On Randers Metrics with Isotropic S-Curvature*

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Abstract

In this paper, we give an explicit formula for those defined on a Riemannian space form. This class contains all Randers metrics of constant flag curvature.

1 Introduction

In Finsler geometry, the S-curvature is originally introduced for the volume comparison theorem [14]. It is a non-Riemannian quantity which interacts with the flag curvature in a delicate way [5] [10] [19]. In this paper, we will study Randers metrics of isotropic S-curvature.

A Randers metric is a Finsler metric in the form $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}y^i y^j}$ and $\beta = b_i y^i$. Early in 2001, the first author discovered that some Randers metrics of constant flag curvature have constant S-curvature [16], [17]. Later on, Bao-Roble show that every Randers metrics of constant flag curvature is of constant S-curvature [1]. This lets further investigation on Randers metrics of isotropic S-curvature (equivalently, $c = c(x)$ is a scalar function in (1)). We prove that a Randers metric $F = \alpha + \beta$ is of isotropic S-curvature $\mathbf{S} = (n+1)cF$ if and only if β satisfies the following equation:

$$r_{ij} + b_i s_j + b_j s_i = 2c(a_{ij} - b_i b_j), \tag{1}$$

where $r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i})$, $s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i})$, $s_i := b^j s_{ji}$. Here $b_{i|j}$ denote the coefficients of the covariant derivative of β with respect to α [6]. However, it is difficult to find explicit solutions of (1). This difficulty can be overcome if we express Randers metrics $F = \alpha + \beta$ in the following form

$$F = \frac{\sqrt{h(x,y)^2 - [h(x,W_x)^2 h(x,y)^2 - \langle y, W_x\rangle_x^2]}}{1 - h(x,W_x)^2} - \frac{\langle y, W_x\rangle_h}{1 - h(x,W_x)^2}, \tag{2}$$

where $h(x,y) = \sqrt{h_{ij}(x)y^i y^j}$ is a Riemannian metric, W is a vector field on M with $h(x,W_x) < 1$ for all $x \in M$ and $\langle\ ,\ \rangle_h$ denotes the inner product defined by h. In [21], the second author shows that a Randers metric $F = \alpha + \beta$ is of isotropic S-curvature $\mathbf{S} = (n+1)cF$ if and only if W satisfies

$$W_{i;j} + W_{j;i} = -4ch_{ij}, \tag{3}$$

*2000 *Mathematics Subject Classification.* Primary 53B40, 53C60

†supported by the National Natural Science Foundation of China (10371138)

where $W_i := h_{ij}W^j$ and $W_{i;j}$ denote the coefficients of the covariant derivative of W with respect to h. Clearly, equation (3) is much simpler than (1).

In this paper, we shall find the general solution W of (3) when h is of constant curvature. Then we obtain a class of Randers metrics of isotropic S-curvature by (2).

As we know, every Riemannian metric h of constant sectional curvature μ is locally isometric to the following metric h_μ on R^n,

$$h_\mu = \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 + \mu|x|^2}, \tag{4}$$

where $y \in T_x\mathrm{R}^n \cong \mathrm{R}^n$ and $|\cdot|$ denotes the standard Euclidean norm. The domain of h_μ is the open ball $\mathrm{B}^n(r_\mu) \subset \mathrm{R}^n$, where r_μ, where $r_\mu := +\infty$ if $\mu \geq 0$ and $r_\mu := 1/\sqrt{-\mu}$ if $\mu < 0$. At any point $x = (x^i) \in \mathrm{B}^n(r_\mu) \subset \mathrm{R}^n$, we can identify a tangent vector $W^i \frac{\partial}{\partial x^i}|_x \in T_x\mathrm{R}^n$ with a vector $(W^i) \in \mathrm{R}^n$ in a canonical way. Thus we may write vector field W on $\mathrm{B}^n(r_\mu)$ in the form $W = (W^i(x))$.

