manifold. Assume that

$$\mathrm{Diam}(M) \leq D. \tag{1.2}$$

It was proved by M. Gromov [G1] that for any Riemannian n-manifold (M, g) satisfying (1.1) and (1.2), the total Betti number (with respect to any field) is bounded, namely,

$$\sum_{i=0}^{n} \beta^i(M) \leq C(n, D).$$

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In this paper, we are interested in a much weaker curvature bound

$$\text{Ric}(M) \geq -(n-1). \tag{1.3}$$

First of all, according to Sha-Yang's examples [SY], the Gromov Betti Number Theorem is not true for Riemannian n-manifolds satisfying (1.2) and (1.3). Nevertheless, the first Betti number is still bounded, i.e.,

$$\beta^1(M) \leq C(n, D). \tag{1.4}$$

This is due to M. Gromov [GLP] and T. Gallot [GT].

Besides the ordinary Betti number, what topological invariants are still controlled by the curvature bounds (1.1) or (1.3) ? To answer this question, we consider the bounded cohomology groups $\hat{\mathrm{H}}^*(M)$. The natural inclusion $I : \hat{C}^*(M) \to C^*(M)$ induces a homomorphism $\iota : \hat{\mathrm{H}}^*(M) \to \mathrm{H}^*(M)$ with image $\tilde{\mathrm{H}}^*(M) := \iota\Big[\hat{\mathrm{H}}^*(M)\Big]$ (see Section 2 below for details). Put

$$\tilde{\beta}^i(M) := \dim \tilde{\mathrm{H}}^i(M).$$

Clearly,

$$\tilde{\beta}^*(M) \leq \beta^*(M).$$

We call $\tilde{\beta}^i(M)$ the ith *bounded Betti numbers* of M. Below are some important examples:

(1)(Gromov) For any closed manifold M, $\hat{\mathrm{H}}^1(M) = 0$, hence $\tilde{\beta}^1(M) = 0$;

(2)(Thurston) For any closed manifold of negative curvature, $\tilde{\mathrm{H}}^k(M) = \mathrm{H}^k(M)$ for all $k \geq 2$, hence $\tilde{\beta}^k(M) = \beta^k(M)$ for $k \geq 2$;

(3) (Trauber) If $\pi_1(M)$ is amenable, then $\hat{\mathrm{H}}^*(M) = 0$, hence $\tilde{\beta}^*(M) = 0$.

The bounded Betti numbers behave quite differently from the ordinary Betti numbers. In particular, the Poincaré duality for $\beta^*(M)$ does not hold for $\tilde{\beta}^*(M)$. By [Gr3], we know that the bounded cohomology groups $\hat{\mathrm{H}}^*(M)$ are completely determined by $\pi_1(M)$. However, the bounded Betti numbers $\tilde{\beta}^*(M)$ depend not only on $\pi_1(M)$, but also on the higher dimensional topological structure of M, when $\pi_1(M)$ is not amenable.

In this paper, we shall prove the following

Theorem A. *There is a constant $C(n, D)$ depending only on n, D such that for closed Riemannian n-manifold (M, g) satisfying Ric$(M) \geq -(n-1)$ and Diam$(M) \leq D$, the total bounded Betti number is bounded*

$$\sum_{i=0}^{n} \tilde{\beta}^i(M) \leq C(n, D). \tag{1.5}$$

Our proof is based on the following conjecture by M. Gromov. For a given n, there is a positive number $r = r(n) > 0$ such that for any metric ball $B(x, \rho)$ of radius $\rho < r$

in a complete Riemannian n-manifold (M, g) with $\mathrm{Ric}(M) \geq -(n-1)$, the image of the natural homomorphism $i : \pi_1(B(x, \rho), x) \to \pi_1(M, x)$ is an amenable group.

It is proved by Gromov that there is a small number $\epsilon(n) > 0$ such that if a closed oriented n-manifold M satisfies (1.3) and

$$\sup_{p \in M} \mathrm{vol}(B(p, 1)) < \epsilon(n), \tag{1.6}$$

then there is a map f from M into an $(n-1)$-dimensional polyhedron P^{n-1} such that the pre-image of any star neighborhood is contained in an *amenable* open subset. Then he concludes that $\|M\| = 0$, which is equivalent to that $\tilde{\beta}^n(M) = 0$. Below is a generalization of Gromov's result.

