A GAUSS-BONNET-CHERN FORMULA FOR FINSLER MANIFOLDS

Zhongmin Shen

Abstract. Let $\pi : E \to M$ be an oriented fiber bundle with $\dim E_x = p$ and $\mathcal{V}TE$ denote the vertical tangent bundle of E. Given a projection $\mathfrak{p} : TE \to \mathcal{V}TE$, a Riemann metric h on $\mathcal{V}TE$ and a metric-compatible connection D on $\mathcal{V}TE$, we construct a $(p+1)$-form Pf and a p-form Π such that $d\Pi = \mathrm{Pf}$ and $\int_{E_x} \Pi = 1, \forall x \in M$. When $E = \mathcal{S}$ is the tangent sphere bundle of M, we establish a Gauss-Bonnet-Chern formula for any triple $\{\mathfrak{p}, h, D\}$ over $\mathcal{S}$. Since every Finsler metric F on M naturally gives a triple $\{\mathfrak{p}, h, D\}$ on $\mathcal{S}$, we establish a Gauss-Bonnet-Chern formula for all Finsler manifolds.

0. Introduction

In 1944, S. S. Chern [Ch3] proves the Gauss-Bonnet theorem for Riemann manifolds. Let M be an n-dimensional oriented C^∞ manifold (n =even). The tangent sphere bundle $\pi : \mathcal{S} \to M$ consists of all rays $[v] = \{tv;\ t > 0\}$. For any Riemann metric on M, Chern constructs an n-form Pf on M from its curvature tensor and an $(n-1)$-form Π on $\mathcal{S}$ such that $d\Pi = \pi^*\mathrm{Pf}$ and $\int_{\mathcal{S}_x} \Pi = 1$. Then he obtains the following formula: $\int_M \mathrm{Pf} = \chi(M)$. We shall call a formula of this type a Gauss-Bonnet-Chern (GBC) formula.

Given a Finsler metric F on M, one would like to establish an analogue of the GBC formula for F. In this case, the problem becomes more difficult, since there is no canonical "metric-compatible" and "torsion-free" linear connection of F on TM. Nevertheless, F naturally induces a Riemann metric g on π^*TM. Here π^*TM denotes the pull-back tangent bundle over $\mathcal{S}$. There is a canonical section ℓ of π^*TM given by $\ell_{[v]} = \frac{1}{F(v)}([v], v)$. We have several important linear connections on π^*TM, such as the Berwald connection, the Chern connection and the Cartan connection, etc. Fix a positive oriented orthonormal basis $\{e_i\}_{i=1}^n$ for π^*TM. Let $\Omega_j^{\ i}$ be the curvature form of an arbitrary linear connection ∇ on π^*TM w.r.t. $\{e_i\}_{i=1}^n$. Put $\ell = \ell^i e_i$ and $\nabla \ell = \theta^i \otimes e_i$. Following [Ch3], one can construct an n-form Pf and an $(n-1)$-form Π on $\mathcal{S}$ by

$$\mathrm{Pf} := (-1)^{\frac{n}{2}} \frac{2}{n!\mathrm{vol}(\mathbb{S}^n)} \sum \epsilon^{i_1 \cdots i_n} \Omega_{i_1}^{\ i_2} \cdots \Omega_{i_{n-1}}^{\ i_n} \tag{0.1}$$

$$\Pi := \sum_{k=0}^{\frac{n}{2}-1} (-1)^k c_k \sum \epsilon^{i_1 \cdots i_n} \Omega_{i_1}^{\ i_2} \cdots \Omega_{i_{2k-1}}^{\ i_{2k}} \theta^{i_{2k+1}} \cdots \theta^{i_{n-1}} \ell^{i_n}. \tag{0.2}$$

1991 *Mathematics Subject Classification.* Primary 53C60, Secondary 53C20, 53B40.

where c_k are determined by $c_k = \frac{(n-2k-1)}{2k} c_{k-1}, c_0 = \frac{1}{(n-1)!\mathrm{vol}(\mathbb{S}^{n-1})}$. When F is Riemannian, according to [Ch3], one has

$$d\Pi = \mathrm{Pf}, \tag{0.3}$$

$$\int_{\mathcal{S}_x} \Pi = 1, \qquad \forall x \in M. \tag{0.4}$$

Then one can prove the Gauss-Bonnet theorem by the Hopf theorem. However, (0.3)(0.4) does not hold simultaneously for a general Finsler metric F, no matter which connection we take.

Take the Cartan connection on π^*TM, which is metric-compatible with g. Lichnerowiz [L] first verifies that $d\Pi = \mathrm{Pf}$. He also notices that $\int_{\mathcal{S}_x} \Pi \neq 1$. Therefore he restricts himself to the class of Finsler metrics with $(\mathcal{S}_x, h_x) = \mathbb{S}^{n-1}, \forall x \in M$ (hence $\int_{\mathcal{S}_x} \Pi = 1$), where h_x denotes the induced Riemann metric on $\mathcal{S}_x$. However, this is a very strong restriction. According to a theorem of Brickell [Br], Finsler metrics with $(\mathcal{S}_x, h_x) = \mathbb{S}^{n-1}, \forall x \in M$ must be Riemannian, provided that $n \geq 3$ and $F(-v) = F(v)$.

It is Bao and Chern [BC2] who first make the following non-trivial observation. Bao and Chern show that the $(n-1)$-form Π of the Chern connection (or any torsion-free connection) has very nice properties. First, $d\Pi = \mathrm{Pf} + \mathfrak{F}$. The additional term $\mathfrak{F}$ occurs, because a torsion-free connection is not metric-compatible. Second, the the restriction of Π to $\mathcal{S}_x$ is a multiple of the volume form of $(\mathcal{S}_x, h_x)$. This observation leads to a GBC formula for Finsler manifolds with $\mathrm{vol}(\mathcal{S}_x, h_x)$ =constant.

After [BC2], the author [S] also establishes several GBC formulas for certain class of Finsler manifolds, by analysing the geometric data of Π on $\mathcal{S}_x$ for the $(n-1)$-form Π of the Cartan connection. Other attempts (in lower dimensions) can be found in [B] [R] [Ch4] [M][BSC2], etc.

