1. Distributions and Subharmonic functions in one variable

In this section we review the theory of distributions and of subharmonic functions on one-variable. Suitable references include [2, 3].

1.1. Distributions. Distributions are very helpful in dynamics because we often deal with dynamically defined (or dynamically invariant) objects, which are often rough.

Heuristically: first we define the “smooth objects” (test functions) and then the “rough objects” (distributions) are defined as dual to them. For our purposes, the smooth objects consist of functions from the space $C_0^\infty (\mathbb{R}^m)$ of infinitely differentiable functions on $\mathbb{R}^m$ having compact support.

Before defining the topology on $C_0^\infty (\mathbb{R}^m)$, let us consider some simpler spaces.

For any compact $K \subset \mathbb{R}^m$, let $C_K^r (\mathbb{R}^m)$ be the space of $r$-times differentiable functions having support in $K$. The topology on $C_K^r (\mathbb{R}^m)$ is defined by the condition that $\phi_n \to \phi$ if the partial derivative $\partial_\alpha \phi_n$ converges uniformly on $K$ to $\partial_\alpha \phi$ for every multi-index $\alpha$ having $|\alpha| \leq r$. Alternatively, this topology can be defined by a norm $\|\phi\|_K^r$ given by the supremum of all derivatives of order less than or equal to $r$ over the compact set $K$.

There are two natural ways to extend $C_K^r (\mathbb{R}^m)$:

Keeping $K$ fixed, we can define the space $C_K^\infty (\mathbb{R}^m)$ of infinitely differentiable functions supported on $K$. We say that $\phi_n \to \phi$ if we have that $\partial_\alpha \phi_n$ converges uniformly to $\partial_\alpha \phi$ for every multi-index $\alpha$ (of arbitrary size $|\alpha|$). The resulting topology is a projective limit of the topologies on $C_K^r (\mathbb{R}^m)$ as $r \to \infty$.

Keeping $r$ fixed, we can define the space $C_0^r (\mathbb{R}^m)$ of $r$-times differentiable functions with compact support. The topology on this space is defined by the condition that $\phi_n \to \phi$ if both

1. There is some compact $L \subset \mathbb{R}^m$ with $\text{supp} \, \phi_n \subset L$ for all $n$, and
2. $\partial_\alpha \phi_n$ converges uniformly on $L$ to $\partial_\alpha \phi$ for every every multi-index $\alpha$ with $|\alpha| \leq r$.

The resulting topology is an inductive limit of the topologies on $C_K^r (\mathbb{R}^m)$ as the compact set $K$ increases.

**Definition 1.1.** The topology on $C_0^\infty (\mathbb{R}^m)$ is defined by the condition that $\phi_n \to \phi$ if

1. there is some compact set $K \subset \mathbb{R}^m$ with $\text{supp} \, \phi_n \subset K$ for all $n$, and
2. the partial derivatives $\partial_\alpha \phi_n$ converge uniformly on $K$ to $\partial_\alpha \phi$ for any multi-index $\alpha$.

This topology is subtle because it depends on both infinitely many derivatives and a possibly arbitrarily large (but compact) support. It is a weird mix of the projective limit topology that is used to define $C_K^\infty (\mathbb{R}^m)$ and the inductive limit topology that is used to define $C_0^\infty (\mathbb{R}^m)$.

For any manifold $M$ the space $C_0^\infty (M)$ is defined similarly.

In the remainder of these notes we will follow Laurent Schwartz (father of distribution theory), introducing the notation $\mathcal{D}(M) \equiv C_0^\infty (M)$.

**Definition 1.2.** The space of distributions $\mathcal{D}'(M)$ on a manifold $M$ (possibly $M = \mathbb{R}^m$) is the space of continuous linear functionals $\mu : \mathcal{D}(M) \to \mathbb{R}$.

Continuous means that $\phi_n \to 0$ in $\mathcal{D}(M)$ implies that $\langle \mu, \phi_n \rangle \to 0$. Here, we use $\langle \mu, \phi \rangle$ to denote the action of $\mu \in \mathcal{D}'(M)$ on $\phi \in \mathcal{D}(M)$. 

\[1\]
Example 1.1. The most famous distribution is the Dirac distribution $\delta_{x_0}$. For any $x_0$, it acts on $\mathcal{D}(\mathbb{R}^n)$ by

$$\langle \delta_0, \phi \rangle := \phi(0).$$

For any $x_0 \in \mathbb{R}^m$, $\delta_{x_0}$ is defined similarly.

Continuity of $\mu$ can alternatively be expressed by requiring that for any compact $K \subseteq M$ there exist $r$ and $C$ so that for every $\phi \in \mathcal{D}(M)$ with $\text{supp}(\phi) \subseteq K$ we have

$$(1.1) \quad |\langle \mu, \phi \rangle| \leq C\|\phi\|_{r,K},$$

where $\|\cdot\|_{r,K}$ is the norm on $C^r_K(\mathbb{R}^m)$.

Both constants implicitly depend on $K$: $C \equiv C(K)$ and $r \equiv r(K)$. In particular, the value of $r$ may increase as $K$ is increased. If the same $r$ works in (1.1) for all compact sets $K$, we say that $\mu$ is a distribution of order $r$.