Theorem B. *Let $n \in \mathbb{Z}$ and X a compact metric space. There is a small constant $\epsilon = \epsilon(n, X) > 0$ such that for any closed Riemannian n-manifold (M, g) satisfying Ric$(M) \geq -(n-1)$, if $d_{GH}(M, X) < \epsilon$, then $\tilde{\beta}^i(M) = 0$ for $i > \dim X$.*

Again, our proof is based on the Gromov conjecture.

Without the Gromov conjecture being true, the upper bound $C(n, D)$ in (1.5) and the small constant $\epsilon(n, X)$ in Theorem B would depend on the amenability radius. See Lemmas 3.1 and 3.2 in Section 3 below. The Gromov conjecture was proved by Fukaya-Yamaguchi [6] under the sectional curvature bound $K(M) \geq -1$. Then it is shown to be affirmative by Cheeger-Colding [2] under the Ricci curvature bound $\mathrm{Ric} \geq -(n-1)$. Cheeger-Colding verifies all listed conditions in [6] for the Gromov conjecture to be true under a lower Ricci curvature bound. At the time when the paper was written, there was no simple direct argument for the Gromov conjecture.

Under a stronger curvature bound, the bounded cohomology groups $\hat{\mathrm{H}}^*(M)$ of M are controlled too. More precisely,

Theorem C. *There are only finitely many isometric isomorphism types of bounded cohomology groups $(\hat{H}^*(M), \|\cdot\|_\infty)$ among closed Riemannian n-manifolds (M, g) satisfying $K(M) \geq -1$ and Diam$(M) \leq D$.*

From Theorem C, we conclude that there is a constant $C(n, D)$ such that if a closed Riemannian n-manifold (M, g) satisfying $K(M) \geq -1$ and $\mathrm{Diam}(M) \leq D$, then

$$\dim \hat{\mathrm{H}}^*(M) \leq C(n, D), \tag{1.7}$$

provided that $\dim \hat{\mathrm{H}}^*(M) < \infty$. However, there are closed manifolds M with $\dim \hat{\mathrm{H}}^*(M) = \infty$. Take $M = S^1 \times S^3 \# S^1 \times S^3$. The fundamental group $\pi_1(M) = Z * Z$. Thus the second bounded cohomology group $\hat{\mathrm{H}}^2(M)$ is not finitely generated. This example is given to us by Fuquan Fang. We do not know whether or not there exist closed manifolds with non-amenable fundamental group such that $\dim \hat{\mathrm{H}}^i(M) < \infty$ for some $i > 1$. Recently, M. Gromov told the first author that he is doubt about the existence of such manifolds.

2. Preliminaries

In this section, we shall summarize some of Gromov's results [G3] which will be needed in our proof.

Let M be a connected topological space. Denote by Σ the set of all singular simplices $\sigma : \Delta \to M$. The standard pseudo-L^∞ norm on $C^*(M)$ is defined by

$$\|c\|_\infty := \sup_{\sigma \in \Sigma} |c(\sigma)|.$$

Consider the subcomplex $\hat{C}^*(M)$ of bounded singular cochains of M. The homology groups $\hat{\mathrm{H}}^*(M)$ of $\hat{C}^*(M)$ is called the *bounded cohomology* of M. Let $\|\cdot\|_\infty^b$ denote the induced norm on $\hat{\mathrm{H}}^*(M)$. Then $(\hat{\mathrm{H}}^*(M), \|\cdot\|_\infty^b)$ becomes a normed space. The natural inclusion $I : \hat{C}^*(M) \to C^*(M)$ induces a homomorphism

$$\iota : \hat{\mathrm{H}}^*(M) \to \mathrm{H}^*(M). \tag{2.1}$$

Put

$$\tilde{\mathrm{H}}^*(M) := \iota\Big[\hat{\mathrm{H}}^*(M)\Big].$$

$\tilde{\mathrm{H}}^*(M)$ is called the *bounded part* of $\mathrm{H}^*(M)$ (see [Br]). Cohomology classes in $\tilde{\mathrm{H}}^*(M)$ are called *bounded classes* of $\mathrm{H}^*(M)$. Put

$$\|\alpha\|_\infty := \inf_{\beta \in I^{-1}(\alpha)} \|\beta\|_\infty^b. \tag{2.2}$$

Then $(\tilde{\mathrm{H}}^*(M), \|\cdot\|_\infty)$ is a normed space.