Our goal in this paper is to establish a GBC formula for *all* Finsler manifolds. Recall that a Finsler metric F on M naturally induces a Riemann metric g on π^*TM. Let ∇ denote the Cartan connection on π^*TM. The Cartan connection has the following property. $\nabla\ell : T\mathcal{S} \to \pi^*TM$ is a bundle map of rank $n-1$ such that $\nabla\ell : \mathcal{V}T\mathcal{S} \to \ell^\perp$ is an isomorphism. Let $\{e_i\}_{i=1}^n$ be an orthonormal frame for π^*TM with $e_n = \ell$. Put $\nabla\ell = \theta^\alpha \otimes e_\alpha$ and $\nabla e_j = \theta_j^{\ i} \otimes e_i$ (hence $\theta^\alpha = \theta_n^{\ \alpha}$). Let $\{f_\alpha\}_{\alpha=1}^{n-1}$ be the basis for $\mathcal{V}T\mathcal{S}$ determined by $\theta^\alpha(f_\beta) = \delta^\alpha_\beta$. Then we get a triple $\{\mathfrak{p}, h, D\}$ over $\mathcal{S}$ by

$$\mathfrak{p} = \theta^\alpha \otimes f_\alpha, \quad h = \theta^\alpha \otimes \theta^\alpha|_{\mathcal{V}T\mathcal{S}}, \quad Df_\beta = \theta_\beta^{\ \alpha} \otimes f_\alpha. \tag{0.5}$$

Here $\mathfrak{p} : T\mathcal{S} \to \mathcal{V}T\mathcal{S}$ is a projection, h is a Riemann metric on $\mathcal{V}T\mathcal{S}$ and D is a metric-compatible connection on $(\mathcal{V}T\mathcal{S}, h)$.

The *curvature form* $(\Omega_j^{\ i})$ of ∇ is defined by by

$$\Omega_j^{\ i} := d\theta_j^{\ i} - \theta_j^{\ k} \wedge \theta_k^{\ i}. \tag{0.6}$$

The *torsion form* (Θ^α) and the *curvature form* $(\Theta_\beta^{\ \alpha})$ of D are defined by

$$\Theta^\alpha : = d\theta^\alpha - \theta^\beta \wedge \theta_\beta^{\ \alpha}. \tag{0.7}$$

$$\Theta_\beta^{\ \alpha} : = d\theta_\beta^{\ \alpha} - \theta_\beta^{\ \tau} \wedge \theta_\tau^{\ \alpha}. \tag{0.8}$$

$\{\Omega_j^{\ i}\}$ and $\{\Theta^\alpha, \Theta_\beta^{\ \alpha}\}$ are related by

$$\Theta^\alpha = \Omega_n^{\ \alpha}, \tag{0.9}$$

$$\Theta_\beta^{\ \alpha} = \Omega_\beta^{\ \alpha} - \theta^\beta \wedge \theta^\alpha. \tag{0.10}$$

In order to establish a GBC formula for a Finsler metric, it suffices to establish a GBC formula for an arbitrary triple $\{\mathfrak{p}, h, D\}$ over $\mathcal{S}$.

Let M be a closed oriented manifold of dimension $n = p+1$. Let $\{\mathfrak{p}, h, D\}$ be an arbitrary triple over $\mathcal{S}$, namely, $\mathfrak{p} : T\mathcal{S} \to \mathcal{V}T\mathcal{S}$ be a projection map, h is a Riemann metric on $\mathcal{V}T\mathcal{S}$ and D is a metric-compatible connection on $(\mathcal{V}T\mathcal{S}, h)$. Let $\{f_\alpha\}$ be a positive orthonormal frame for $\mathcal{V}T\mathcal{S}$. Let (θ^α), $(\theta_\beta^{\ \alpha})$, (Θ^α) and $(\Theta_\beta^{\ \alpha})$ be given by (0.5) (0.7) and (0.8), respectively .

The vertical parts Q^α and $Q_\beta^{\ \alpha}$ of Θ^α and $\Theta_\beta^{\ \alpha}$ are determined by

$$Q^\alpha \equiv \Theta^\alpha, \quad Q_\beta^{\ \alpha} \equiv \Theta_\beta^{\ \alpha}, \quad \text{over } \mathcal{V}T\mathcal{S}. \tag{0.11}$$

Put

$$\mathcal{O}_\beta^{\ \alpha} = dQ_\beta^{\ \alpha} + Q_\beta^{\ \mu} \wedge \theta_\mu^{\ \alpha} - \theta_\beta^{\ \mu} \wedge Q_\mu^{\ \alpha}. \tag{0.12}$$

The following forms are well-defined on $\mathcal{S}$.

$$\Phi_k = \sum \epsilon^{\alpha_1\cdots\alpha_p} \Theta_{\alpha_1}^{\ \alpha_2} \cdots \Theta_{\alpha_{2k-1}}^{\ \alpha_{2k}} \theta^{\alpha_{2k+1}} \cdots \theta^{\alpha_p}, \tag{0.13}$$

$$\Psi_k = \sum \epsilon^{\alpha_1\cdots\alpha_p} \Theta_{\alpha_1}^{\ \alpha_2} \cdots \Theta_{\alpha_{2k-1}}^{\ \alpha_{2k}} \Theta^{\alpha_{2k+1}} \theta^{\alpha_{2k+2}} \cdots \theta^{\alpha_p}, \tag{0.14}$$

$$\Phi_k^{(v)} = \sum \epsilon^{\alpha_1\cdots\alpha_p} Q_{\alpha_1}^{\ \alpha_2} \cdots Q_{\alpha_{2k-1}}^{\ \alpha_{2k}} \theta^{\alpha_{2k+1}} \cdots \theta^{\alpha_p}, \tag{0.15}$$

$$\Psi_k^{(v)} = \sum \epsilon^{\alpha_1\cdots\alpha_p} Q_{\alpha_1}^{\ \alpha_2} \cdots Q_{\alpha_{2k-1}}^{\ \alpha_{2k}} \Theta^{\alpha_{2k+1}} \theta^{\alpha_{2k+2}} \cdots \theta^{\alpha_p}, \tag{0.16}$$

$$F_k^{(v)} = \sum \epsilon^{\alpha_1\cdots\alpha_p} \mathcal{O}_{\alpha_1}^{\ \alpha_2} Q_{\alpha_3}^{\ \alpha_4} \cdots Q_{\alpha_{2k-1}}^{\ \alpha_{2k}} \theta^{\alpha_{2k+1}} \cdots \theta^{\alpha_p}, \tag{0.17}$$

Here we put $F_0^{(v)} = 0$ and $\Psi_k = \Psi_k^{(v)} = 0$ if $2k = p$.