Theorem 1.1 *Let $h = h_\mu$ be the Riemannian metric in (4) and $W = (W^i)$ be a vector field on the open ball $\mathrm{B}^n(r_\mu) \subset \mathrm{R}^n$. Let $F = \alpha + \beta$ be the Randers metric on $\mathrm{B}^n(r_\mu) \subset \mathrm{R}^n$ which is expressed in terms of h and W by (2).*

Assume that $n \geq 3$. Then F has isotropic S-curvature, $\mathbf{S} = (n+1)cF$ for some scalar function $c = c(x)$, if and only if

$$c = \frac{\lambda + \langle a, x\rangle}{\sqrt{1 + \mu|x|^2}}. \tag{5}$$

where λ is a constant and $a \in \mathrm{R}^n$ is a constant vector, and $W = (W^i)$ of (3) is given by

$$W = -2\Big\{\Big(\lambda\sqrt{1+\mu|x|^2} + \langle a, x\rangle\Big)x - \frac{|x|^2 a}{\sqrt{1+\mu|x|^2}+1}\Big\} + xQ + b + \mu\langle b, x\rangle x, \tag{6}$$

where $Q = ({q_j}^i)$ is an anti-symmetric matrix and $b = (b^i) \in \mathrm{R}^n$ is a constant vector.

The two-dimensional case is more delicate. See section 4 below.

We should point out that the class of Randers metrics defined by (2) using $h = h_\mu$ in (4) and $W = (W^i)$ in (6) includes all Randers metrics of constant flag curvature. More precisely, $F = \alpha + \beta$ is of constant flag curvature if and only if $c = (\lambda + \langle a, x\rangle)/\sqrt{1+\mu|x|^2}$ is a constant [3]. Recent study shows that all Randers metrics defined by (2) using $h = h_\mu$ in (4) and $W = (W^i)$ in (6) are of scalar flag curvature [8].

Acknowledgment: After the paper is submitted, the referee informed us that the general solution of (6) when h has constant curvature in the conformal form has been obtained by Y. B. Shen ([13]). His solutions for c and W are expressed in different forms. The authors would like to thank the referee for his valuable comments.

2 Preliminaries

Finsler metrics under our consideration are always positive definite, namely, if $F = F(x, y)$ is a Finsler metric on an n-dimensional manifold M, then in any standard local coordinate system (x^i, y^i) in TM, the matrix $g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}(x, y) > 0$ is positive definite for any $y \neq 0$. The Hausdorff-Busemann volume form $dV = \sigma_F(x)dx^1 \wedge \cdots \wedge dx^n$ is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathrm{B}^n)}{\mathrm{Vol}\Big\{(y^i) \in \mathrm{R}^n \mid F(y^i \frac{\partial}{\partial x^i}|_x) < 1\Big\}}.$$

The Finsler metric F induces a vector field $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ on TM defined by

$$G^i = \frac{1}{4} g^{il}\Big\{[F^2]_{x^ky^l}(x, y)y^k - [F^2]_{x^l}(x, y)\Big\},$$

where $(g^{ij}) := (g_{ij})^{-1}$. The S-curvature is defined by

$$\mathbf{S} := \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m}\Big(\ln \sigma_F\Big).$$

For Riemannian metrics, the S-curvature vanishes. $\mathbf{S}$ is said to be *isotropic* if there is a scalar function $c = c(x)$ on M such that

$$\mathbf{S} = (n + 1)cF.$$

As we mentioned early in the introduction, for a Randers metric $F = \alpha + \beta$ expressed by (2), equation (1) is equivalent to (3). Equation (3) has been studied a long time ago in [9]. The following lemma is staright forward by Ricci identity. See (69.2) in [9].

Lemma 2.1 *Let (M, h) be an n-dimensional Riemannian manifold. Let $W = W^i \frac{\partial}{\partial x^i}$ be a vector field on M satisfying (3) where $c = c(x)$ is a scalar function on M, $W_i := h_{ij}W^j$ and $W_{i;j}$ denote the coefficients of the covariant derivatives of W with respect to h. Then*

$$W_{k;i;j} = 2\Big\{c_{;k}h_{ij} - c_{;i}h_{jk} - c_{;j}h_{ki}\Big\} - W_m R_j{}^m{}_{ki}. \tag{7}$$

By Lemma 2.1, we can prove the following

Lemma 2.2 *Let W satisfy (3) for some scalar function $c = c(x)$. Assume that h has constant curvature μ. Then c satisfies*

$$\nabla^2 c + \mu c h = 0, \qquad (n \geq 3) \tag{8}$$

$$\Delta c + 2\mu c h = 0, \qquad (n = 2) \tag{9}$$

where ∇ and Δ denote the Hessian and Laplacian of h respectively.

Proof: It is easy to prove this lemma. For readers' convenience, we give the details below.