The most important fact in the bounded cohomology theory is that the normed space $(\hat{\mathrm{H}}^*(M), \|\cdot\|_\infty^b)$ actually depends only on the fundamental group $\pi_1(M)$ ([G3]) and $\hat{\mathrm{H}}^*(M) = 0$ for connected closed manifolds with $\pi_1(M)$ amenable.

Below, we shall sketch Gromov's ideas to prove the above fact. One is referred to [Br][I] for different arguments. Although Gromov's theory is for general topological spaces, we shall focus on closed manifolds.

First, Gromov introduced a notion of multipcomplex. A *(simplicial) multicomplex* is defined as a set K divided into the union of closed affine simplices $\Delta_\sigma \subset K$, $\sigma \in I$ such that the intersection of any two simplices $\Delta_\sigma \cap \Delta_\tau$ is a subcomplex in both Δ_σ and Δ_τ. The set K with the weakest topology which agrees with the decomposition $K = \cup_{\sigma \in I} \Delta_\sigma$ is denoted by $|K|$. The union of all i-dimensional simplices in K is called the *i-skeleton* of K, denoted by $K^i \subset K$.

Let M be an n-dimensional closed manifold. Denote by Σ the set of all singular simplices $\sigma : \Delta^i \to M$, $i = 0, 1, 2, \cdots$, which are injective on the vertices of the standard (oriented) i-simplex Δ^i. Take one copy of Δ^i for each σ, denoted by Δ^i_σ, and put $K := \cup_{\sigma \in \Sigma} \Delta^i_\sigma$. This union has a natural structure of a multicomplex such that the canonical map: $S : |K| \to M$

defined by $S|_{\Delta^i_\sigma} := \sigma : \Delta^i_\sigma \to M$, is continuous. Gromov proves that S is a weak homotopy equivalence. The multicomplex K is *large and complete* in the sense that every component of K has infinitely many vertices, and every continuous map $f : \Delta^i \to K$ is homotopic, relative to $\partial\Delta^i$, to a simplicial embedding $g : \Delta^i \to K$, provided $f|_{\partial\Delta^i} : \partial\Delta^i \to K$ is a simplicial embedding.

For the multicomplex K constructed above, there is another natural notion of bounded cohomology $\hat{\mathrm{H}}^*_a(K)$. Let $\hat{C}^*_a(K)$ denote the complex of bounded antisymmetric real cochains c, that is, $c(\Delta^i_\sigma) = -c(\Delta^i_{\sigma\circ\delta})$ for any orientation-preserving affine isomorphism $\delta : \Delta^i \to \Delta^i$. Then $\hat{\mathrm{H}}^*_a(K)$ is defined to be the homology group of $\hat{C}^*_a(K)$ with the natural pseudo-norm L_∞. Define a homomorphism $h : \hat{C}^i(M) \to \hat{C}^i_a(K)$ by

$$h(\Delta^i_\sigma) := \frac{1}{(i+1)!}\sum_\delta [\delta] c(\sigma\circ\delta). \tag{2.3}$$

Gromov asserts that h induces an isometric isomorphism

$$h^* : \hat{\mathrm{H}}^i(M) \to \hat{\mathrm{H}}^i_a(K). \tag{2.4}$$

In order to prove the fact that $\hat{\mathrm{H}}^*(M)$ actually depend only on $\pi_1(M)$, Gromov introduces a large and complete subcomplex $i : \tilde{K} \hookrightarrow K$ with the following properties:

(i) each continuous map of a simplex Δ^i into $\tilde{K}$, whose restriction to the boundary is a simplicial embedding, is homotopic relative to the boundary $\partial\Delta^i$ to at most one simplicial embedding $\Delta^i \to K$.

(ii) the natural inclusion $i : \tilde{K} \hookrightarrow K$ is a homotopy equivalence. Hence it induces an isometric isomorphism

$$i^* : \hat{\mathrm{H}}^*(K) \to \hat{\mathrm{H}}^*_a(\tilde{K}). \tag{2.5}$$

A subcomplex $\tilde{K}$ with these properties exists and is uniquely determined, up to an simplicial isomorphism, by the homotopy type of K. $\tilde{K}$ is called a *minimal model* of K.

