MATH 261
TEST 3 30 points

YOU MUST SHOW YOUR WORK TO GET A CREDIT

Problem 1 3 points Find the Jacobian of transformation

\[ x = v \quad w, \quad y = u + v^2, \quad z = u + v^2 \]

\[
\begin{vmatrix}
0 & 0 & 0 \\
2u & 0 & 1 \\
1 & 2v & 0 \\
\end{vmatrix}
= 0 + 4u(0-1) + 2u \cdot 2v v 
= 8uvw + 1
\]

Answer

Problem 2 3 points Evaluate the triple integral \( \iiint_H (9 - x^2 - y^2) \, dv \)

where \( H \) is the solid hemisphere \( x^2 + y^2 + z^2 \leq 9, z \geq 0 \).

\[
\int_0^{2\pi} \int_0^{\pi/2} \int_0^3 (9 - r^2\sin^2 \phi) r^2 \sin \phi \, dr \, d\phi \, d\theta =
\]

\[
\int_0^{2\pi} \int_0^{\pi/2} \int_0^3 (9 - r^2\sin^2 \phi) r^2 \sin \phi \, dr \, d\phi \, d\theta =
\]

\[
= 2\pi \left[ \int_0^{\pi/2} \left[ \frac{9}{3} (1 - \sin^2 \phi) - \frac{9}{5} \right] d\phi \right]
\]

\[
= 2\pi \left[ \int_0^{\pi/2} \frac{9}{3} \, d\phi \right]
\]

\[
= 2\pi \left[ \frac{9}{3} \cdot \frac{\pi}{2} \right]
\]

Answer

Check your work:

\[
= 2\pi \cdot 9 \cdot \left( 1 - \frac{2}{3} \right) = 2\pi \left( 9 - \frac{2 \cdot 9}{3} \right)
\]

\[
= 2\pi \left( 9 - \frac{18}{3} \right) = 2\pi \left( 9 - 6 \right) = 2\pi \cdot 3 = 6\pi
\]

\[
= \frac{2\pi \cdot 3^3}{5} = \frac{2\pi \cdot 27}{5} = \frac{54\pi}{5}
\]
Problem 3 3 points Evaluate the line integral 
\[ \int_C \sin x \, dx + \cos y \, dy, \]
where \( C \) consists of the top half of the circle \( x^2 + y^2 = 1 \) from \((1,0)\) to \((-1,0)\).

\[
x = \cos t, \quad y = \sin t, \quad \frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t.
\]

\[
\int_C \sin x \, dx + \cos y \, dy = \int_0^\pi \sin(x) \, dx + \cos(y) \, dy
\]

\[
= \int_0^\pi \sin(\cos t) \cdot (-\sin t) \, dt + \cos(\sin t) \cos t \, dt
\]

\[
= \left[ -\cos(\cos t) + \sin(\sin t) \right]_0^\pi
\]

\[
= -\cos(-1) + \cos 1 + 0 - 0 = \cos 1 - \cos 1 = 0
\]

Answer

Problem 4 3 points. Find the area of the surface. The part of the surface 
\( z = x^2 + 2y \) that lies above the triangle with vertices \((0,0)\), \((1,0)\) and \((1,2)\).

\[
\iint_D \sqrt{1 + (\frac{\partial z}{\partial x})^2 + \frac{\partial z}{\partial y}^2} \, dA
\]

\[
= \int_0^1 \int_0^{\sqrt{x}} \sqrt{1 + 4x^2 + 4} \, dy \, dx
\]

\[
= \int_0^1 2 \sqrt{5 + 4x^2} \, dx
\]

\[
= \frac{2}{8} \left[ \frac{3}{2} (5 + 4x^2)^{3/2} \right]_0^1
\]

\[
= \frac{1}{4} \left[ (3^{3/2}) - 5^{3/2} \right] = \frac{1}{4} \left[ (3\sqrt{3}) - 5\sqrt{5} \right]
\]

Answer
Problem 5 4 points (a) Determine whether or not the vector field is conservative. If it is conservative, find \( f \) such that \( \mathbf{F} = \nabla f \) and (b) evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \).

\[
\mathbf{F} = e^y \mathbf{i} + xe^y \mathbf{j} + (z + 1)e^z \mathbf{k}; \quad C: \ x = t, \ y = t^2, \ z = t^3, \ 0 \leq t \leq 1
\]

\[
\begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
e^y & xe^y & (z + 1)e^z \\
e^y & xe^y & (z + 1)e^z \\
\end{vmatrix} = \mathbf{i} \cdot 0 - \mathbf{j} \cdot 0 + \mathbf{k}(e^y - e^y) = 0
\]

\[
\frac{\partial f}{\partial x} = e^y \quad \frac{\partial f}{\partial y} = 0 \quad \frac{\partial f}{\partial z} = (z + 1)e^z
\]

\[
f(x, y, z) = e^y x + e(x, y, z)
\]

\[
f_y = xe^y + e(y, z) \quad xe^y + e(y, z) = c(z)
\]

\[
f_z = c'(z) = (z + 1)e^z
\]

\[
S(z) = ze^{z^2} - e^2 + e^2 = ze^z + c
\]

\[
u = z, \ du = dz
\]

\[
v = e^2
\]

\[
f(x, y, z) = xe^y + ze^z + c = (1, 1, 1)
\]

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \left. f(x, y, z) \right|_{(0, 0, 0)}^{(1, 1, 1)} = e + e - 0 = 2e
\]

Answer (a) \( xe^y + ze^z + c \)