The Dirac distribution $\delta_{x_0}$ has order 0. An example of distribution $\mu \in \mathcal{D}'(\mathbb{R})$ of order $r$ (but not lower) is given by

$$\langle \mu, \phi \rangle := \phi^{(r)}(0),$$

where $\phi^{(r)}(x)$ denotes the $r$-th derivative of $\phi$.

Exercise: construct a distribution of infinite order.

Equivalently, a distribution $\mu$ is of order $r$ if and only if it defines a continuous linear functional on $C^0_0(M)$. Distributions of order 0 are particularly nice: According to the Rietz Representation Theorem, see [5, Thm 2.14], any distribution $\mu$ of order 0 is represented by a signed Radon measure $\nu$:

$$(1.2) \quad \langle \mu, \phi \rangle = \int \phi \, d\nu.$$  

Recall that a Radon measure is measure on the Borel sigma algebra that is locally finite and for which the measure of any set can be approximated as the supremum over the measures of its compact subsets.

A distribution $\mu \in \mathcal{D}'(M)$ is positive if $\langle \mu, \phi \rangle \geq 0$ for every test function $\phi$ that satisfies $\phi(x) \geq 0$ for all $x$.

Lemma 1.2. Any positive distribution $\mu$ is represented by a positive Radon measure.

Proof. Let $K \subseteq \mathbb{R}^m$ be some compact set. It is standard to construct a bump function $\chi \in \mathcal{D}(M)$ with $\chi \geq 0$ and $\chi(x) \equiv 1$ on $K$.

For any $\phi \in \mathcal{D}(M)$ having support in $K$, we have

$$(1.3) \quad -\|\phi\|_K^0 \cdot \chi \leq \phi \leq \|\phi\|_K^0 \cdot \chi.$$

Since $\mu$ is positive, it preserves the sense of the inequalities from (1.3), giving

$$-\|\phi\|_K^0 \cdot \langle \mu, \chi \rangle \leq \langle \mu, \phi \rangle \leq \|\phi\|_K^0 \cdot \langle \mu, \chi \rangle.$$

In particular, (1.1) is satisfied for $r = 0$ with $C_K = \langle \mu, \chi \rangle$ depending only on $K$. Thus, $\mu$ is of order 0 and, by Reitz, it is representable by a Radon measure $\nu$. Moreover, $\mu \geq 0$ implies $\nu \geq 0$. \hfill $\Box$

Definition 1.3. The space of distributions $\mathcal{D}'(M)$ is equipped with the weak-* topology: we say that $\mu_n \to \mu \in \mathcal{D}'(M)$ if for every $\phi \in \mathcal{D}(M)$ we have $\langle \mu_n, \phi \rangle \to \langle \mu, \phi \rangle$.
Exercise: show that the space of positive distributions is closed in the weak-\* topology.

Remark 1.1. We can embed the space $L^1_{loc}(\mathbb{R}^m)$ of measurable, locally integrable, functions into $\mathcal{D}'(\mathbb{R}^m)$ by sending $f \in L^1_{loc}(\mathbb{R}^m)$ to $\mu_f \in \mathcal{D}'(\mathbb{R}^m)$ defined by

$$\langle \mu_f, \phi \rangle := \int \phi \cdot f \, d\text{Leb}. \quad (1.4)$$

Here, Leb denotes the Lebesgue measure on $\mathbb{R}^m$. For this reason, elements of $\mathcal{D}'(M)$ are often called “generalized functions”. Moreover, we will typically blur the distinction between $f$ and $\mu_f$, often writing $\langle f, \phi \rangle \equiv \langle \mu_f, \phi \rangle$.

Proposition 1.3. The embedding of $L^1_{loc}(\mathbb{R}^m)$ to $\mathcal{D}'(\mathbb{R}^m)$ is continuous.

Proof. We suppose that $f_n \to f$ in $L^1_{loc}(\mathbb{R}^m)$ in order to show that $\langle \mu_{f_n}, \phi \rangle \to \langle \mu_f, \phi \rangle$ for any $\phi \in \mathcal{D}(\mathbb{R}^m)$. Hölder’s Inequality gives

$$|\langle \mu_{f_n}, \phi \rangle - \langle \mu_f, \phi \rangle| = \left| \int_K (f_n - f) \phi \, d\text{Leb} \right| \leq \int_K |f_n - f| \phi \, d\text{Leb} \leq \|f_n - f\|_{L^1(K)} \cdot \|\phi(x)\|_{L^\infty(K)} \to 0,$$

where $K = \text{supp} \phi \subseteq \mathbb{R}^m$. \hfill \Box

For any $\mu \in \mathcal{D}'(\mathbb{R}^m)$ the partial derivative $\partial_i \mu \in \mathcal{D}'(\mathbb{R}^m)$ by

$$\langle \partial_i \mu, \phi \rangle := -\langle \mu, \partial_i \phi \rangle. \quad (1.5)$$

If $\mu$ were represented by a smooth function, then (1.5) is just a restatement of integration by parts, keeping in mind that there are no boundary terms since $\phi$ is compactly supported. Using this definition, any distribution $\mu$ can be differentiated arbitrarily many times. (If $\mu$ is of finite order, the order may increase under differentiation.) In particular, considering any $f \in L^1_{loc}(\mathbb{R}^m)$ as a distribution, allows us to differentiate it.

Moreover, any partial differential operator (with smooth coefficients) extends to act on $\mathcal{D}'(M)$ in the analogous way. The most interesting for us will be $\Delta = \partial^2_{xx} + \partial^2_{yy}$.