First, we assume that (M,h) is an n-dimensional Riemannian manifold. Let $W = W^i \frac{\partial}{\partial x^i}$ be a vector field on M and $c = c(x)$ be a scalar function on M satisfying (3) for some scalar function $c = c(x)$. By (7),

$$W_{i;j;k} = 2\Big\{c_{;i}h_{jk} - c_{;j}h_{ki} - c_{;k}h_{ij}\Big\} - W_m R_{k\ ij}^{\ m}. \tag{10}$$

Differentiating (10) yields

$$W_{i;j;k;l} = 2\Big\{c_{;i;l}h_{jk} - c_{;j;l}h_{ki} - c_{;k;l}h_{ij}\Big\} - W_{m;l} R_{k\ ij}^{\ m} - W_m R_{k\ ij;l}^{\ m}. \tag{11}$$

Exchanging the indices k and l yields

$$W_{i;j;l;k} = 2\Big\{c_{;i;k}h_{jl} - c_{;j;k}h_{li} - c_{;l;k}h_{ij}\Big\} - W_{m;k} R_{l\ ij}^{\ m} - W_m R_{l\ ij;k}^{\ m}. \tag{12}$$

Note that $c = c(x)$ is a scalar function, thus $c_{;k;l} = c_{;l;k}$. It follows from (11) and (12) that

$$\begin{aligned} W_{i;j;k;l} - W_{i;j;l;k} &= 2\Big\{c_{;i;l}h_{jk} + c_{;j;k}h_{li}\Big\} - 2\Big\{c_{;j;l}h_{ki} + c_{;i;k}h_{jl}\Big\} \\ &\quad + W_{m;k} R_{l\ ij}^{\ m} - W_{m;l} R_{k\ ij}^{\ m} + W_m\Big\{R_{l\ ij;k}^{\ m} - R_{k\ ij;l}^{\ m}\Big\}. \end{aligned}$$

Applying the Ricci identity $W_{i;j;k;l} - W_{i;j;l;k} = W_{m;j} R_{i\ kl}^{\ m} + W_{i;m} R_{j\ kl}^{\ m}$ and the identity (3) to the above identity, one obtains

$$\begin{aligned} & 2\Big\{c_{;i;l}h_{jk} + c_{;j;k}h_{li}\Big\} - 2\Big\{c_{;j;l}h_{ki} + c_{;i;k}h_{jl}\Big\} \\ & = 4cR_{ijkl} + W_m\Big\{R_{k\ ij;l}^{\ m} - R_{l\ ij;k}^{\ m}\Big\} \\ & \quad + W_{m;j} R_{i\ kl}^{\ m} - W_{m;i} R_{j\ kl}^{\ m} + W_{m;l} R_{k\ ij}^{\ m} - W_{m;k} R_{l\ ij}^{\ m}. \end{aligned} \tag{13}$$

Now we assume that h has constant curvature μ, i.e.,

$$R_{k\ ij}^{\ m} = \mu\Big\{\delta_i^m h_{jk} - \delta_j^m h_{ik}\Big\}.$$

By (3), we obtain from (13) that

$$\Big\{c_{;i;l}h_{jk} + c_{;j;k}h_{li}\Big\} - \Big\{c_{;j;l}h_{ki} + c_{;i;k}h_{jl}\Big\} = 2\mu c\Big\{h_{jl}h_{ik} - h_{jk}h_{il}\Big\}. \tag{14}$$

At a point, we may choose an orthonormal basis so that $h_{ij} = \delta_{ij}$. In (13), letting $k = j$ and $l = i$ ($i \neq j$) yields

$$c_{;i;i} + c_{;j;j} + 2\mu c = 0 \qquad (i \neq j). \tag{15}$$

When $n \geq 3$, it follows from (15) that

$$c_{;i;i} + \mu c = 0. \tag{16}$$

For any i, l, there is $m \neq i, l$. In (13), letting $j = k = m$, one obtains

$$c_{;i;l} + c_{;m;m}\delta_{il} + 2\mu c\delta_{il} = 0. \tag{17}$$

By (16), $c_{;m;m} = -\mu c$. Plugging it into (17) yields (8).

In dimension $n = 2$, (9) follows from (15) directly. Q.E.D.

Remark 2.3 Tashiro has shown that if a *complete* Riennian manifold (M, h) with a vector field W and scalar function c satisfies (3) and (8) with $\mu = 1$, then it is isometric to the standard unit sphere ([20]).

3 Theorems 1.1

In this section, we are going to prove Theorem 1.1. Throughout this section, we always assume that the dimension is greader than two.