Let $h_x = h_{|\mathcal{S}_x}$. The function $V(x) := \mathrm{vol}(\mathcal{S}_x, h_x)$ is called the *volume function* on M. In general, $V(x) \neq$constant.

For arbitrary constants c_k, define

$$\begin{aligned} \mathrm{Pf} = &\frac{1}{p!V(x)}\{\sum_{k=0}^{[\frac{p}{2}]} c_k[(p-2k)(\Psi_k - \Psi_k^{(v)}) - kF_k^{(v)} \\ &- \pi^* d(\log V) \wedge (\Phi_k - \Phi_k^{(v)})] + [p\Psi_0 - \pi^* d(\log V) \wedge \Phi_0]\}. \end{aligned} \tag{0.18}$$

The following is the main theorem.

Theorem 0.1. *Let M be an oriented closed manifold of dimension $n = p + 1$. Given a projection $\mathfrak{p} : T\mathcal{S} \to \mathcal{V}T\mathcal{S}$, a Riemann metric h on $\mathcal{V}T\mathcal{S}$, and a metric-compatible connection on $\mathcal{V}T\mathcal{S}$. For any vector field X on M with isolated zeros, the n-form Pf in (0.18) satisfies*

$$\int_M [X]^* Pf = \chi(M), \tag{0.19}$$

where $[X] : M \setminus \{zeros\} \to \mathcal{S}$ denotes the section defined by X.

The proof of Theorem 0.1 will be given in §1-§3. Note that when $V(x) = constant$, (0.18) reduces to

$$\mathrm{Pf} = \frac{1}{p!V(x)}\{\sum_{k=0}^{[\frac{p}{2}]} c_k[(p-2k)(\Psi_k - \Psi_k^{(v)}) - kF_k^{(v)} + p\Psi_0\}. \tag{0.20}$$

Applying (0.19) to the special case when the triple $\{\mathfrak{p}, h, D\}$ over $\mathcal{S}$ is given by a Finsler metric F, one obtains a GBC formula for F (Theorem 4.2). In §5, we shall derive the Gauss-Bonnet-Chern formula for Riemann manifolds from Theorem 4.2, by choosing a suitable set of constants $\{c_k\}$ in (0.18).

Acknowledgements. The author would like to thank David Bao and S.S.Chern for many valuable discussions.

1. Exact forms on a fiber bundle

In §0, we briefly describe how to get the triple $\{\mathfrak{p}, h, D\}$ over $\mathcal{S}$ from a Finsler metric F on a smooth manifold M of dimension $n = p + 1$. Then we define an n-forms Pf (0.18) and state Theorem 0.1. In this section we shall study a general triple $\{\mathfrak{p}, h, D\}$ over an arbitrary fiber bundle $\pi : E \to M$ with $\dim E_x = p$, where $\mathfrak{p} : TE \to \mathcal{V}TE$ is a projection, g is a Riemann metric on $\mathcal{V}TE$ and D is a metric-compatible connection on $(\mathcal{V}TE, h)$.

Let $\{f_\alpha\}_{\alpha=1}^p$ be a positive orthonormal frame for $(\mathcal{V}TE, h)$. Let $\{\theta^\alpha\}$ and $\{\theta_\beta{}^\alpha\}$ be given by

$$\mathfrak{p} = \theta^\alpha \otimes f_\alpha, \qquad Df_\beta = \theta_\beta{}^\alpha \otimes f_\alpha.$$

Let (Θ^α), $(\Theta_\beta{}^\alpha)$, Φ_k and Ψ_k be given by (0.7), (0.8), (0.13) and (0.14), respectively. We have

Lemma 1.1.

$$d\Phi_k = (p-2k)\Psi_k, \qquad 0 \le k \le [\frac{p}{2}] \tag{1.1}$$

Here we put $\Psi_k = 0$ when $2k = p$.

Proof. We have the following Bianchi identities

$$d\Theta^\alpha = -\Theta^\beta \wedge \theta_\beta{}^\alpha + \theta^\beta \wedge \Theta_\beta{}^\alpha \tag{1.2}$$

$$d\Theta_\beta{}^\alpha = -\Theta_\beta{}^\mu \wedge \theta_\mu{}^\alpha + \theta_\beta{}^\mu \wedge \Theta_\mu{}^\alpha \tag{1.3}$$

We first prove (1.1) for $k=0$.

$$\begin{aligned} d\Phi_0 = &\sum \epsilon^{\alpha_1\cdots\alpha_p} \sum_i (-1)^{i-1}\theta^{\alpha_1}\cdots(\sum_\beta \theta^\beta)\wedge\theta_\beta^{\ \alpha_i}\cdots\theta^{\alpha_p} \\ &+\sum \epsilon^{\alpha_1\cdots\alpha_p}\sum_i(-1)^{i-1}\theta^{\alpha_1}\cdots\Theta^{\alpha_i}\cdots\theta^{\alpha_p} \\ &=\sum \epsilon^{\alpha_1\cdots\alpha_p}\theta^{\alpha_1}\cdots(\sum_i\Theta^{\alpha_i})\cdots\theta^{\alpha_p} \\ &=p\Psi_0. \end{aligned}$$