There is a standard procedure for approximating a distribution $\mu$ by a smooth object $\mu_\epsilon \in C^\infty(M)$ that will satisfy $\mu_\epsilon \to \mu$ (weakly) as $\epsilon \to 0$. We present this procedure in §1.3, convolving $\mu$ with a suitable bump-function $\phi_\epsilon \in \mathcal{D}(M)$. As a consequence, $C^\infty(M)$ will form a dense subset of $\mathcal{D}'(M)$.

Therefore, the theory of distributions gets developed as follows: we prove a desired statement for smooth objects first, and then “take limits” to obtain the result for all distributions. In §1.4 we will use this technique is used in the context of subharmonic functions.

1.2. Subharmonic functions in $\mathbb{R}^2$. It will be convenient to use $z = x + iy$ to describe a point in $\mathbb{R}^2$ having Cartesian coordinates $(x, y)$.

Definition 1.4. Let $U \subset \mathbb{R}^2$ be open. A function $h : U \to [\pm \infty)$ is called subharmonic if

(1) For any closed disc $\overline{B}_r(z) \subset U$, we have the sub-mean value property (SMVP):

$$h(z) \leq \frac{1}{2\pi} \int h(z + re^{i\theta}) \, d\theta, \quad (1.6)$$
(2) $h$ is upper semi-continuous: for any $z \in \mathbb{R}^2$ we have
\[ h(z) \geq \limsup_{\xi \to z} h(\xi), \]
and
(3) $h(z)$ is not identically equal to $-\infty$ on any component of $U$.

It is important to allow $h(z)$ to be $-\infty$ for some points $z \in U$ (at these points the SMVP is trivially satisfied). Subharmonic functions are particularly convenient to work with because condition (1) gives an upper estimate for $h(z)$ and condition (2) a lower estimate.

**Lemma 1.4. (Equivalent formulations of the SMVP)**

Suppose that $f : U \to [-\infty, \infty)$ is upper-semicontinuous and not identically equal to $-\infty$ on any components of $U$ and that $\overline{D}_r(z) \subset U$. The following are equivalent:

1. $f$ satisfies the classical SMVP (on boundaries of discs):
\[ h(z) \leq M(z, r) := \frac{1}{2\pi} \int h(z + re^{i\theta})d\theta, \]
2. the mean value $M(z, r)$ is a non-decreasing function of $0 < \rho \leq r$,
3. $f$ satisfies the SMVP on discs:
\[ h(z) \leq \frac{1}{\pi r^2} \int_{\overline{D}_r(z)} h(z)d\text{Leb}, \]
and
4. $f$ satisfies the maximum principle: For any compact $K \Subset U$ and every $z \in K$
\[ f(z) \leq \sup_{z \in \partial K} f(z) \]

We omit the proof of Lemma 1.4. See Theorem 4.5 and Proposition 4.9 from [2, Ch. 19].

A basic example is $h(z) = \log |z|$. The result is harmonic on the complement of $\{z = 0\}$ and upper-semi-continuous everywhere.

Why does it satisfy the SMVP on a disc containing $z = 0$ that is not centered there?

More generally, if $f : U \to \mathbb{C}$ is analytic, then $h(z) = \log |f(z)|$ is subharmonic on $U$. It follows from the Poisson-Jensen Formula; see [1].

Three other ways to construct subharmonic functions are as follows:

- the point-wise maximum $h(z) = \max(h_1(z), h_2(z))$ of any two subharmonic functions is subharmonic,
- the composition $\phi \circ h(z)$ of a any increasing convex function $\phi$ with a subharmonic $h$ is subharmonic, and
- if $h$ is subharmonic on $U$ and $\overline{D}_r(z_0) \subset U$, then one can replace $h$ with the harmonic function on $\overline{D}_r(z_0)$ that agrees with $h$ on $\partial \overline{D}_r(z_0)$.

These examples show that the class of subharmonic functions is much larger than that of analytic functions. They are much less rigid than the class of analytic functions or, equivalently, of harmonic functions.

**Lemma 1.5.** If $h$ is subharmonic, then $h \in L^1_{\text{loc}}(\mathbb{R}^2)$. In particular, $h$ defines a distribution $\mu_h \equiv h \in D'(\mathbb{R}^2)$ as defined in Remark 1.1.
Sketch of proof: It suffices to show that $h$ is integrable on any compact set $K \subseteq U$. Since $h$ is upper semi-continuous, on $K$ we have $h(z) \leq C := \sup_{z \in K} h(z)$. In particular,

$$
\int_K h(z) d\text{Leb} \in \mathbb{R} \cup \{-\infty\}
$$


We will now check that the integral is not equal to $-\infty$. Let

$$
\mathcal{P} := \{z \in U : h(z) = -\infty\}
$$

be the polar set.

Suppose, in order to obtain a contradiction, that $\mathcal{P}$ has non-empty interior. Since $h \neq -\infty$ on any component of $U$, we can pick a point $z_0 \in U \setminus \mathcal{P}$ and a radius $r > 0$ so that $\overline{D}_r(z_0) \subset U$ and intersects $\mathcal{P}$ in an open subset. The SMVP on discs gives that $h(z_0) = -\infty$, a contradiction.