Fix a minimal model $\tilde{K}$ of K. Let $\Gamma_1 = \Gamma_1(\tilde{K})$ denote the group of simplicial automorphisms of $\tilde{K}$ which are homotopic to the identity and keeps the 1-skeleton of $\tilde{K}$ fixed. Then $\tilde{K}_1 := \tilde{K}/\Gamma_1$ is a $K(\pi, 1)$ multicomplex with $\pi = \pi_1(\tilde{K}) = \pi_1(M)$ and the projection $p : \tilde{K} \to \tilde{K}_1$ induces an isomorphism between fundamental groups. In particular, the projection $p : \tilde{K} \to \tilde{K}_1$ induces an isometric isomorphism

$$p^* : \hat{\mathrm{H}}^*_a(\tilde{K}_1) \to \hat{\mathrm{H}}^*_a(\tilde{K}). \tag{2.6}$$

In virtue of (2.4)-(2.6), one can conclude that

$$\Phi := p^{*-1} \circ i^* \circ h^* : \hat{\mathrm{H}}^*(M) \to \hat{\mathrm{H}}^*_a(\tilde{K}_1) \tag{2.7}$$

is an isometric isomorphism. Thus Gromov concludes that the normed cohomology groups $\hat{\mathrm{H}}^*(M)$ depend only on $\pi_1(M)$.

Let $\tilde{\Gamma} := \oplus_{x\in\tilde{K}_1^0}\pi_1(\tilde{K}_1, x)$. The group $\tilde{\Gamma}$ acts on $\tilde{K}_1$ in a natural way. Assume that $\pi_1(M)$ is amenable, then $\tilde{\Gamma}$ is amenable. The standard averaging process leads to the following remarkable conclusion: $\hat{\mathrm{H}}^*(M) = \hat{\mathrm{H}}^*_a(\tilde{K}) = 0$. By a similar argument, one can show that the amenable normal subgroups of $\pi_1(M)$ make no contributions to the bounded cohomology $\hat{\mathrm{H}}^*(M)$. More precisely, we have the following

Lemma 2.1. *Let $\Gamma \subset \pi_1(M)$ be a normal amenable subgroup. Then Γ induces an action $\tilde{\Gamma}$ on $\tilde{K}_1$ such that $\tilde{K}_1/\tilde{\Gamma}$ is a multicomplex of $K(\pi, 1)$ type with $\pi = \pi_1(M)/\Gamma$ and $\hat{H}^*(M)$ is isometric isomorphic to $\hat{H}^*_a(\tilde{K}_1/\tilde{\Gamma})$.*

Consider a class $\mathcal{M}$ of certain closed n-manifolds. Let

$$\mathcal{M}_\pi := \Big\{\pi_1(M),\ M \in \mathcal{M}\Big\}/\sim$$
$$\hat{\mathcal{M}}^i := \Big\{(\hat{\mathrm{H}}^i(M), \|\cdot\|^b_\infty),\ M \in \mathcal{M}\Big\}$$

where $\pi_1(M) \sim \pi_1(M')$ if and only if there are normal amenable subgroups $N \triangleleft \pi_1(M)$ and $N' \triangleleft \pi_1(M')$ such that $\pi_1(M)/N \approx \pi_1(M')/N'$. Suppose that there are only finitely many isomorphism types of $\pi_1(M)$ in $\mathcal{M}_\pi$. By Lemma 2.1, one can conclude that there are only finitely isometric isomorphism types of normed spaces $(\hat{\mathrm{H}}^*(M), \|\cdot\|^b_\infty)$ in $\hat{\mathcal{M}}^i$ for each i.

We now consider the bounded part $\tilde{\mathrm{H}}^*(M)$ of $\mathrm{H}^*(M)$. Although $\tilde{\mathrm{H}}^*(M)$ is the image of $\hat{\mathrm{H}}^*(M)$, it is not clear how does it depend on the fundamental group. In certain cases, the bounded cohomology group is very large, while the bounded part is trivial. Look at a closed integral homology 3-spheres M with a hyperbolic metric. In this case, $\tilde{\mathrm{H}}^*(M) = \mathrm{H}^*(M) = 0$, but $\hat{\mathrm{H}}^2(M) \neq 0$.

It is natural to consider the case when a compact manifold M is covered by a number of open amenable subsets. Here a subset U is said to be amenable if for any $x \in U$, the image of the inclusion $i_* : \pi_1(U, x) \to \pi_1(M, x)$ is an amenable subgroup. One expects that $\tilde{\mathrm{H}}^*(M)$ might be controlled by an amenable covering of the manifold. Based on Gromov's bounded cohomology theory, N. V. Ivanov [I] has made an important observation. He proved an analog of Leray's theorem on amenable coverings.