Then we deal with the general case, $1\le k\le[\frac{p}{2}]$. Observe that

$$\begin{aligned} d\Phi_k =& -2k\sum \epsilon^{\alpha_1\cdots\alpha_p}(\sum_{\beta=1}^{p}\Theta_{\alpha_1}^{\ \beta}\wedge\theta_\beta^{\ \alpha_2})\Theta_{\alpha_3}^{\ \alpha_4}\cdots\Theta_{\alpha_{2k-1}}^{\ \alpha_{2k}}\theta^{2k+1}\cdots\theta^{\alpha_p} \\ &+(p-2k)\sum\epsilon^{\alpha_1\cdots\alpha_p}\Theta_{\alpha_1}^{\ \alpha_2}\cdots\Theta_{\alpha_{2k-1}}^{\ \alpha_{2k}}(\sum_{\beta=1}^{p}\theta^\beta\wedge\theta_\beta^{\ \alpha_{2k+1}})\theta^{\alpha_{2k+2}}\cdots\theta^{\alpha_p} \\ =& -2k\sum \epsilon^{\alpha_1\cdots\alpha_p}(\sum_{t=1}^{2k}\Theta_{\alpha_1}^{\ \alpha_t}\wedge\theta_{\alpha_t}^{\ \alpha_2})\Theta_{\alpha_3}^{\ \alpha_4}\cdots\Theta_{\alpha_{2k-1}}^{\ \alpha_{2k}}\theta^{2k+1}\cdots\theta^{\alpha_p} \\ &-2k\sum \epsilon^{\alpha_1\cdots\alpha_p}(\sum_{t=2k+1}^{p}\Theta_{\alpha_1}^{\ \alpha_t}\wedge\theta_{\alpha_t}^{\ \alpha_2})\Theta_{\alpha_3}^{\ \alpha_4}\cdots\Theta_{\alpha_{2k-1}}^{\ \alpha_{2k}}\theta^{2k+1}\cdots\theta^{\alpha_p} \\ &+(p-2k)\sum\epsilon^{\alpha_1\cdots\alpha_p}\Theta_{\alpha_1}^{\ \alpha_2}\cdots\Theta_{\alpha_{2k-1}}^{\ \alpha_{2k}}(\sum_{t=1}^{2k}\theta^{\alpha_t}\wedge\theta_{\alpha_t}^{\ \alpha_{2k+1}})\theta^{\alpha_{2k+2}}\cdots\theta^{\alpha_p} \\ &+(p-2k)\Psi_k \\ &=-2kA-2kB+(p-2k)C+(p-2k)\Psi_k. \end{aligned}$$

We assert that $A=0$.

$$\begin{aligned} A =& \sum \epsilon^{\alpha_1\cdots\alpha_p}(\sum_{t=1}^{2k}\Theta_{\alpha_1}^{\ \alpha_t}\wedge\theta_{\alpha_t}^{\ \alpha_2})\Theta_{\alpha_3}^{\ \alpha_4}\cdots\Theta_{\alpha_{2k-1}}^{\ \alpha_{2k}}\theta^{2k+1}\cdots\theta^{\alpha_p} \\ &=2k\sum\epsilon^{\alpha_1\cdots\alpha_4\cdots\alpha_p}\Theta_{\alpha_1}^{\ \alpha_3}\theta_{\alpha_3}^{\ \alpha_2}\Theta_{\alpha_3}^{\ \alpha_4}\cdots\Theta_{\alpha_{2k-1}}^{\ \alpha_{2k}}\theta^{2k+1}\cdots\theta^{\alpha_p} \\ &=2k\sum\epsilon^{\alpha_4\cdots\alpha_1\cdots\alpha_p}\Theta_{\alpha_4}^{\ \alpha_3}\theta_{\alpha_3}^{\ \alpha_2}\Theta_{\alpha_3}^{\ \alpha_1}\cdots\Theta_{\alpha_{2k-1}}^{\ \alpha_{2k}}\theta^{2k+1}\cdots\theta^{\alpha_p} \\ &=-2k\sum\epsilon^{\alpha_1\cdots\alpha_4\cdots\alpha_p}\Theta_{\alpha_4}^{\ \alpha_3}\theta_{\alpha_3}^{\ \alpha_2}\Theta_{\alpha_3}^{\ \alpha_1}\cdots\Theta_{\alpha_{2k-1}}^{\ \alpha_{2k}}\theta^{2k+1}\cdots\theta^{\alpha_p} \\ &=-A. \end{aligned}$$

Thus $A=0$. It is not difficult to verify that

$$\begin{aligned} B &= (p-2k)\sum\epsilon^{\alpha_1\cdots\alpha_p}\Theta_{\alpha_1}^{\ \alpha_2}\cdots\Theta_{\alpha_{2k-1}}^{\ \alpha_{2k}}\theta^{\alpha_{2k}}\theta_{\alpha_{2k}}^{\ \alpha_{2k+1}}\theta^{\alpha_{2k+2}}\cdots\theta^{\alpha_p}. \\ C &= 2k\sum\epsilon^{\alpha_1\cdots\alpha_p}\Theta_{\alpha_1}^{\ \alpha_2}\cdots\Theta_{\alpha_{2k-1}}^{\ \alpha_{2k}}\theta^{\alpha_{2k}}\theta_{\alpha_{2k}}^{\ \alpha_{2k+1}}\theta^{\alpha_{2k+2}}\cdots\theta^{\alpha_p}. \end{aligned}$$

Remark: If $p = 2m$, then

$$\Phi_m := \sum \epsilon^{\alpha_1 \cdots \alpha_p} \Theta_{\alpha_1}^{\ \alpha_2} \cdots \Theta_{\alpha_{p-1}}^{\ \alpha_p}$$

is a closed form representing the Euler class of $\mathcal{V}TE$.

Let $\{c_k\}$ be an arbitrary set of constants. Define

$$\tilde{\Pi} : = \sum_{k=0}^{[\frac{p}{2}]} c_k \Phi_k, \tag{1.4}$$

$$\widetilde{\mathrm{Pf}} : = \sum_{i=0}^{[\frac{p}{2}]} (p - 2k) c_k \Psi_k. \tag{1.5}$$

It follows from Lemma 1.1 that $d\tilde{\Pi} = \widetilde{\mathrm{Pf}}$. However, $\int_{E_x} \tilde{\Pi} \neq \textit{constant}$. In §2, we shall modify $\tilde{\Pi}$ as well as $\widetilde{\mathrm{Pf}}$ to get the desired Π and Pf satisfying (0.3)(0.4).