We can cover the compact set $K$ by a finite union of open discs $\bigcup D_{r_i}(z_i)$ so that each $\overline{D}_{r_i}(z_i) \subset U$. Since $\mathcal{P}$ has empty interior, we can choose the centers $z_1, \ldots, z_n \in U \setminus \mathcal{P}$. The SMVP on discs gives that

$$
\int_{\bigcup D_{r_i}(z_i)} h(z) d\text{Leb} > -\infty.
$$

Since $K \subset \bigcup D_{r_i}(z_i)$, we also have $\int_K h d\text{Leb} > -\infty$. □

A more detailed proof can be found in [2].

Example 1.6. We compute the distributional Laplacian $\Delta$ of $h(z) = \log |z|$. For any test function $\phi$ we must compute

$$
\langle \Delta h, \phi \rangle = \langle h, \Delta \phi \rangle = \int_C \Delta \phi \ h \ d\text{Leb}.
$$

The first equality follows from the definition of distributional derivatives (1.5). (The minus signs cancel since two derivatives are taken.)

There is some $R > 0$ so that $\text{supp} \phi \subset \overline{D}_R(0)$. For any $0 < r < R$, let $D := \overline{D}_R(0) \setminus \overline{D}_r(0)$.

Gauss' Theorem gives that for any $C^\infty$ vector field $v$ defined in a neighborhood of $D$, we have

$$
\int_{\partial D} v \cdot \hat{n} dS = \int_D \nabla \cdot v dA,
$$

where $\hat{n}$ is the outward pointing normal vector on the boundary. Applied to $v = \phi \nabla h - h \nabla \phi$ we obtain

$$
(1.7) \quad \int_{\partial D} \left( \phi \frac{\partial h}{\partial \hat{n}} - h \frac{\partial \phi}{\partial \hat{n}} \right) dS = \int_D (\phi \Delta h - h \Delta \phi) dA.
$$

Since $\text{supp} \phi \subset \overline{D}_R(0)$, we have that $\phi$ and $\frac{\partial \phi}{\partial \hat{n}}$ vanish on $|z| = R$. Moreover, $\Delta h = 0$ on $D$. Thus (1.7) simplifies to

$$
\int_D h \Delta \phi = \int_{|z|=r} h \frac{\partial \phi}{\partial \hat{n}} dS - \int_{|z|=r} \phi \frac{\partial h}{\partial \hat{n}} dS
$$

We take the limit as $r \to 0$ so that the left hand side converges to $\langle \Delta h, \phi \rangle = \int_C h \Delta \phi$. The first term in the right hand side tends to 0 since $\phi$ is bounded, $h(re^{i\theta}) = \log r$, and the circle has length $2\pi r$. (We are using that $\lim_{r \to 0} r \log r = 0.$)
At $|z| = r$, we have
\[
\frac{\partial \log |z|}{\partial \hat{n}} = - \frac{\partial \log x}{\partial x} \bigg|_{x=r} = - \frac{1}{r}
\]
Therefore,
\[
-\int_{|z|=r} \phi \frac{\partial h}{\partial \hat{n}} \, dS = -2\pi r \int \phi(re^{i\theta}) \left( -\frac{1}{r} \right) d\theta \xrightarrow{r \to 0} 2\pi \phi(0).
\]
We conclude that for any $\phi \in \mathbb{D}(\mathbb{R}^2)$, we have $\langle \Delta \log |z|, \phi \rangle = 2\pi \phi(0)$. In other words, (1.8)
\[
\Delta \log |z| = 2\pi \delta_0
\]
where $\delta_0$ is the Dirac $\delta$-distribution that was introduced in Example 1.1.

Using terminology from the theory of partial differential equations, (1.8) states that $\frac{1}{2\pi} \log |z|$ is the fundamental solution to Laplace’s equation $\Delta u = v$. I.e. for whatever $v$ we are given, the convolution $u = \frac{1}{2\pi} \log |z| * v$ will be a solution.

More generally, Laplace’s equation can also be solved with a measure $\nu$ on the right hand side: $\Delta u = \nu$. The solution is the function $u(z) = \frac{1}{2\pi} \log |z| * \nu$, where the convolution is taken in the sense of measures. For example, this is how one can reconstruct the Green’s function $G(z)$ from the harmonic measure $\mu$, as given in (??).

**Example 1.7.** For any polynomial $p(z) = \prod(z - a_i)$ we have
\[
\log |p(z)| = \sum \log |z - a_i|
\]
so that
\[
\Delta \log |p(z)| = 2\pi \sum \delta_{a_i} = 2\pi \left[ \text{div}(p) \right].
\]
Here $\text{div}(p)$ denotes the divisor of $p$ which, in this context, is just the set of zeros, each counted with multiplicity. Square brackets $[\cdot]$ around a divisor indicates the distribution obtained by integrating a test function over that divisor.

If $r(z) = p(z)/q(z)$ is a rational function with zeros $a_1, \ldots, a_k$ and poles $b_1, \ldots, b_l$, each counted with multiplicity, then
\[
\Delta \log |r(z)| = 2\pi \left( \sum \delta_{a_i} - \sum \delta_{b_j} \right) = 2\pi \left[ \text{div}(r) \right].
\]
This time, $\text{div}(r)$ denotes the zeros and poles of $r$ with multiplicities (and poles are given a negative sign.)

