Math 542 - Topology II, Homework 4
Due date: Tuesday, April 19.

**Problem 1:** Calculate the singular homology of the torus $S^1 \times S^1$ using the long exact sequence of the pair $(S^1 \times S^1, A \times S^1)$ where $A$ is a closed arc on the circle. (Note: you can do this without any explicit analysis of the boundary maps. Said another way, the boundary maps can be calculated algebraically, given the other information you know about the groups in the sequence.)

Problems from old comprehensive exams:

**Problem 2:** Let $C = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1, \text{and} z = 0\}$ and let $A = \mathbb{R}^3 - C$. Calculate the homology groups of $A$. (Hint: use a Mayer-Vietoris sequence.)

**Problem 3:** i) Prove that $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic unless $n = m$, by calculating the relative homology groups $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\})$ for an arbitrary vector $x \in \mathbb{R}^n$.

ii) Using similar methods, prove that any homeomorphism $D^n \rightarrow D^n$ must restrict to a homeomorphism $S^{n-1} \rightarrow S^{n-1}$ (where $S^n$ is the boundary of $D^n$), and show that the Mobius band is not homeomorphic to $S^1 \times I$.

**Problem 4:** Let $K$ denote the Klein bottle. Calculate the homology groups of the all possible connected sums of $\mathbb{R}P^2$, $K$ and $T^2$. (Hint: use the Mayer-Vietoris sequence arising from the connected sum decomposition. You may want to use the fact that $H_1(X)$ is the abelianization of $\pi_1 X$, and that the map $H_1 X \rightarrow H_1 Y$ induced by a map $f : X \rightarrow Y$ is the map on abelianizations induced by $f_* : \pi_1 X \rightarrow \pi_1 Y$.) (Turn in any 3 of the computations, including $T^2 \# T^2$.)

**Problem 5:** Let $f : S^n \rightarrow S^n$ be a continuous mapping, and assume $n \geq 1$. Show that if $f_* : H_n S^n \rightarrow H_n S^n$ is non-zero, then $f$ is surjective. (Don’t turn in).

Problems from Hatcher:

- Section 2.1 # 21. Hints: You can define a map $s : C_*(X) \rightarrow C_{*+1}(SX)$ by using functoriality of the cone construction. What I mean is, each singular simplex $\Delta^n \rightarrow X$ gives rise to a corresponding map $C(\Delta^n) = \Delta^{n+1} \rightarrow CX$ between the cones. If you think of $SX$ as the union of two cones, you should be able to come up with the desired map; it will send a singular simplex $\sigma$ to the difference between an “upper cone” on $\sigma$ and a “lower cone” on $\sigma$.

  To show that $s$ induces an isomorphism on homology show that $\partial \circ s_* = \text{Id}_{H_* (X)}$, where $\partial$ denotes the boundary map for the long exact sequence associated to the pair $(CX,X)$ (note: you don’t need to separately show that $s_* \circ \partial$ is the identity). Since we were thinking of $SX$ as a union of two cones above, it’s best to think of $CX$ here as $(X \times [-1,1]) / X \times \{1\}$, and then you can think of $SX$ as $CX / X \times \{-1\}$. Now to compute $\partial s(\sigma)$ you can think about the boundaries of the upper and lower cones separately. The upper cone is relatively easy to deal with directly from the definition of the boundary map. For the lower cone, some more work is required. One possibility is to compare this lower cone, which is a map $C(\Delta^n) \rightarrow [-1,0] \times X / X \times \{-1\}$, to the map $\Delta^n \times I \rightarrow X \times [-1,0] / X \times \{-1\}$. These are very similar, and we saw a way of breaking up the latter into a sum of $n+1$ dimensional simplices (in the proof of homotopy invariance). You should be able to compute the boundary of this (squashed) prism.

  One thing to keep in mind, when you’re computing these boundaries, is that the boundary map is really only defined on singular cycles, not on arbitrary singular chains.

- Section 2.2, # 29 (don’t turn in),
- Section 2.2. # 30 (don’t turn in).