Math 462 Homework Assignment #12

Your Name: __________________________

Due Date: May 6, 2003

1. Let $D^2$ denote the unit disk in $E^2$. The first fundamental form of a coordinate patch $\mathbf{x}: D^2 \to E^n$ is given by

$$I = \frac{1 - v^2}{1 - u^2 - v^2} du^2 + \frac{-2uv}{1 - u^2 - v^2} du dv + \frac{1 - u^2}{1 - u^2 - v^2} dv^2.$$  

Find the Gauss curvature $K$.

2. Let $\mathcal{U} = \{(u, v) \mid u > 0, v > 0\}$. The first fundamental form of a coordinate patch $\mathbf{x}: \mathcal{U} \to E^n$ is given by

$$I = \psi_{uu}(u, v) du^2 + 2\psi_{uv}(u, v) du dv + \psi_{vv}(u, v) dv^2,$$

where

$$\psi(u, v) = \frac{u^2}{4v} + \ln \sqrt{\frac{\pi}{v}}.$$  

Find the Gauss curvature $K$.

3. Let $\mathbf{x}: R^2 \to S^2 \subset E^3$ be a coordinate patch of the unit sphere $S^2$ given by

$$\mathbf{x}(u, v) = \left[ \frac{u}{\sqrt{1 + u^2 + v^2}}, \frac{v}{\sqrt{1 + u^2 + v^2}}, \frac{1}{\sqrt{1 + u^2 + v^2}} \right].$$  

Find the first fundamental form of the above coordinate patch. Then compute the Gauss curvature.

4. Let $\sigma, \gamma : (a, b) \to E^3$ be unit speed curves with same curvature $k(s) = |\dot{\sigma}(s)| = |\dot{\gamma}(s)|$. Let $S$ and $M$ be surfaces covered by the following coordinate patches $\mathbf{x} = \mathbf{x}(s, v)$ and $\mathbf{y} = \mathbf{y}(s, v)$, respectively,

$$\mathbf{x}(s, v) := \sigma(s) + v\dot{\sigma}(s), \quad \mathbf{y}(s, v) := \gamma(s) + v\dot{\gamma}(s).$$  

Show that the map $\mathbf{f} : S \to M$ defined by $\mathbf{f}(\mathbf{x}(s, v)) := \mathbf{y}(s, v)$ is an isometry.

5. Let $\mathbf{x} = \mathbf{x}(u, v)$ be a coordinate patch of some surface and $\mathbf{N} = \mathbf{N}(u, v)$ denote the unit normal vector to the surface at $p = \mathbf{x}(u, v)$. Show that the Gauss curvature $K$ can be expressed by

$$K^2 = \frac{(\mathbf{N}_u \cdot \mathbf{N}_u)(\mathbf{N}_v \cdot \mathbf{N}_v) - (\mathbf{N}_u \cdot \mathbf{N}_v)^2}{(\mathbf{x}_u \cdot \mathbf{x}_u)(\mathbf{x}_v \cdot \mathbf{x}_v) - (\mathbf{x}_u \cdot \mathbf{x}_v)^2}.$$
Hint: Refer to Problem # 6 in Homework 10.

6. Assume that the first fundamental form of some coordinate patch \( \mathbf{x}: \mathcal{U} \to E^n \) is given by

\[
I = \frac{v}{u^2} du^2 + \frac{1}{\alpha u^2} dv^2,
\]

where \( \alpha > 0 \) is an arbitrary positive constant. Show that the Gauss curvature \( K = -\frac{1}{\alpha} \). Find the geodesic equations.

7. Take the following coordinate patch for the torus \( T^2 \),

\[
\mathbf{x} = \left[ (R + r \cos u) \cos v, (R + r \cos u) \sin v, r \sin u \right].
\]

Show that the geodesic equations are

\[
u'' + \frac{(R + r \cos u)}{r} \sin u \left( v' \right)^2 = 0
\]

\[
v'' - 2 \frac{r \sin u}{(R + r \cos u)} u' v' = 0.
\]

8. Find the geodesics on the cylinder \( x^2 + y^2 = 1 \) in \( E^3 \) by using the geodesic equations.

9. Recall that the shape operator \( S_p: T_pM \to T_pM \) is given by

\[
S_p(\lambda \mathbf{x}_u + \mu \mathbf{x}_v) = -\lambda \mathbf{N}_u - \mu \mathbf{N}_v.
\]

Show that the shape operator \( S \) for the saddle surface \( z = xy \) parameterized by

\[
\mathbf{x}(u, v) = (u, v, uv)
\]

is given on a basis by

\[
S(\mathbf{x}_u) = \frac{-uv}{(1 + u^2 + v^2)^{3/2}} \mathbf{x}_u + \frac{1 + u^2}{(1 + u^2 + v^2)^{3/2}} \mathbf{x}_v
\]

\[
S(\mathbf{x}_v) = \frac{1 + u^2}{(1 + u^2 + v^2)^{3/2}} \mathbf{x}_u - \frac{uv}{(1 + u^2 + v^2)^{3/2}} \mathbf{x}_v.
\]

10. The right circular cone is parameterized by

\[
\mathbf{x} = (u \sin \alpha \cos \theta) \mathbf{e}_1 + (u \sin \alpha \sin \theta) \mathbf{e}_2 + (u \cos \alpha) \mathbf{e}_3,
\]

where \( \alpha = \text{constant}, \ 0 < \alpha < \pi/2, \ u > 0 \). Find the geodesic curvature \( k_g \) of the curve \( u = u_0 \) and the geodesic equations.