Equation (1.9) is a baby version of the Poincaré-Lelong formula, which relates the (pluri)-Laplacian of a rational function $r : \mathbb{C}P^n \to \mathbb{C}$ to integration over its divisor of zeros and poles.

We finish the subsection by considering subharmonic functions that are of class $C^\infty$:

**Proposition 1.8.** A $C^\infty$ function $h : U \to \mathbb{R}$ is subharmonic if and only if $\Delta h(z) \geq 0$ at every $z \in U$.

**Proof.** We will show that $h$ satisfies the sub-mean value property for (SMVP) (1.6) if and only if $\Delta h(z) \geq 0$ at every $z \in U$.

Suppose that $\Delta h(z) \geq 0$ at every $z \in U$. In order to prove the SMVP, we will show that
\[
M(z, r) = \frac{1}{2\pi} \int h(z + re^{i\theta}).
\]
is an increasing function of $r > 0$. This is sufficient because $M(z, r) \to h(z)$ as $r \to 0$. Without loss of generality, we can suppose that $z = 0$.

Let

$$u(z) = \frac{1}{2\pi} (\log R - \log |z|),$$

which is harmonic on $D := D_R(0) - D_r(0)$, non-negative, and vanishes on $|z| = R$.

We will apply Gauss’ Theorem to $v = h \nabla u - u \nabla h$. Since $\Delta u = 0$ and $u$ vanishes on $|z| = R$, the result simplifies to (1.10)

$$M(0, R) - M(0, r) = \int_{|z|=R} h \frac{\partial u}{\partial n} dS - \int_{|z|=r} h \frac{\partial u}{\partial n} dS = \int_D u \Delta h + \int_{|z|=r} u \frac{\partial h}{\partial n} \geq 0.$$

The first term on the right hand side of (1.10) is non-negative because $\Delta h \geq 0$ and $u \geq 0$.

To see that the second term on the right hand side is positive, note that $u(z)$ is a positive constant on $|z| = r$ and that

$$\int_{|z|=r} h \frac{\partial h}{\partial n} = \int_D \Delta h \text{Leb} \geq 0$$

again follows from Gauss’ Theorem.

Now suppose that $h$ satisfies the SMVP (1.6). Take a Taylor expansion in $r e^{i\theta}$ centered at $z \in U$:

$$h(z + r e^{i\theta}) = h(z) + r \left( \frac{\partial h}{\partial x} \cos \theta + \frac{\partial h}{\partial y} \sin \theta \right) + \frac{r^2}{2} \left( \frac{\partial^2 h}{\partial x^2} \cos^2 \theta + \frac{\partial^2 h}{\partial x \partial y} \cos \theta \sin \theta + \frac{\partial^2 h}{\partial y^2} \sin^2 \theta \right) + O(r^3)$$

The terms containing $\cos \theta$, $\sin \theta$, and $\cos \theta \sin \theta$ will vanish when we integrate $\theta$ over $[0, 2\pi]$. We get

$$\frac{1}{2\pi} \int h(z + r e^{i\theta}) d\theta = h(z) + \frac{r^2}{4} \Delta h(z) + O(r^3).$$

Since

$$\frac{1}{2\pi} \int h(z + r e^{i\theta}) \geq h(z)$$

for $r$ arbitrarily close to 0, we get that $\Delta h(z) \geq 0$. \hfill \Box

1.3. Regularization. We will see how to approximate an arbitrary distribution $\mu$ by smooth distributions $\mu_\epsilon$ that converge to it (in the weak-$\ast$ topology) as $\epsilon \to 0$. By smooth distributions, we mean those represented by smooth functions under the inclusion $f \mapsto \mu_f$ described in Remark 1.1. In the end of the section we apply the results to subharmonic functions.

Let’s recall the simplest setting. If $f \in L^1_{\text{loc}}(\mathbb{R}^m)$ and $\phi \in D(\mathbb{R}^m)$, then for every $x \in \mathbb{R}^m$ the integral

$$(1.11) \quad f * \phi(x) := \int_{\mathbb{R}^m} f(\xi) \phi(x - \xi) d\text{Leb}(\xi)$$

is defined. Consequently, it defines the function $f * \phi(x)$, called the convolution of $f$ with $\phi$. Derivatives $\frac{\partial}{\partial x^\alpha} f * \phi(x)$ can be computed by differentiating $\phi$ under the integral sign. Since $\phi$ is of class $C^\infty$, we can do this for any multi-index $\alpha$; i.e. $f * \phi(x)$ is $C^\infty$. 
In order to approximate \( f \in L^1_{\text{loc}} \) by a \( C^\infty \) function, one chooses a sequence of test functions \( \phi_n \) with \( \text{supp} \phi_n \to \{0\} \) with the normalization \( \int_{\mathbb{R}^m} \phi_n d\text{Leb} = 1 \). The resulting convolutions \( f * \phi_n \) converges to \( f \) within \( L^1_{\text{loc}} \) as \( n \to \infty \). Moreover, if \( f \) happens to be continuous, then \( f * \phi_n(x) \) converge uniformly to \( f \) on compact subsets of \( \mathbb{R}^m \). See, for example, [2, §18.3].

