Solutions to Exam 2

[15 points] 1. Calculate the derivative of each of the following functions. It is not necessary to simplify your answers.

a) \( g(x) = (x^4 - 3x^2 + 1)(x - \sqrt{x} + 7x^{-2}) \)

\[ g'(x) = \left(4x^3 - 6x + 1\right)(x - \sqrt{x} + 7x^{-2}) + \left(x^4 - 3x^2 + 1\right)(1 - \frac{1}{2}x^{-1/2} - 14x^{-3}) \]

b) \( f(t) = \frac{e^t + t^2}{t^4 - 5t + 8} \)

\[ f'(t) = \frac{(e^t - 2t)(t^4 - 5t^3 + 8) - (e^t + t^2)(4t^3 - 5)}{(e^t + t^2)^2} \]

c) \( y = \ln(\sin^{-1}x^2) \)

\[ y' = \frac{2x}{\sin^{-1}(x^2)\sqrt{1-x^4}} \]

[10 points] 2. Calculate the derivative of each of the following functions. It is not necessary to simplify your answers.

a) \( f(x) = \sin(e^x) \)

\[ f'(x) = \cos(e^x) \cdot e^x \]

b) \( y = \tan^{-1}(x^3 + 1) \)

\[ y' = \frac{3x^2}{(x^3 + 1)^2 + 1} \]

[10 points] 3. Calculate \( f''(4) \), where \( f(x) = 2x^3 - 3x + 32x^{1/2} \). SIMPLIFY YOUR ANSWER.

\[ f''(x) = 12x - 8x^{-3/2} \]

\[ f''(4) = 48 - \frac{8}{4^{3/2}} = 48 - \frac{8}{(\sqrt{4})^3} = 48 - \frac{8}{4} = 47. \]

[10 points] 4. Calculate \( \frac{dy}{dx} \) at \( x = 1 \) if \( y = \sin\left(\frac{\pi(x+1)}{x+5}\right) \). SIMPLIFY YOUR ANSWER.

\[ \frac{dy}{dx} = \cos\left(\frac{\pi(x+1)}{x+5}\right) \cdot \frac{\pi(x+1) - \pi(x+5)}{(x+5)^2}, \quad \text{at} \quad x = 1 \]

\[ = \cos\left(\frac{2\pi}{6}\right) \cdot \frac{4\pi}{36} = \frac{1}{2} \cdot \frac{\pi}{6} = \frac{\pi}{12}. \]

[10 points] 5. Find an equation for the tangent line to \( y = x^{\sin x} \) at the point \( x = \frac{\pi}{2} \).

\[ \ln y = \sin x \ln x, \quad \frac{\partial y}{y} = \cos x \ln x + \frac{\sin x}{x} \quad \text{and} \quad y'\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \cdot \left(0 \cdot \frac{\pi}{2} + \frac{1}{\pi\frac{\pi}{2}}\right) = 1. \]

Hence the equation of the tangent line is \( y - \left(\frac{\pi}{2}\right)^{\frac{1}{2}} = 1\left(x - \frac{\pi}{2}\right) \), i.e., \( y = x \).
[10 points] 6. Use implicit differentiation to calculate $dy/dx$ at $(2,1)$ if

$$x^2y + xy^2 = x^3 - 2.$$  

$$2xy + x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} = 3x^2,$$

so

$$\frac{dy}{dx} = \frac{3x^2 - y^2 - 2xy}{x^2 + 2xy}.$$

Hence when $(x,y) = (2,1)$

$$\frac{dy}{dx}\bigg|_{x=2, y=1} = \frac{3\cdot 2^2 - 1^2 - 2\cdot 2}{4^2 + 2\cdot 2} = \frac{7}{8}.$$

[15 points] 7. A television camera is positioned 40 miles from the base of a rocket launching pad. The angle of elevation of the camera has to change in order to keep the rocket in sight. If the rocket rises vertically and its speed is 6 miles/minute, find the rate at which the angle of elevation of the camera is increasing 5 seconds after lift-off. Make sure to draw a picture describing this situation.

We know that $\frac{dh}{dt} = 6 \text{ miles/minute}$, and we need $h$ to find $\frac{d\theta}{dt}$ when $t = 5 \text{ seconds}$.

From the picture, we have $\tan \theta = \frac{h}{40}$, so $\theta = \tan^{-1} \left( \frac{h}{40} \right)$ and

$$\frac{d\theta}{dt} = \frac{1}{(\frac{h}{40})^2 + 1} \cdot \frac{dh}{dt},$$

when $t = 5 \text{ seconds}$, $h = 5\cdot \frac{1}{10} = \frac{1}{2}$, so

$$\frac{d\theta}{dt} \bigg|_{t=5 \text{ seconds}} = \frac{1}{\left( \frac{1/2}{40} \right)^2 + 1} \cdot \frac{1/10}{40} = \frac{1}{10 \left( \frac{1/4}{40} + 40 \right)} = \frac{1}{16} \frac{1}{400} = \frac{16}{6400} \text{ rad/second}.$$

[10 points] 8. Let $f(x) = \ln x$. Estimate $\Delta f$ at $a = 1$ using Linear Approximation, where $\Delta x = 0.02$.

$$\Delta f \approx f'(1) \Delta x = \frac{1}{1} \cdot 0.02 = 0.02.$$

[10 points] 9. Find the absolute maximum and the absolute minimum values of $f(x) = x^3 - \frac{3}{2}x^2 + 1$ on the interval $[0,2]$.

$$f'(x) = 3x^2 - 3x = 3x(x - 1),$$

so the critical points of $f$ are $x = 0$ and $x = 1$. Thus the absolute max and min must occur at $x = 0$, $x = 1$, or $x = 2$.

$$f(0) = 1, \ f(1) = 1 - \frac{3}{2} + 1 = \frac{1}{2}, \ f(2) = 8 - \frac{3}{2} \cdot 4 + 1 = 3.$$

So the absolute minimum value is $\frac{1}{2}$ and the absolute max is $3$. 