Lemma 2.2. ([I]) *Let M be an n-dimensional manifold , $\mathcal{U}$ be an amenable covering of M, N be the nerve of this covering, and $|N|$ be the geometric realization of the nerve. Then the canonical map $\iota : \hat{H}^*(M) \to H^*(M)$ factors through the map $\phi : H^*(|N|) \to H^*(M)$. In other words, there is a homomorphism $\psi : \hat{H}^*(M) \to H^*(|N|)$ such that $\iota = \phi \circ \psi$.*

3. Proofs of Theorems A and B

Let AmenRad(M) denote the *amenability radius* which is defined to be the infimum $r > 0$ such that for every metric ball $B(x, \rho)$ of radius $\rho \leq r$, the image of the natural homomorphism $i : \pi_1(B(x,\rho), x) \to \pi_1(M, x)$ is an amenable group. We first prove the following

Lemma 3.1. *There is a constant $C(n,r,D)$ depending only on n, r, D such that for closed Riemannian n-manifold (M,g) satisfying Ric$(M) \geq -(n-1)$, AmenRad$(M) \geq r > 0$ and Diam$(M) \leq D$, the total bounded Betti number is bounded*

$$\sum_{i=0}^{n} \tilde{\beta}^i(M) \leq C(n,r,D).$$

Proof: Let (M,g) be a closed Riemannian n-manifold satisfying Ric $\geq -(n-1)$, AmenRad$(M) \geq r > 0$ and Diam$(M) \leq D$. Take a maximal set of disjoint $r/2$-balls $B(p_i, r/2)$, $i = 1, \cdots, m$. Then $\mathcal{U} := \{B(p_i, r)\}_{i=1}^m$ cover M. Assume that $B(p_{i_o}, r/2)$ has the smallest volume among $B(p_i, r/2)$. By the Bishop-Gromov volume comparison, we obtain

$$m \leq \frac{\mathrm{vol}(M)}{\mathrm{vol}(B(p_{i_o}, r/2))} \leq \frac{\int_0^D \sinh^{n-1}(t)dt}{\int_0^{r/2} \sinh^{n-1}(t)dt} = C(n,r,D).$$

Let N be the nerve of this covering $\mathcal{U}$ and $|N|$ be the geometric realization of the nerve. Since the number of the simplices in N is bounded by $C(n,r,D)$, there is constant $C'(n,r,D)$ depending on $C(n,r,D)$ such that

$$\dim \mathrm{H}^*(|N|) \leq C'(n,r,D).$$

Note that each ball $B(x_i, r)$ in $\mathcal{U}$ is amenable. By Lemma 2.2, we conclude that

$$\tilde{\beta}^*(M) \leq \dim \mathrm{H}^*(|N|) \leq C'(n,r,D).$$

This proves Lemma 3.1. □

Now we prove the following

Lemma 3.2. *Let $n \in \mathbb{Z}, r > 0$ and X a compact metric space. There is a small constant $\epsilon = \epsilon(n,r,X) > 0$ such that for any closed n-manifold M satisfying Ric$(M) \geq -(n-1)$ and AmenRad$(M) \geq r > 0$, if $d_{GH}(M,X) < \epsilon$, then $\tilde{\beta}^i(M) = 0$ for $i > \dim X$.*

Proof: Since X is compact, we can take a finite open covering $\{W_j\}$ of X with mesh $< r/8$ and order $\leq \dim X + 1$, that is, Diam$(W_j) < r/4$ for all j and every point x is contained no more than $\dim X + 1$ subsets W_j.

Claim 1. There is a positive number δ such that every geodesic ball $B(x,\delta)$ in X is contained in some W_j.

Indeed, if this is not true, we can find a sequence of points x_i in X and positive numbers $\delta_i \to 0$ such that the geodesic ball $B(x_i, \delta_i)$ is not totally contained in any W_j for all i. Since X is compact, we can find, by taking a subsequence if necessary, a limit point x of x_i in X. But now, the point x must be in some W_j and hence W_j contains a geodesic ball $B(x, \rho)$ for some positive radius $\rho > 0$. Then the triangle inequality implies that the geodesic ball $B(x_i, \delta_i)$ is contained in W_j for large i. This gives a contradiction and Claim 1 holds.