Such a sequence of test functions can be constructed nicely as follows. Choose a single \( C^\infty \) function \( \psi(x) \equiv \psi(||x||) \geq 0 \) depending only on \( ||x|| \) with \( \text{supp} \psi \subset \{ ||x|| \leq 1 \} \) and \( \int \psi d\text{Leb} = 1 \). Then, for any \( \epsilon > 0 \),

\[
\psi_\epsilon := \frac{1}{\epsilon^m} \psi \left( \frac{x}{\epsilon} \right)
\]

will be supported in \( \{ ||x|| \leq \epsilon \} \) and have normalized integral. We will call such a family \( \psi_\epsilon \) a normalized family of bump functions.

If \( \mu \in \mathcal{D}'(\mathbb{R}^m) \) and \( \phi \in \mathcal{D}(\mathbb{R}^m) \), the convolution is defined by

\[
\mu * \phi(x) := \langle \mu(\xi), \phi(x - \xi) \rangle.
\]

Although \( \mu \) may not typically be represented by a function, we use the notation \( \mu(\xi) \) to denote that it is being paired with functions of \( \xi \) (not of \( x \)). Note that \( \mu * \phi \) is an “honest function” since it assigns some value to every \( x \in \mathbb{R}^m \). Moreover, if we consider \( \phi(x - \xi) \) as a function of \( \xi \), it depends continuously (in the topology of \( \mathcal{D}(\mathbb{R}^m) \)) on \( x \in \mathbb{R}^m \). This gives that \( \mu * \phi(x) \) is a continuous function.

Suppose we take a partial derivative \( \partial_{x_i} \):

\[
(1.12) \quad \partial_{x_i} (\mu * \phi(x)) = \lim_{\Delta_i \to 0} \left\langle \mu, \frac{\phi(x + \Delta x_i - \xi) - \phi(x - \xi)}{||\Delta x_i||} \right\rangle = \langle \mu, \partial_{x_i} \phi(x - \xi) \rangle.
\]

The last equality holds since \( \mu \) is continuous and the difference quotients converge in \( \mathcal{D}(\mathbb{R}^m) \) to the partial derivative \( \partial_{x_i} \). Since \( \phi \in \mathcal{D}(\mathbb{R}^m) \) arbitrarily many further partial derivatives can be taken, so that \( \mu * \phi(x) \in C^\infty \).

**Lemma 1.9. (Equivariance)** If \( D \) is a linear differential operator (with \( C^\infty \) coefficients) then

\[
D(\mu * \phi) = (D\mu) * \phi
\]

**Proof.** It suffices to check this for any first order partial derivative \( \partial_{x_i} \). We have

\[
\partial_{x_i} (\mu * \phi(x)) = \langle \mu, \partial_{x_i} \phi(x - \xi) \rangle = -\langle \mu, \partial_{\xi_i} \phi(x - \xi) \rangle = \langle \partial_{\xi_i} \mu, \phi(x - \xi) \rangle,
\]

with the first equality coming from (1.12) and the last equality being the definition of \( \partial_{\xi_i} \mu \).

**Lemma 1.10. (Convergence)** If \( \mu_n \to \mu \) in \( \mathcal{D}'(U) \), then \( \mu_n * \psi_\epsilon \) converges uniformly on compacts subsets of \( U \) to \( \mu * \psi_\epsilon \).

**Proof.** Since \( \mu_n \to \mu \) in the weak-* topology, we have the pointwise convergence

\[
(1.13) \quad \mu_n * \psi_\epsilon(x) = \langle \mu_n(\xi), \psi_\epsilon(x - \xi) \rangle \to \langle \mu(\xi), \psi_\epsilon(x - \xi) \rangle = \mu * \psi_\epsilon(x)
\]

for any \( x \in U \). Using the same reasoning together with (1.12), we find pointwise convergence of the derivatives \( \partial_{x_i} (\mu_n * \psi_\epsilon(x)) \) to \( \partial_{x_i} (\mu * \psi_\epsilon) \). This is sufficient for the derivatives \( \{ \partial_{x_i} (\mu_n * \psi_\epsilon) \} \) to be uniformly bounded on compact sets, in turn giving that \( \{ \mu_n * \psi_\epsilon \} \) is equicontinuous. Therefore, the Azerla-Ascoli Theorem allows us to replace the pointwise convergence in (1.13) with uniform convergence on compact sets.
Lemma 1.11. (Positivity) If $\mu \in \mathcal{D}'(\mathbb{R}^m)$ and $\phi \in \mathcal{D}(\mathbb{R}^m)$ with $\mu \geq 0$ and $\phi \geq 0$, then $\mu * \phi \geq 0$.

Proof. It follows directly from the definition. \hfill $\square$

Proposition 1.12. (Approximation of general distributions) If $\psi_\epsilon$ is a normalized family of bump functions and $\mu$ is any distribution, then $\mu_\epsilon := \mu * \psi_\epsilon$ is $C^\infty$ and $\mu_\epsilon \to \mu$ in $\mathcal{D}'(\mathbb{R}^m)$.