By assumption $h = \mu_\mu$ is given by (4) and the Randers metric $F = \alpha + \beta$ is defined by (2). We first determine the scalar function $c = c(x)$. According to Lemma 2.2, c satisfies

$$c_{;i;j} + \mu c h_{ij} = 0, \tag{18}$$

where

$$h_{ij} := \frac{\delta_{ij}}{1 + \mu|x|^2} - \frac{\mu x^i x^j}{(1 + \mu|x|^2)^2}.$$

If $T = T_i dx^i$ is a 1-form, then

$$T_{i;j} = \frac{\partial T_i}{\partial x^j} + \mu \frac{x^i T_j + x^j T_i}{1 + \mu|x|^2}. \tag{19}$$

By (19), we obtain

$$c_{;i;j} = c_{x^i x^j} + \mu \frac{x^i c_{x^j} + x^j c_{x^i}}{1 + \mu|x|^2},$$

where $c_{x^i} = \frac{\partial c}{\partial x^i}$ and $c_{x^i x^j} = \frac{\partial^2 c}{\partial x^i \partial x^j}$ denote the partial derivatives of c. Let $f := \sqrt{1 + \mu|x|^2}\, c$. We have

$$f_{x^i x^j} = \sqrt{1 + \mu|x|^2}\Big\{ c_{;i;j} + \mu c h_{ij} \Big\} = 0.$$

Thus

$$f = \lambda + \langle a, x \rangle,$$

where λ is a constant and $a \in \mathrm{R}^n$ is a constant vector. We obtain a general formula for c,

$$c = \frac{\lambda + \langle a, x \rangle}{\sqrt{1 + \mu|x|^2}}.$$

For the above c, we can solve (3) for W.

Case 1: $\mu = 0$. Let

$$P_i := W_i - |x|^2 a^i + 2(\lambda + \langle a, x\rangle)x^i.$$

Then (3) is equivalent to

$$\frac{\partial P_i}{\partial x^j} + \frac{\partial P_j}{\partial x^i} = 0.$$

By an elementary argument (see [3]), we get

$$P_i = x^j {q_j}^i + b^i,$$

where $Q = ({q_j}^i)$ is an anti-symmetric matrix and $b = (b^i) \in \mathrm{R}^n$ is a constant vector. We obtain

$$W^i = W_i = -2\Big\{(\lambda + \langle a, x\rangle)x^i - \frac{|x|^2}{2}a^i\Big\} + x^j {q_j}^i + b^i.$$

Case 2: $\mu \neq 0$. Let

$$P_i := W_i - \frac{2}{\mu}c_{;i}.$$

Then P_i satisfy

$$P_{i;j} + P_{j;i} = 0. \tag{20}$$

Using (19), we can rewrite (20) as follows:

$$\frac{\partial P_i}{\partial x^j} + \frac{\partial P_j}{\partial x^i} + 2\mu\frac{x^i P_j + x^j P_i}{1 + \mu|x|^2} = 0.$$

Let $H_i := (1 + \mu|x|^2)P_i$. We obtain

$$\frac{\partial H_i}{\partial x^j} + \frac{\partial H_j}{\partial x^i} = (1 + \mu|x|^2)\Big\{\frac{\partial P_i}{\partial x^j} + \frac{\partial P_j}{\partial x^i} + 2\mu\frac{x^j P_i + x^i P_j}{1 + \mu|x|^2}\Big\} = 0.$$

By a similar argument for P_i in the case when $\mu = 0$, we obtain

$$H_i = x^j {q_j}^i + v^i,$$

where $Q = ({q_j}^i)$ is an anti-symmetric matrix and $v = (v^i) \in \mathrm{R}^n$ is a constant vector. Thus

$$P_i = (1 + \mu|x|^2)^{-1}\Big\{x^j {q_j}^i + v^i\Big\}.$$

A direct computation yields

$$c_{;i} = \frac{a^i}{\sqrt{1 + \mu|x|^2}} - \frac{\mu(\lambda + \langle a, x\rangle)x^i}{(1 + \mu|x|^2)^{3/2}}. \tag{21}$$

We obtain

$$\begin{aligned} W_i &= P_i + \frac{2}{\mu}c_{;i} \\ &= (1 + \mu|x|^2)^{-1}\Big\{x^j {q_j}^i + v^i\Big\} + \frac{2a^i}{\mu\sqrt{1 + \mu|x|^2}} - \frac{2(\lambda + \langle a, x\rangle)x^i}{(1 + \mu|x|^2)^{3/2}}. \end{aligned}$$