Next we consider the closed complement F_j of W_j in X, $F_j := X - W_j$, and set

$$E_j := \Big\{ x \in X | d(x, F_j) \geq \frac{\delta}{2} \Big\}.$$

The set E_j is closed and the triangle inequality implies that $\{E_j\}$ is a closed covering of X due to our choice of δ.

We take the positive number $\delta(X)$ to be the minimum of $\delta/8$ and $r/8$.

Assuming the Gromov-Hausdorff distance between M and X is less than $\delta(X)$, we can find an admissible metric d on the disjoint union $M \amalg X$ such that the classical Hausdorff distance of M and X in $M \amalg X$ is less than $\delta(X)$. Then we define an open covering $\mathcal{U} = \{U_j\}$ of M by setting

$$U_j := \{p \in M : d_{M \amalg X}(p, E_j) < 2\delta(X)\}.$$

The triangle inequality then gives that $\mathcal{U}$ has mesh less than r and it covers M.

Claim 2. The order of this open covering $\mathcal{U}$ of M is at most $\dim X + 1$.

Indeed, if there is a point p in $m = (\dim X + 2)$ different open sets in $\mathcal{U}$, say, $U_j, j = 1, \cdots, m$, then we can find a point $x \in X$ with $d_{M \amalg X}(p, x) < \delta(X)$ and the triangle inequality gives $d(x, E_j) < 3\delta(X)$. Hence, one has $d(p, F_j) \geq \frac{\delta}{2} - \frac{3\delta}{8} = \frac{\delta}{8} > 0$ and thus $p \in W_j$ for $j = 1, 2, \ldots, m$. This contradicts to the order of the covering $\{W_j\}$ since $m = \dim X + 2$ and Claim 2 follows.

Therefore, we obtain an amenable open covering $\mathcal{U}$ of M when $d_H(M, X) < \delta(X)$. Let N be the nerve of this covering $\mathcal{U}$ and $|N|$ be the geometric realization of the nerve. By our construction,

$$\dim |N| \leq \dim X.$$

Thus

$$\mathrm{H}^i(|N|) = 0, \qquad i > \dim X.$$

By Lemma 2.2, we conclude that $\tilde{\beta}^i(M) = 0$ for all $i > \dim X$. □

Now we prove Theorems A and B. It was conjectured by M. Gromov that the amenability radius of any complete Riemannian n-manifold (M, g) with $\mathrm{Ric}(M) \geq -(n-1)$ is bounded below by a positive number $r = r(n) > 0$. This conjecture was proved by Fukaya-Yamaguchi [FY1] under the sectional curvature bound $K(M) \geq -1$. Then it is shown to be affirmative under the Ricci curvature bound $\mathrm{Ric} \geq -(n-1)$ in [CC1]. Thus, the extra condition on amenability radius in Lemmas 3.1 and 3.2 can be removed. Then Theorems A and B follow. □

4. Proof of Theorem C

In this section we shall prove Theorem C. First, we recall the notion about the equivariant Hausdorff distance from [FY1]. Let $\mathcal{M}_{met}$ denote the set of all isometry classes of pointed inner metric spaces (X, p) such that for each ρ the ball $B(p, \rho)$ is relatively compact in X. Let $\mathcal{M}_{eq}$ be the set of triples (X, G, p) where (X, p) is in $\mathcal{M}_{met}$ and G is a closed group of isometries of X. For $\rho > 0$, put

$$G(\rho) = \{g \in G \mid d(gp, p) < \rho\}.$$

Definition 4.1. Let (X, G, x), (Y, H, y) be in $\mathcal{M}_{eq}$. An ϵ-equivariant pointed Hausdorff approximation stands for a triple (f, ϕ, ψ) of maps $f : B(x, \frac{1}{\epsilon}) \to Y$, $\phi : G(\frac{1}{\epsilon}) \to H(\frac{1}{\epsilon})$ and $\psi : H(\frac{1}{\epsilon}) \to G(\frac{1}{\epsilon})$ such that

1. $f(x) = y$,
2. the ϵ-neighborhood of $f(B(x, \frac{1}{\epsilon}))$ contains $B(y, \frac{1}{\epsilon})$,
3. if $p, q \in B(x, \frac{1}{\epsilon})$, then $|d(f(p), f(q)) - d(p, q)| < \epsilon$,
4. if $p \in B(x, \frac{1}{\epsilon})$, $g \in G(\frac{1}{\epsilon})$, $gp \in B(x, \frac{1}{\epsilon})$, then $d(f(gp), \phi(g)(f(p))) < \epsilon$,
5. if $p \in B(x, \frac{1}{\epsilon})$, $h \in H(\frac{1}{\epsilon})$, $\psi(h)(p) \in B(x, \frac{1}{\epsilon})$, then $d(f(\psi(h)(p)), h(f(p))) < \epsilon$.