Proof. We need to show that $\langle \mu_\epsilon, \phi \rangle \to \langle \mu, \phi \rangle$ for any test function $\phi \in \mathcal{D}(\mathbb{R}^m)$. As $\mu_\epsilon$ is a $C^\infty$ function, the pairing on the left hand side is given by integration:

$$
\langle \mu_\epsilon, \phi \rangle = \int_{\mathbb{R}^m} \mu_\epsilon(x) \phi(x) d\text{Leb}(x) = \int_{\mathbb{R}^m} \langle \mu(\xi), \psi(x-\xi) \rangle \phi(x) d\text{Leb}(x)
$$

The linearity of the pairing $\langle \cdot, \cdot \rangle$ and continuity allow is to move the integral and $\psi$ into the pairing as:

$$
(1.14) \quad \int_{\mathbb{R}^m} \langle \mu(\xi), \psi(x-\xi) \rangle \phi(x) d\text{Leb}(x) = \left\langle \mu(\xi), \int_{\mathbb{R}^m} \psi(x-\xi) \phi(x) d\text{Leb}(x) \right\rangle.
$$

(To give a completely rigorous proof of (1.14) one needs to approximate the integral by Riemann sums and take limits; see [3, §4.1].)

The term on the right is the standard convolution $\psi_\epsilon * \phi$ that was described at the introduction to this subsection. Hence, $\psi_\epsilon * \phi \to \phi$ in $\mathcal{D}(\mathbb{R}^m)$ as $\epsilon \to 0$. Then, since $\mu$ acts continuously on $\mathcal{D}(\mathbb{R}^m)$, we get that

$$
\langle \mu, \psi_\epsilon * \phi \rangle \to \langle \mu, \phi \rangle.
$$

\hfill $\square$

Proposition 1.13. (Approximation of subharmonic functions) Suppose that $h$ is subharmonic on $\mathbb{R}^2$, then for the smooth approximations $h_\epsilon := h * \psi_\epsilon$ are subharmonic. Moreover, for each fixed $z$, $h_\epsilon(z) \searrow h(z)$ as $\epsilon \searrow 0$.

Note: it will be important in the proof that the normalized family of bump functions $\psi_\epsilon(z)$ depends only on $|z|$.

Proof. Since $h_\epsilon$ is $C^\infty$, we need only check that it satisfies one of the four conditions given in Lemma 1.4 that are equivalent to the SMVP.

Since $h \in L^1_{\text{loc}}$ the convolution is just

$$
h_\epsilon(z) := \int_{B_\epsilon(z)} h(\xi) \psi_\epsilon(z-\xi) d\text{Leb}(\xi) = \int_{B_\epsilon(z)} h(\xi) \psi \left( \frac{z-\xi}{\epsilon} \right) \frac{d\text{Leb}(\xi)}{\epsilon^2} = \int_{\Delta(0)} h(z-\xi \epsilon) \psi(\xi) d\text{Leb}(\xi).
$$

Therefore,

$$
(1.15) \quad M_{h_\epsilon}(z,r) = \frac{1}{2\pi} \int h_\epsilon(z + r e^{i\theta}) = \frac{1}{2\pi} \int h(z + r e^{i\theta} - \xi) d\theta d\text{Leb}(\xi)
$$

Since $h$ is subharmonic, the average

$$
M_h(z-\xi, r) = \frac{1}{2\pi} \int h(z + r e^{i\theta} - \xi) d\theta
$$

is a non-decreasing function of $r$. Thus, the integral in (1.15) gives that $M_{h_\epsilon}(z, r)$ is also a non-decreasing function of $r$. 

We will now show for each fixed \( z \) that \( h_\epsilon(z) \downarrow h(z) \) as \( \epsilon \downarrow 0 \). It will follow from \( M(z,r) \) being non-decreasing. If \( 0 < \epsilon < \delta \), then

\[
h_\epsilon(z) = \int_D h(z - \xi \epsilon) \psi(\xi) d\text{Leb}(\xi)
= \int_0^1 r \phi(r) \int_0^{2\pi} h(z - \epsilon re^{i\theta}) d\theta dr \leq \int_0^1 r \phi(r) \int_0^{2\pi} h(z - \delta re^{i\theta}) d\theta dr = h_\delta(z).
\]

Therefore, for fixed \( z \) we have that \( h_\epsilon(z) \) non-increases as \( \epsilon \) decreases. A use of the Monotone convergence theorem can then be used to show that \( h_\epsilon(z) \to h_0(z) = h(z) \) as \( \epsilon \to 0 \). \( \square \)

### 1.4. Fundamental Equivalence for Subharmonic functions.

**Lemma 1.14.** Let \( \{h_n\} \) be a sequence of subharmonic functions on a connected open \( U \subset \mathbb{C} \). If \( h_n(z) \geq h_{n+1}(z) \) for every \( z \in U \) and \( n \in \mathbb{N} \), then the pointwise limit \( h(z) := \lim h_n(z) \) is either subharmonic on \( U \) or identically equal to \(-\infty\).

**Proof.** We check that \( h \) satisfies Properties 1 and 2 from Definition 1.4. The latter follows from the fact that the point-wise infimum over a family of upper semi-continuous functions is again upper semi-continuous.