Finally, we completely determine $W^i = h^{ij} W_j$.

$$W^i = 2\sqrt{1+\mu|x|^2}\Big\{\mu^{-1}a^i - \lambda x^i\Big\} + x^j q_j{}^i + v^i + \mu\langle v, x\rangle x^i.$$

We express

$$\begin{aligned} \mu^{-1}\sqrt{1+\mu|x|^2} &= \mu^{-1}\Big\{\sqrt{1+\mu|x|^2} - 1\Big\} + \mu^{-1} \\ &= \frac{|x|^2}{\sqrt{1+\mu|x|^2}+1} + \mu^{-1}. \end{aligned}$$

Let

$$b^i := v^i + 2\mu^{-1}a^i.$$

We obtain

$$\begin{aligned} W^i &= -2\Big(\lambda\sqrt{1+\mu|x|^2} + \langle a, x\rangle\Big)x^i + \frac{2|x|^2 a^i}{\sqrt{1+\mu|x|^2}+1} \\ &\quad + x^j q^i{}_j + b^i + \mu\langle b, x\rangle x^i. \end{aligned}$$

Q.E.D.

Note that $(\mathrm{B}^n(r_\mu), h_\mu)$ is incomplete when $\mu > 0$. The global version of the above results can be stated as follows.

Theorem 3.1 *Let* S^n *denote the standard unit n-sphere in* R^{n+1} $(n > 2)$ *with the induced Riemannian metric* $h = \sqrt{h_{ij}y^i y^j}$ *of constant curvature* $\mu = 1$*. Let* W *be a vector field on* S^n *with* $\|W_x\|_h < 1$ *for all* $x \in M$*. Let* $F = \alpha + \beta$ *be defined by (2). If F has isotropic S-curvature,* $\mathbf{S} = (n+1)cF$ *for some scalar function* $c = c(x)$ *on* M*, then c is an eigenfunction corresponding to the first eigenvalue* $\lambda_1 = n$*. Hence it satisfies*

$$c_{;i;j} + c h_{ij} = 0, \tag{22}$$

In this case, $W = W^i \frac{\partial}{\partial x^i}$ *is given by*

$$W^i = P^i + 2c^i, \tag{23}$$

where $P = P^i \frac{\partial}{\partial x^i}$ *is a Killing vector field on* S^n *and* $\nabla c = c^i \frac{\partial}{\partial x^i}$ *is the gradient of c on* S^n*.*

Conversely, if c is an eigenfunction on S^n *satisfying (22) and W is given by (23), then the Randers metric* $F = \alpha + \beta$ *defined by (2) using W has isotropic S-curvature,* $\mathbf{S} = (n+)cF$*.*

The Killing vector field P in (23) can be locally expressed as well as the gradient $\nabla c = c^i \frac{\partial}{\partial x^i}$ on S^n.

Now let us find out the local expressions for $P = P^i \frac{\partial}{\partial x^i}$ and $\nabla c = c^i \frac{\partial}{\partial x^i}$ when the underlying metric $h = h_{+1}$ in (4) is expressed by

$$h = \frac{\sqrt{|y|^2 + (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1+|x|^2},$$

where $y \in T_x\mathrm{R}^n \cong R^n$. From the proof of Theorem 1.1, we have

$$\begin{aligned} c_{;i} &= (1+|x|^2)^{-1/2}a^i - (1+|x|^2)^{-3/2}\Big\{\lambda + \langle a, x\rangle\Big\}x^i, \\ P_i &= (1+|x|^2)^{-1}\Big\{x^j q_j{}^i + v^i\Big\}. \end{aligned}$$

Then $P = P^i \frac{\partial}{\partial x^i}$ and $\nabla c = c^i \frac{\partial}{\partial x^i}$ are given by

$$\begin{aligned} P^i &= x^j q_j{}^i + v^i + \langle v, x\rangle x^i, \\ c^i &= \sqrt{1+|x|^2}\Big\{a^i - \lambda x^i\Big\}. \end{aligned}$$

4 The two-dimensional case

The two-dimensional case is more delicate. We can characterize the vector fields $W = W^1\frac{\partial}{\partial x} + W^2 \frac{\partial}{\partial y}$ satisfying (3) as follows.