2. The construction of Pf and Π on a fiber bundle

Let $\pi : E \to M$ be an oriented fiber bundle with $\dim E_x = p$. Given a triple $\{\mathfrak{p}, h, D\}$ over E. In this section we shall construct a $(p+1)$-form Pf and a p-form Π, satisfying (0.3)(0.4) on E.

Let $\{f_\alpha\}$ be a positive orthonormal basis for $(\mathcal{V}TE, h)$. Put

$$\mathfrak{p} := \theta^\alpha \otimes f_\alpha, \qquad D f_\beta = \theta_\beta^{\ \alpha} \otimes f_\alpha.$$

Let $\{\omega^i\}_{i=1}^q$ be a positive co-frame for $\pi^* T^* E \subset T^* E$. Then $T^* E$ has the following direct decomposition

$$T^* E = \mathrm{span}\{\omega^i\} \oplus \mathrm{span}\{\theta^\alpha\}.$$

Let (Θ^α) and $(\Theta_\beta^{\ \alpha})$ be given by (0.7)(0.8). (Θ^α) and $(\Theta_\beta^{\ \alpha})$ can be expressed as follows

$$\Theta^\alpha = \frac{1}{2} R^\alpha_{\ ij} \omega^i \wedge \omega^j + P^\alpha_{\ i\mu} \omega^i \wedge \theta^\mu + \frac{1}{2} Q^\alpha_{\ \lambda\mu} \theta^\lambda \wedge \theta^\mu \tag{2.1}$$

$$\Theta_\beta^{\ \alpha} = \frac{1}{2} R^{\ \alpha}_{\beta\ ij} \omega^i \wedge \omega^j + P^{\ \alpha}_{\beta\ i\mu} \omega^i \wedge \theta^\mu + \frac{1}{2} Q^{\ \alpha}_{\beta\ \lambda\mu} \theta^\lambda \wedge \theta^\mu \tag{2.2}$$

Let (Q^α) and $(Q_\beta^{\ \alpha})$ be given by (0.11). we have

$$Q^\alpha = \frac{1}{2} Q^\alpha_{\ \lambda\mu} \theta^\lambda \wedge \theta^\mu \tag{2.3}$$

$$Q_\beta^{\ \alpha} = \frac{1}{2} Q^{\ \alpha}_{\beta\ \lambda\mu} \theta^\lambda \wedge \theta^\mu. \tag{2.4}$$

Let $h_x = (i_x)^* h$, where $i_x : E_x \to E$ denotes the natural embedding. The volume function V is defined by $V(x) = \mathrm{vol}(E_x, h_x)$. Let $\{\dot{f}_\alpha\}$ be the local frame for TE_x such that $(i_x)_*(\dot{f}_\alpha) = f_\alpha$. The induced linear connection $\dot{D}$ on TE_x is given by

$$\dot{D} \dot{f}_\beta = (i_x)^* \theta_\beta^{\ \alpha} \otimes \dot{f}_\alpha. \tag{2.5}$$

Since $(i_x)^*\omega^i = 0$,

(2.6) $(i_x)^*\Theta^\alpha = (i_x)^*Q^\alpha$ is the torsion form of $\dot{D}$, and
(2.7) $(i_x)^*\Theta_\beta{}^\alpha = (i_x)^*Q^\alpha_\beta$ is the curvature form of $\dot{D}$.

From (2.6)(2.7), one can see that if $Q^\alpha = 0$, then $(i_x)^*Q^\alpha_\beta$ is the Riemann curvature form of h_x w.r.t. $\{\dot{f}_\alpha\}$.

Let $\mathcal{O}_\beta{}^\alpha$, Φ_k, Ψ_k, $\Phi_k^{(v)}$, $\Psi_k^{(v)}$ and $F_k^{(v)}$ be given by (0.12)-(0.17). By the same argument as for Lemma 1.1, we get the following

Lemma 2.1. *For $0 \le k \le [\frac{p}{2}]$, the following hold*

(2.8) $$d\Phi_k^{(v)} = (p-2k)\Psi_k^{(v)} + kF_k^{(v)}.$$

Let c_k be arbitrary constants. Define Pf as in (0.18) and Π by

(2.9) $$\Pi = \frac{1}{p!V(x)}\{\sum_{k=0}^{[\frac{p}{2}]} c_k(\Phi_k - \Phi_k^{(v)}) + \Phi_0\}$$

Proposition 2.2. *Let Pf and Π be constructed by (0.18) and (2.9). Then*

(2.10) $$d\Pi = Pf$$

and Π satisfies

(2.11) $$\int_{E_x} (i_x)^*\Pi = 1, \qquad \forall x \in M.$$

Proof. Define dV on E by

$$dV = \theta^1 \cdots \theta^p.$$

Clearly, $dV_x := (i_x)^*dV$ is the Riemann volume form of (E_x, h_x). By Lemmas 1.1 and 2.1, one can easily verify (2.10). It follows from (2.6)(2.7) that

(2.12) $$(i_x)^*\Phi_k = (i_x)^*\Phi_k^{(v)}, \qquad (i_x)^*\Phi_0 = p!(i_x)^*dV.$$

Thus

$$\int_{E_x} (i_x)^*\Pi = \frac{1}{V(x)}\int_{E_x}(i_x)^*dV = 1.$$

□

A natural question arises: Under what curvature condition $V(x) = constant$? We the following

Proposition 2.5. *If $P^\alpha_{k\beta} + P^\beta_{k\alpha} = 0$, then all fibers (E_x, h_x) are isometric to each other. If $P^\alpha_{k\alpha} = 0$, then $V(x) = constant$.*

Since the proof is quite simple (compare [S]) , so is omitted here. Let

(2.13) $$\mathcal{P} := P^\alpha_{k\alpha}\omega^k.$$

We have, in general

(2.14) $$d(\log V)(u) = \frac{1}{V(x)}\int_{E_x}(i_x)^*\mathcal{P}(X_u)dV, \qquad \forall u \in TM$$

where X_u denote the horizontal lift of u. Thus if $P^\alpha_{k\alpha} = 0$, then $V(x) = constant$ (compare [BS]).

