Exercises 1.3

1. Let \( A = \begin{pmatrix} 4 & 3 & -2 \\ 2 & -5 & 6 \end{pmatrix} \) and let \( B = \begin{pmatrix} 0 & -1 & 3 \\ 2 & 1 & 6 \\ 5 & 2 & 1 \end{pmatrix} \).
   
   (a) Find \( AB \) (from the definition) as a \( 2 \times 3 \) matrix.

   (b) Partition \( A \) as \( (A_{11} \mid A_{12}) \) and \( B \) as \( \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \), where both \( A_{11} \) and \( B_{11} \) are \( 2 \times 2 \) matrices, that is, say what each of \( A_{11}, A_{12}, \ldots, B_{22} \) are.

   (c) Determine each of the relevant products from (b) above and find \( AB \) as a partitioned matrix.

2. Use partitioned matrices to show that if \( A \) is an \( m \times n \) matrix and \( B \) is an \( n \times p \) matrix whose \( k^{th} \) column is zero, then the \( k^{th} \) column of \( AB \) is zero.

3. Explore how your software handles block matrices.
   
   (a) Enter the matrices
   \[
   A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}
   \]
   \[
   C = \begin{pmatrix} 4 & 2.6 & 0 \\ 3 & -3 & 8 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & -2 \\ 1.5 & 4 \end{pmatrix}
   \]

   (b) Make a \( 4 \times 5 \) matrix \( E \) from the matrices \( A, B, C, \) and \( D \) to get
   \[
   E = \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & -1 & 4 & 3 & 1 \\ 4 & 2.6 & 0 & 3 & -2 \\ 3 & -3 & 8 & 1.5 & 4 \end{pmatrix}
   \]
   You probably do not need to retype all the entries! Note that \( E = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \)

   (c) Make a \( 4 \times 4 \) matrix \( F \) from \( E \) by deleting its last column \( F = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & 4 & 3 \\ 4 & 2.6 & 0 & 3 \\ 3 & -3 & 8 & 1.5 \end{pmatrix} \)

4. Prove: If a matrix is multiplied on the right by a diagonal matrix, the \( j^{th} \) column of the product is the \( j^{th} \) diagonal entry times the \( j^{th} \) column of the original matrix. (Compare with Exercise ??)

5. Suppose \( A \) is a square matrix partitioned as
   \[
   A = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}
   \]
   where \( X \) and \( Z \) are square invertible matrices and \( 0 \) is a zero matrix.

   (a) Find formulas for \( P, Q, R, \) and \( S \) so that the block matrix
   \[
   \begin{pmatrix} P & Q \\ R & S \end{pmatrix}
   \]
   is \( A^{-1} \). (Caution: matrix multiplication is not commutative!) If you are successful, you will have shown that matrices with the given block form are invertible.

   (b) Use your formula to find \( A^{-1} \) when \( X = \begin{pmatrix} -1 \end{pmatrix}, Y = \begin{pmatrix} 1 & -1 \end{pmatrix} \), and \( Z = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \) (Note that \( Z^{-1} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \))
Exercises 2.1

1. \[
\begin{align*}
&\{ \begin{array}{rcl}
w & = & 5 \\
2w + x & = & 2 \\
w + x + y & = & -1 \\
w - x + 2y + z & = & 4
\end{array} \} & \text{3. } \begin{array}{rcl}
w & = & 2w - x + 2y + z \\
& = & 1 \\
x + y - z & = & 0 \\
3y + z & = & 0 \\
2z & = & 6
\end{array}
\end{align*}
\]

2. \[
\begin{align*}
&\{ \begin{array}{rcl}
a + 2b + c & = & 2 \\
- a + 3b - c + d & = & 3 \\
a & = & 4 \\
2a + b & = & -1
\end{array} \} & \text{4. } \begin{array}{rcl}
w & = & 2w - x + 2y + z \\
& = & 0 \\
x + y - z & = & 0 \\
y - z & = & 0
\end{array}
\end{align*}
\] (Hint: solve for \(w, x,\) and \(y\) in terms of \(z\). There will be infinitely many solutions, one for each value of \(z\).)

5. Write each system in Problems 6–9 as a matrix equation.

Use your software to solve the following systems. Be sure to check your answers!

6. \[
\begin{align*}
&\{ \begin{array}{rcl}
x + 2y & = & 3 \\
3x + 4y & = & -2
\end{array} \} & \text{8. } \begin{array}{rcl}
w & = & w - y + 2z \\
& = & 0 \\
w + x + 3y - z & = & 5 \\
2w & = & 5z \\
w + x + y + 2z & = & 4
\end{array}
\end{align*}
\]

7. \[
\begin{align*}
&\{ \begin{array}{rcl}
x - y + z & = & 1 \\
x + 3y + 3z & = & 5 \\
2x & = & 4 \\
2x & = & 4 \\
3x + y & = & 4 \\
w - x & = & y + 2z
\end{array} \} & \text{9. } \begin{array}{rcl}
w & = & 2w + 3x + y - z \\
& = & 1 \\
w & = & -w + 2x + 3y + z \\
2w & = & 2w + x - 2y + 3z \\
& = & 0 \\
w & = & w - x + y + 2z
\end{array}
\end{align*}
\]

10. The five-tuples \((2, 2, 1, -1, 1)\) and \((1, 1, 2, -1, -1)\) are both solutions of the system:

\[
\begin{align*}
&\{ \begin{array}{rcl}
a + b + 4c + d + e & = & 8 \\
- a - b + 2c + 2d + e & = & 1 \\
2a + b - c - d - 2e & = & 4 \\
& = & 5 \\
& = & 0
\end{array} \}
\end{align*}
\]

(a) Without using Gaussian elimination or a machine, write down two non-trivial solutions of the associated homogeneous system.

(b) Write down two other solutions of the given system.

11. Let \(A\) be the matrix

\[
\begin{pmatrix}
1 & -1 & 2 & 1 \\
2 & 1 & -3 & -1 \\
1 & 1 & 3 & -2 \\
-1 & 2 & -2 & 3
\end{pmatrix}
\]

and let \(b = (3, -1, 3, 2)\) and let \(c = (0, 4, -4, 4)\).

(a) Check that \(Y = (1, 1, 1, 1)\) solves the system \(AX = b\) and that \(Z = (1, 0, -1, 1)\) solves the system \(AX = c\).

(b) Without using Gaussian Elimination or a machine, find a solution of the system \(AX = (6, -2, 6, 4) = 2b\).

(c) Without using Gaussian Elimination or a machine, find a solution of the system \(AX = (3, 3, -1, 6) = b + c\).

(d) Without using Gaussian Elimination or a machine, find a solution of the system \(AX = (9, 5, 1, 14) = 3b + 2c\).