We remark that it is required neither that f is continuous nor that ϕ, ψ are homomorphisms. The equivariant pointed Hausdorff distance $d_{eH}((X, G, x), (Y, H, y))$ is defined to be the infimum of the positive numbers ϵ such that there exist ϵ equivariant Hausdorff approximations from (X, G, x) to (Y, H, y) and from (Y, H, y) to (X, G, x). By d_H we denote the pointed Hausdorff distance, which is the case when the groups are trivial. The notion

$$\lim_{i\to\infty} (X_i, G_i, x_i) = (Y, G, y)$$

means

$$\lim_{i\to\infty} d_{eH}((X_i, G_i, x_i), (Y, H, y)) = 0.$$

Now we proceed to prove Theorem C by the method of absurdity as in [W2]. Suppose Theorem C were false. Then, there exists a sequence of Riemannian n- manifolds M_j satisfying $K(M) \geq -1$ and $\mathrm{Diam}(M) \leq D$ such that all of their bounded cohomology $\hat{\mathrm{H}}^*(M_j)$ are different.

Choose a base point x_j in M_j and a corresponding point $\tilde{x}_j$ in its universal covering $\tilde{M}_j$. The fundamental group $\pi_1(M_j)$ acts on $\tilde{M}_j$ as deck transformation. Applying [F] Theorem 2.1, [FY1] Proposition 3.6 and [FY2] Theorem 4.1 for our sequence (M_j, x_j) and their universal coverings $(\tilde{M}_j, \tilde{x}_j)$ and fundamental groups $G_j = \pi_1(M_j)$, one has

Lemma 4.1. *There exist an Alexandrov space (Y, y) and a Lie group G which is a closed subgroup of isometries of Y such that one has $Y/G = X$ and*

$$\lim_{i\to\infty} (\tilde{M}_i, G_i, \tilde{x}_i) = (Y, G, y).$$

Moreover, for any normal subgroup H of G with the properties

1. *G/H is discrete, and*
2. *H is generated by $H(\rho)$ with $\rho > 0$,*

there exists a sequence of normal subgroups H_i of G_i such that

1. $\lim_{i\to\infty}(\tilde{M}_i, H_i, \tilde{x}_i) = (Y, H, y)$,
2. G_i/H_i *is isomorphic to* G/H *for sufficiently large* i,
3. H_i *is generated by* $H_i(\rho+\epsilon_i)$ *for some* ϵ_i *with* $\epsilon_i \to 0$.

Next we take the normal subgroup G_0 of the connected component of the identity element of G. Since G_0 is generated by $G_0(\epsilon)$ for any positive number ϵ. According to [FY1], for any complete Riemannian n-manifold (M, g), the amenability radius $\mathrm{AmenRad}(M) \geq \epsilon(n) > 0$. Thus we can choose ϵ to be $\epsilon(n)/4$.

Since G is a Lie group, G/G_0 is discrete. Lemma 4.1 then implies that there exists a sequence of normal subgroups E_j of G_j such that G_j/E_j is isomorphic to G/G_0 for sufficiently large i. Moreover, E_j is generated by $E_j(2\epsilon)$ for large j.

From our choice of the number ϵ, Margulis' Lemma implies that the normal subgroup E_j is almost nilpotent. Then, Lemma 2.1 yields that $\hat{\mathrm{H}}^*(M_j)$ is isometrically isomorphic to $\hat{\mathrm{H}}^*_a(K_j)$ for a multicomplex K_j of $K(\pi, 1)$ type with $\pi = G_j/E_j \simeq G/G_0$.

Since any two $K(\pi, 1)$ multicomplices with isomorphic π's are homotopy equivalent, thus $\hat{\mathrm{H}}^*(M_j)$ is isometrically isomorphic to each other for sufficiently large j. This leads to a contradiction and Theorem C follows. □

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