Answer (b) \( 2e \)
Problem: 5 points Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem. \( C \) consists of the parabola \( y = x^2 \) from \((-1,1)\) to \((1,1)\) and the line segment from \((1,1)\) to \((-1,1)\).

\[
\oint_C xy^2 \, dx - x^2 y \, dy
\]

(a) \( S = \int_C xy^2 \, dx - x^2 y \, dy = 0 + 0 = 0 \)

\( C_1 \):
\[
t = x \quad dx = dt, \quad y = t^2 \quad dy = 2t \, dt
\]

\(-1 \leq t \leq 1\)

\( C_2 \):
\[
x = 1 \quad (1-t) + (-1)t = 1 - 2t \quad dx = -2 \, dt
\]
\[
y = 1 \quad (1-t) + 1t = 1 \quad dy = 0
\]

\[
S = \int_{-1}^{1} t \cdot t^4 \, dt - t^2 \cdot t^2 \cdot 2t \, dt = -\int_{-1}^{1} t^5 \, dt = 0
\]

\( S_{C_2} = \int_{0}^{1} (1-2t)(-2) \, dt = -2 \left[ \frac{t - 2t^2}{2} \right]_{0}^{1} = 0
\]

\[
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -2x \cdot y - 2x \cdot y = -4x y
\]

(6) \( S_c = -\iint_{C} 4xy \, dy \, dx = -\int_{-1}^{1} \int_{x^2}^{1-x^4} dy \, dx = -\int_{-1}^{1} \left[ \frac{y}{2} \right]_{x^2}^{1-x^4} dx = -\int_{-1}^{1} \left( \frac{1-x^4}{2} - \frac{x^2}{2} \right) dx = 0
\]

Answer (a)

Answer (b)

\[
12 + 9 = 21
\]
Problem 7 3 points. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$

$$\mathbf{F}(x, y, z) = x \mathbf{i} - z \mathbf{j} + y \mathbf{k}$$

$S$ is the part of the sphere $x^2 + y^2 + z^2 = 4$ in the first octant with orientation toward the origin.

$$x = 2 \sin \phi \cos \theta$$
$$y = 2 \sin \phi \sin \theta$$
$$z = 2 \cos \phi$$

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = (-2 \sin \theta \cos \phi \mathbf{i} + 2 \cos \theta \sin \phi \mathbf{j} + 2 \sin \phi \mathbf{k})$$

$$d\mathbf{S} = \mathbf{r}_\phi \times \mathbf{r}_\theta \ dA$$

$$= \mathbf{i} (0 + \sin^2 \phi \cos \theta) - \mathbf{j} (0 - \sin^2 \phi \sin \theta) + \mathbf{k} \sin \phi \cos \phi$$

$$= \mathbf{i} \sin \phi \cos \theta, \mathbf{j} \sin \phi \sin \theta, \mathbf{k} \sin \phi \cos \phi$$

$$S \mathbf{E} \cdot d\mathbf{S} = \int \int \left( 2 \sin \phi \cos \theta + \sin^2 \phi \cos \theta + \sin^2 \phi \sin \theta \right)$$

Answer:

Problem 8 3 points. Use the Divergence Theorem to evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $S$ is the surface of the solid bounded by the hyperboloid $x^2 + y^2 - z^2 = 1$ and the planes $z = -2$ and $z = 2$

$$\mathbf{F}(x, y, z) = x^3 y \mathbf{i} - x^2 y^2 \mathbf{j} - x^2 y \mathbf{k}$$

$$\text{div} \mathbf{F} = 3x^2 y - 2x^2 y - x^2 y = 0$$

$$S \mathbf{F} \cdot d\mathbf{S} = 0$$

Answer:

Problem 9 3 points. Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. $C$ is the curve of intersection of the plane $x + z = 5$ and the cylinder $x^2 + y^2 = 9$ oriented counterclockwise as viewed from above.

$$\mathbf{F}(x, y, z) = xy \mathbf{i} + 2z \mathbf{j} + 3y \mathbf{k}$$

$$21 + 9 = \boxed{30}$$

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\[
\mathbf{S} \mathbf{F} \cdot d\mathbf{r} = \sum_{s} \mathbf{SS} \text{curl} \mathbf{F} \cdot d\mathbf{s}
\]

\[
\mathbf{r} = \langle x, y, 5-x \rangle
\]
\[
\mathbf{r}_x = \langle 1, 0, -1 \rangle
\]
\[
\mathbf{r}_y = \langle 0, 1, 0 \rangle
\]
\[
\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix}
 1 & 0 & -1 \\
 0 & 1 & 0 \\
 1 & 0 & 1
\end{vmatrix} = \mathbf{i} (0+1) - \mathbf{j} \cdot 0 + \mathbf{k} \cdot 1 = \langle 1, 0, 1 \rangle
\]
\[
\text{curl} \mathbf{F} = \begin{vmatrix}
 0 & \frac{2}{y} & \frac{2}{z} \\
 \frac{2}{x} & 0 & 2z \\
 2y & 3y & 0
\end{vmatrix} = \langle 3-x, 0, -x \rangle
\]
\[
\mathbf{SS} \text{curl} \mathbf{F} \cdot d\mathbf{s} = \mathbf{SS} (1+0-x) dA = 2\pi \int_{0}^{2\pi} \frac{x^2}{2} d\theta
\]
\[
= \int_{0}^{2\pi} \frac{\pi r^2}{2} d\theta = \pi r^2 \left[ \frac{x^2}{2} \right]_{0}^{2\pi} = \pi r^2
\]