3. Proof of Theorem 0.1

Let M be as in Theorem 0.1. Let X be an arbitrary vector field with isolated singularilites $\{x_i\}_{i=1}^q$. If follows from (2.10) that

$$\int_{M\setminus(\cup_{i=1}^q B_\epsilon(x_i))}[X]^*\mathrm{Pf} = \int_{M\setminus(\cup_{i=1}^q B_\epsilon(x_i))} d[X]^*\Pi = \sum_{i=1}^q \int_{\partial B_\epsilon(x_i)}[X]^*\Pi.$$

Here $B_\epsilon(x) := \varphi^{-1}(\mathbb{B}^n(\epsilon))$ for some coordinate system $\varphi : U \to \mathbb{R}^n$ with $\varphi(x) = 0$. Using (2.11), we can easily get

$$\lim_{\epsilon\to 0^+}\int_{\partial B_\epsilon(x_i)}[X]^*\Pi = \mathrm{ind}_{x_i}(X),$$

where $\mathrm{ind}_x(X)$ denotes the index of X at x. Thus

$$\int_M [X]^*\mathrm{Pf} = \sum_{i=1}^q \mathrm{ind}_{x_i}(X).$$

Theorem 0.1 follows from the Hopf theorem that the Euler number

$$\chi(M) = \sum_{i=1}^q \mathrm{ind}_{x_i}(X).$$

4. Finsler manifolds

In this section, we shall apply Theorem 0.1 to Finsler manifolds.

Let M be a n-dimensional manifold ($n = p + 1$). Let $\pi : \mathcal{S} \to M$ denote the tangent sphere bundle of M and π^*TM denote the pull-back tangent bundle over $\mathcal{S}$. The vectors in π^*TM are denoted by $([v], w)$, where $[v] \in \mathcal{S}_x, w \in T_xM$. There is a canonical bundle map $\rho : T\mathcal{S} \to \pi^*TM$ defined by

$$\rho(\hat{X}) = ([v], \pi_*(\hat{X})), \qquad \forall \hat{X} \in T_{[v]}\mathcal{S}.$$

Let (x^i) be a locall coordinate system in M and (x^i, y^j) the standard coordinate system in TM. Denote by $\partial_i|_{[v]} = ([v], \frac{\partial}{\partial x^i}|_x)$ the natural local basis for π^*TM at $[v] \in \mathcal{S}_x$. Let F be a Finsler metric on M and write $F(x, y) = F(y^i \frac{\partial}{\partial x^i}|_x)$. The induced Riemann metric $g : \pi^*TM \otimes \pi^*TM \to \mathbb{R}$ and the Cartan tensor $A : \pi^*TM \otimes \pi^*TM \otimes \pi^*TM \to \mathbb{R}$ are defined by

$$g(\partial_i, \partial_i)_{[v]} = \frac{1}{2}\frac{\partial^2[F^2]}{\partial y^i \partial y^j}(x, y)$$

$$A(\partial_i, \partial_j, \partial_k)_{[v]} = \frac{1}{4}F\frac{\partial^3[F^2]}{\partial y^i \partial y^j \partial y^k}(x, y).$$

Here $v = y^i\frac{\partial}{\partial x^i}|_x$. The canonical section ℓ of π^*TM is defined by

$$\ell := ([v], \frac{v}{F(v)}).$$

Let $\{b_i\}_{i=1}^n$ be an arbitrary local frame for π^*TM. Write $g_{ij} = h(b_i, b_j), A_{ijk} = A(b_i, b_j, b_k)$, and $\ell = \ell^i b_i$. We have

$$g_{ij}\ell^i\ell^j = 1, \quad A_{ijk}\ell^i = 0. \tag{4.1}$$

Let $\{\omega^i\}_{i=1}^n$ be defined by

$$\rho = \omega^i \otimes b_i.$$

S. S. Chern proves the following theorem ([Ch1][Ch2][BC1]).

Theorem 4.1(Chern). *There is a unique set of local* 1*-forms* $\{\omega_j{}^i\}$ *on* $\mathcal{S}$ *satisfying*

$$d\omega^i = \omega^j \wedge \omega_j{}^i \tag{4.2}$$

$$dh_{ij} = h_{kj}\omega_i{}^k + h_{ik}\omega_j{}^k + 2A_{ijk}\theta^k, \tag{4.3}$$

where

$$\theta^i := d\ell^i + \ell^j \omega_j{}^i. \tag{4.4}$$

Define a set of 1-forms $(\theta_j{}^i)$ by

$$\theta_j{}^i := \omega_j{}^i + A^i_{jk}\theta^k,$$

where $A^i_{jk} = g^{il}A_{jkl}$. It is easy to verify that

$$d\omega^i = \omega^j \wedge \theta_j{}^i - A^i_{jk}\omega^j \wedge \theta^k \tag{4.5}$$

$$dh_{ij} = h_{kj}\theta_i{}^k + h_{ik}\theta_j{}^k. \tag{4.6}$$

By (4.1) we also have

$$\theta^i = d\ell^i + \ell^j \theta_j{}^i. \tag{4.7}$$

The Cartan connection ∇ and the Chern connection ∇' on π^*TM are given by

$$\nabla b_j = \theta_j{}^i \otimes b_i, \quad \nabla' b_j = \omega_j{}^i \otimes b_i.$$

(4.2) means that ∇' is torsion-free and (4.6) means that ∇ is metric-compatible.

Define a bundle map $\mu : T\mathcal{S} \to \pi^*TM$ by

$$\mu := \nabla\ell = \theta^i \otimes b_i.$$

It is easy to check that rank$\mu = n - 1$ and

$$\mu|_{\mathcal{V}T\mathcal{S}} : \mathcal{V}T\mathcal{S} \to \ell^\perp$$

is a bundle isomorphism.

From now on, we always let $\{e_i\}_{i=1}^n$ be an *orthonormal* frame for π^*TM such that $e_n = \ell$. Let $\{f_\alpha\}_{\alpha=1}^{n=1}$ be the orthonormal basis for $\mathcal{V}T\mathcal{S}$ such that $\mu(f_\alpha) = e_\alpha$.