Assume that a 1-form $W^* := W_1 dx + W_2 dy$ satisfies (3). Let

$$P := \{1+\mu(x^2+y^2)\}W_1, \qquad Q := \{1+\mu(x^2+y^2)\}W_2$$

and $x := x^1, y := x^2$. Then P and Q satisfy

$$\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = \frac{4\mu c x y}{1+\mu(x^2+y^2)}, \tag{24}$$

$$(1+\mu x^2)\frac{\partial P}{\partial x} = -2c\frac{(1+\mu x^2)(1+\mu y^2)}{1+\mu(x^2+y^2)} = (1+\mu y^2)\frac{\partial Q}{\partial y}. \tag{25}$$

It follows from (24) and (25) that

$$\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = -\mu x y \Big\{ \frac{1}{1+\mu y^2}\frac{\partial P}{\partial x} + \frac{1}{1+\mu x^2}\frac{\partial Q}{\partial y}\Big\}, \tag{26}$$

$$\frac{1}{1+\mu y^2}\frac{\partial P}{\partial x} - \frac{1}{1+\mu x^2}\frac{\partial Q}{\partial y} = 0. \tag{27}$$

Conversely, if P and Q satisfy (26) and (27), respectively, then they satisfy (24) and (25) with

$$c := -\frac{1+\mu(x^2+y^2)}{2(1+\mu y^2)}\frac{\partial P}{\partial x} = -\frac{1+\mu(x^2+y^2)}{2(1+\mu x^2)}\frac{\partial Q}{\partial y}. \tag{28}$$

Let $W = W^1\frac{\partial}{\partial x} + W^2\frac{\partial}{\partial y}$ be defined by

$$W^1 := (1+\mu x^2)P + \mu x y Q, \qquad W^2 := (1+\mu y^2)Q + \mu x y P. \tag{29}$$

Then Randers metric $F = \alpha + \beta$ defined by (2) or (**??**) using W has isotropic S-curvature $\mathbf{S} = 3cF$, where c is given by (28).

We have proved the following

Theorem 4.1 *Let $W = W^1 \frac{\partial}{\partial x} + W^2 \frac{\partial}{\partial y}$ be a vector field on the Riemannian space form $(\mathrm{B}^2(r_\mu), h_\mu)$ and $F = \alpha + \beta$ be the Randers metric on $\mathrm{B}^2(r_\mu)$ defined by (2) or (**??**) using W. Then F has isotropic S-curvature, $\mathbf{S} = 3cF$, if and only if the components W^1 and W^2 are given by (29), where P, Q satisfy (26) and (27) respectively. In this case, the scalar function $c = c(x)$ is given by (28).*

The case when $\mu = 0$ is very interesting. In this case, $W^1 = P$ and $W^2 = Q$. (26) and (27) become

$$\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = 0, \tag{30}$$

$$\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} = 0. \tag{31}$$

If $W := W^1 \frac{\partial}{\partial x} + W^2 \frac{\partial}{\partial y}$ satisfies (30) and (31), then the Randers metric $F = \alpha + \beta$ defined by (**??**) using W has isotropic S-curvature, $\mathbf{S} = 3cF$, where

$$c := -\frac{1}{2}\frac{\partial P}{\partial x} = -\frac{1}{2}\frac{\partial Q}{\partial y}.$$

Remark 4.2 Let $z := x + iy$ and define

$$f(z) := P(x, y) + iQ(x, y).$$

Then (30) and (31) are equivalent to that f is a holomorphic function on the complex plane.

By the above remark, we can easily construct some examples. Let

$$f = A + Bz + Cz^2 + Dz^3,$$

where $A = A_1 + iA_2, B = B_1 + iB_2, C = C_1 + iC_2$ and $D = D_1 + iD_2$ are constants. Then

$$\begin{aligned} P &= A_1 + B_1 x - B_2 y + C_1(x^2 - y^2) - 2C_2 xy + D_1(x^3 - 3xy^2) - D_2(3x^2 y - y^3), \\ Q &= A_2 + B_1 y + B_2 x + 2C_1 xy + C_2(x^2 - y^2) + D_1(3x^2 y - y^3) + D_2(x^3 - 3xy^2). \end{aligned}$$

Then P, Q satisfy (30) and (31). The Randers metric $F = \alpha + \beta$ defined by (2) using $W = W^1 \frac{\partial}{\partial x} + W^2 \frac{\partial}{\partial y}$, where $W^1 = P$ and $W^2 = Q$, has isotropic S-curvature, $\mathbf{S} = 3cF$, where c is given by

$$c = -\frac{1}{2}\Big\{ B_1 + 2C_1 x - 2C_2 y + 3D_1(x^2 - y^2) - 6D_2 xy \Big\}.$$

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