Put $\rho = \omega^i \otimes e_i$ and $\mu = \theta^\alpha \otimes e_\alpha$. We have a direct decomposition for $T^*\mathcal{S}$

$$T^*\mathcal{S} = \mathrm{span}\{\omega^i\} \oplus \mathrm{span}\{\theta^\alpha\}. \tag{4.8}$$

Put $\nabla e_j = \theta_j^{\ i} \otimes e_i$. Hence $\theta^\alpha = \theta_n^{\ \alpha}$. Then we get a triple $\{\mathfrak{p}, h, D\}$ given by (0.5), that is,

$$\mathfrak{p} = \theta^\alpha \otimes f_\alpha, \quad h = \theta^\alpha \otimes \theta^\alpha|_{\mathcal{V}T\mathcal{S}}, \quad Df_\beta = \theta_\beta^{\ \alpha} \otimes f_\alpha. \tag{4.9}$$

The curvature form $(\Omega_j^{\ i})$ of ∇ is given by (0.6). It can be expressed by

$$\Omega_j^{\ i} = \frac{1}{2}\tilde{R}_j^{\ i}{}_{kl}\omega^k \wedge \omega^l + \tilde{P}_j^{\ i}{}_{k\alpha}\omega^j \wedge \theta^\alpha + \frac{1}{2}\tilde{Q}_j^{\ i}{}_{\alpha\beta}\theta^\alpha \wedge \theta^\beta.$$

Let (Θ^α) and $(\Theta_\beta^{\ \alpha})$ be given by (0.7)(0.8). From the definition, it is easy to show that

$$\Theta^\alpha = \Omega_n^{\ \alpha}, \tag{4.10}$$

$$\Theta_\beta^{\ \alpha} = \Omega_\beta^{\ \alpha} - \theta^\beta \wedge \theta^\alpha. \tag{4.11}$$

$$Q^\alpha = \tilde{Q}_n^{\ \alpha} \tag{4.12}$$

$$Q_\beta^{\ \alpha} = \tilde{Q}_\beta^{\ \alpha} - \theta^\beta \wedge \theta^\alpha. \tag{4.13}$$

Let (Θ^α) and $(\Theta_\beta^{\ \alpha})$ be expressed by (2.1)(2.2). It follows from (4.10)-(4.13) that

$$R^\alpha_{\ ij} = \tilde{R}_n^{\ \alpha}{}_{ij}, \quad P^\alpha_{\ i\mu} = \tilde{P}_n^{\ \alpha}{}_{i\mu}, \quad Q^\alpha_{\ \lambda\mu} = \tilde{Q}_n^{\ \alpha}{}_{\lambda\mu}. \tag{4.14}$$

$$R_\beta^{\ \alpha}{}_{ij} = \tilde{R}_\beta^{\ \alpha}{}_{ij}, \quad P_\beta^{\ \alpha}{}_{i\mu} = \tilde{P}_\beta^{\ \alpha}{}_{i\mu}, \quad Q_\beta^\alpha{}_{\lambda\mu} = \tilde{Q}_\beta^\alpha{}_{\lambda\mu} + \delta^\alpha_\lambda\delta^\beta_\mu - \delta^\alpha_\mu\delta^\beta_\lambda. \tag{4.15}$$

By the well-known fact (see e.g. [BSC1]), we have

$$\tilde{Q}_j^{\ i}{}_{\alpha\beta} = A^l_{j\alpha}A^i_{l\beta} - A^l_{j\beta}A^i_{l\alpha}. \tag{4.16}$$

$$Q^\alpha_{\ \lambda\mu} = \tilde{Q}_n^{\ \alpha}{}_{\lambda\mu} = 0. \tag{4.17}$$

Remark. It follows from (4.17) that $Q^\alpha = 0$. By (2.5)-(2.7), one can see that the induced $\dot{D}$ on $\mathcal{S}_x$ is the Christoffel (Levi-Civita) connection of h_x. Further, $(i_x)^*Q_\beta^{\ \alpha}$ is the Riemann curvature of h_x.

Define a form Pf as in (0.18) in terms of $\Phi_k, \Psi_k, \Phi_k^{(v)}, \Psi_k^{(v)}, F_k^{(v)}$, which are related to θ^i and $\Omega_j^{\ i}$ by (0.11)-(0.17) and (4.10)-(4.13). The following is just a corollary of Theorem 0.1.

Theorem 4.2. *Let (M, F) be an oriented closed Finsler manifold of dimension $n = p+1$. For any vector field X with isolated zeros on M,*

$$\int_M [X]^* Pf = \chi(M). \tag{4.11}$$

5. Riemannian manifolds

In this section we shall briefly derive the Gauss-Bonnet-Chern formula for Riemann manifolds from Theorem 4.2, by choosing a suitable set of constants $\{c_k\}$ in (0.18).

Let $(M, \bar{g})$ be an oriented Riemannian manifold of dimension $n = p+1$. Thus $F(v) := \sqrt{\bar{g}(v,v)}$ is a special Finsler metric. We shall continue to use the notations in §4. Since $A = 0$, it follows from (4.15) (4.16) that

$$Q_\beta^{\ \alpha} = -\theta^\beta \wedge \theta^\alpha. \tag{5.1}$$

By (2.7), we know that $(\mathcal{S}_x, h_x)$ has constant curvature $= 1$ (if $n \geq 3$). Thus all $(\mathcal{S}_x, h_x)$ are naturally isometric to the standard unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, in particular, $V(x) = \mathrm{vol}(\mathbb{S}^{n-1})$. It follows from (5.1) that

$$\mathcal{O}_\beta^{\ \alpha} = -\Theta^\beta \wedge \theta^\alpha + \theta^\beta \wedge \Theta^\alpha. \tag{5.2}$$

Substituting (5.1) and (5.2) into $\Psi_k^{(v)}$ and $F_k^{(v)}$ yields

$$\Psi_k^{(v)} = (-1)^k \Psi_0, \qquad F_k^{(v)} = (-1)^k 2\Psi_0. \tag{5.3}$$

Suppose that constants c_k satisfy

$$\sum_{i=0}^{[\frac{n-1}{2}]} (-1)^i c_i = 1. \tag{5.4}$$

For such set of c_k, define Pf as in (0.18). It follows from (5.3)(5.4) that

$$\mathrm{Pf} = \frac{1}{(n-1)!\mathrm{vol}(\mathbb{S}^{n-1})} \sum_{i=0}^{[\frac{n-1}{2}]} (n-1-2i) c_i \Psi_i. \tag{5.5}$$

Define

$$\mathrm{Pf}_o = -n \sum \epsilon^{\alpha_1 \cdots \alpha_{n-1}} \Omega_{\alpha_1}^{\ \alpha_2} \cdots \Omega_n^{\ \alpha_{n-1}}. \tag{5.6}$$

Without much difficulty, one can find constants c_i satisfying (5.4), for which, the n-form Pf in (5.5) has the following form

$$\mathrm{Pf} = (-1)^{\frac{n}{2}} \frac{2}{n!\mathrm{vol}(\mathbb{S}^n)} \mathrm{Pf}_o. \tag{5.7}$$

Let $\{\bar{e}_i\}_{i=1}^n$ be an arbtrary orthnormal frame for $(TM, \bar{g})$, and $\{e_i = ([v]; \bar{e}_i)\}_{i=1}^n$ be the corresponding frame for π^*TM. The induced Riemann metric g on π^*TM is given by $g(e_i, e_j) = \bar{g}(\bar{e}_i, \bar{e}_j)$. Let $\{\bar{\omega}^i\}_{i=1}^n$ be the dual co-frame frame for T^*M. Then $\{\omega^i\}$ defined by ρ (see §4) satisfy

$$\omega^i = \pi^* \bar{\omega}^i.$$

Let $(\bar{\omega}_j^{\ i})$ be the Levi-Civita connection form on TM and $(\bar{\Omega}_j^{\ i})$ denote the curvature form. Then the Chern/Cartan connection form $(\omega_j^{\ i}) = (\theta_j^{\ i})$ satisfies

$$\omega_j^{\ i} = \theta_j^{\ i} = \pi^* \omega_j^{\ i}.$$

The curvature form $(\Omega_j^{\ i})$ has the following form

$$\Omega_j^{\ i} = \pi^* \bar{\Omega}_j^{\ i}. \tag{5.8}$$

Then Pf_o in (5.6) can also expressed by

$$\mathrm{Pf}_o = \sum \epsilon^{i_1 \cdots i_n} \Omega_{i_1}^{\ i_2} \cdots \Omega_{i_{n-1}}^{\ i_n} = \pi^* \sum \epsilon^{i_1 \cdots i_n} \bar{\Omega}_{i_1}^{\ i_2} \cdots \bar{\Omega}_{i_{n-1}}^{\ i_n} \tag{5.10}$$

It follows from (5.10) that the n-form Pf in (5.7) has the following form

$$\mathrm{Pf} = (-1)^{\frac{n}{2}} \frac{2}{n! \mathrm{vol}(\mathbb{S}^n)} \pi^* \sum \epsilon^{i_1 \cdots i_n} \bar{\Omega}_{i_1}^{\ i_2} \cdots \bar{\Omega}_{i_{n-1}}^{\ i_n}.$$

Thus for any vector filed with isolated zeros on M,

$$[X]^* \mathrm{Pf} = (-1)^{\frac{n}{2}} \frac{2}{n! \mathrm{vol}(\mathbb{S}^n)} \sum \epsilon^{i_1 \cdots i_n} \bar{\Omega}_{i_1}^{\ i_2} \cdots \bar{\Omega}_{i_{n-1}}^{\ i_n}.$$

It follows from (4.18) that

$$(-1)^{\frac{n}{2}} \frac{2}{n! \mathrm{vol}(\mathbb{S}^n)} \int_M \sum \epsilon^{i_1 \cdots i_n} \bar{\Omega}_{i_1}^{\ i_2} \cdots \bar{\Omega}_{i_{n-1}}^{\ i_n} = \chi(M). \tag{5.11}$$

This is just the Gauss-Bonnet-Chern theorem proved by S. S. Chern in [Ch3].

References

[B] H. Busemann, *On geodesic curvature in two dimensional Finsler spaces*, Ann. Mat. Pura Appl. **31(4)** (1950), 281-295.

[BC1] D. Bao and S.S. Chern, *On a notable connection in Finsler geometry*, Houston J. Math. **19(1)** (1993), 135-180.

[BC2] ———, *A note on the Gauss-Bonnet theorem for Finsler spaces*, Ann. Math. **143** (1966), 233-252.

[BCS1] D. Bao, S.S. Chern and Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Spring-Verlag, to appear.

[BCS2] ———, *On the Gauss-Bonnet integrand for 4-dimensional Landsberg spaces*, Contemporary Math. (to appear).

[BS] D. Bao and Z. Shen, *On the volume of unit tangent spheres in a Finsler manifold*, Results in Math. **26** (1994), 1-17.

[Ch1] S.S. Chern, *On the Euclidean connections in a Finsler space*, Proc. Nat'l Acad. Sc. **29(1)** (1943), 33-37; or Selected Papers, vol II, 107-111, Springer 1989..

[Ch2] ———, *On Finsler geometry*, C. R. Acad. Sc. Paris **314** (1992), 757-761.

[Ch3] ———, *A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds*, Ann. Math. **45** (1944), 747-752.

[Ch4] ______, *Historical remarks on Gauss-Bonnet*, In Analysis et Cetera, volume dedicated to Jürgen Moser,, 209-217, Academic Press, 1990.

[L] A. Lichnerowicz, *Quelques théorèmes de géométrie différentielle globale*, Comm. Math. Helv. **22** (1949), 271-301.

[M] M. Matsumoto, *On three important problems in Finsler geometry*, Differential Geometry and applications, Proceedings of the Conference, 1988, Dubrovnik, Yugoslavia.

[R] H. Rund, *A divergence theorem for Finsler metrics*, Monatsh. Math. **79** (1975), 233-252.

[S] Z. Shen, *Some formulas of Gauss-Bonnet-Chern type in Riemann-Finsler geometry*, J. für die reine und angewandte Mathematik **475** (1996), 149-165.

Department of Mathematical Science, Indiana University-Purdue University at Indianapolis, Indianapolis, IN 46202–3216, U.S.A.

E-mail address: zshen@math.iupui.edu