We check the SMVP (Property 1) at some \( z \) for which \( \overline{D}_r(z) \subset U \). For each \( n \) we have that

\[
(1.16) \quad h(z) \leq h_n(z) \leq \frac{1}{2\pi} \int h_n(z + re^{i\theta}) d\theta.
\]

It follows from the Monotone Convergence Theorem (see, e.g. [5, Thm 1.26]) that the sequence of limits on the right hand side converge to

\[
\frac{1}{2\pi} \int h(z + re^{i\theta}) d\theta.
\]

Therefore, the inequality (1.16) holds in the limit, giving the SMVP for \( h(z) \). \( \square \)

We now prove a generalization of Proposition 1.8, dropping the smoothness hypotheses:

**Proposition 1.15.** (Fundamental Equivalence)

1. If \( h \) is subharmonic, then its distributional Laplacian is non-negative: \( \Delta h \geq 0 \).
2. If \( \mu \) is any distribution for which \( \Delta \mu \geq 0 \), then \( \mu \) is represented by a subharmonic function. I.e there is a subharmonic function \( h \) so that

\[
\langle \mu, \phi \rangle = \int_{\mathbb{R}^2} \phi \ h \ d\text{Leb}
\]

for every \( \phi \in \mathcal{D}(\mathbb{R}^2) \).

**Proof.** Suppose that \( h \) is subharmonic and let \( h_\epsilon = h * \psi_\epsilon \) be the sequence of regularizations described in Propositions 1.12 and 1.13. They are \( C^\infty \) subharmonic functions that converge to \( h \) (as distributions) when \( \epsilon \to 0 \). Proposition 1.8 gives that \( \Delta h_\epsilon \geq 0 \) for each \( \epsilon > 0 \). Since the positive distributions form a weak-* closed subset of \( \mathcal{D}'(\mathbb{R}^2) \), we also have \( \Delta h \geq 0 \).

Now suppose that \( \mu \in \mathcal{D}'(\mathbb{R}^2) \) satisfies \( \Delta \mu \geq 0 \). For any \( \epsilon > 0 \), the regularization \( \mu_\epsilon = \mu * \psi_\epsilon \) is \( C^\infty \) and satisfies \( \Delta \mu_\epsilon = (\Delta \mu) * \psi_\epsilon \geq 0 \) (both \( \Delta \mu \geq 0 \) and \( \psi_\epsilon \geq 0 \)). Therefore, Proposition 1.8 gives that \( \mu_\epsilon \) is subharmonic.
By construction, $\mu_\epsilon \to \mu$ within $\mathcal{D}'(\mathbb{R}^2)$. We will show for any fixed $z \in \mathbb{R}^2$, $\mu_\epsilon(z)$ non-increases as $\epsilon \searrow 0$. This allows us to take a pointwise limit
\[ h(z) = \lim_{\epsilon \searrow 0} \mu_\epsilon(z), \]
which, according to Lemma 1.14, will either be subharmonic or identically equal to $-\infty$.

We would like to use Lemma 1.13 to get the desired monotonicity in $\epsilon$ for $\mu_\epsilon(z)$. However, since $\mu$ is not subharmonic (we are trying to prove that it is), we must use this Lemma in the following indirect manner.

We convolve twice, considering $(\mu * \psi_\delta) * \psi_\epsilon$ for $\delta, \epsilon > 0$. For fixed $\delta > 0$, $\mu * \psi_\delta$ is subharmonic (same reason as for $\mu * \psi_\epsilon$.) Therefore, Lemma 1.13 gives that $\mu * \psi_\delta * \psi_\epsilon(z)$ is monotone in $\epsilon$. Lemma 1.13 also gives the pointwise convergence
\[ \lim_{\delta \to 0} \mu * \psi_\delta * \psi_\epsilon(z) = \lim_{\delta \to 0} \mu * \psi_\delta(z) = \mu * \psi_\epsilon(z). \]

Since monotonicity is preserved under pointwise limits, we see that that $\mu * \psi_\epsilon(z)$ is also monotone in $\epsilon$.

Therefore, $h(z) = \lim_{\epsilon \searrow 0} \mu_\epsilon(z)$ is either subharmonic identically equal to $-\infty$. For any test function $\chi \geq 0$ from $\mathcal{D}(\mathbb{R}^2)$ we have
\[ (1.17) \langle \mu, \chi \rangle = \lim_{\epsilon \to 0} \langle \mu_\epsilon, \chi \rangle = \lim_{\epsilon \to 0} \int \mu_\epsilon \chi \, d\text{Leb} = \int (\lim_{\epsilon \to 0} \mu_\epsilon) \chi \, d\text{Leb} = \int h \chi \, d\text{Leb}, \]
with the second to last equality following from the Monotone Convergence Theorem. Since $\langle \mu, \chi \rangle \in \mathbb{R}$, we see that $h$ is not identically equal to $-\infty$, hence $h$ is subharmonic.

Equation (1.17) can also be obtained for general $\phi \in \mathcal{D}(\mathbb{R}^2)$ by splitting each integral to be over sets where $\phi$ is of constant sign. Therefore, $\mu$, is represented by the subharmonic function $h$. \qed

1.5. Compactness and convergence theorems for subharmonic functions. First, we consider different modes of convergence for subharmonic functions:

**Proposition 1.16. (Equivalent modes of convergence)** Let $\{h_n\}$ be a sequence of subharmonic functions on $U \subset \mathbb{R}^2$ and $h$ a subharmonic function on $U$. Then $h_n \to h$ in $\mathcal{D}'(U)$ if and only if $h_n \to h$ in $L^1_{\text{loc}}(U)$.

**Proof.** Proposition 1.3 gives that convergence in $L^1_{\text{loc}}(U)$ implies convergence in $\mathcal{D}'(U)$. (Subharmonicity is not needed--this is true for any convergent sequence of $L^1_{\text{loc}}$ functions.)

Now suppose that $h_n \to h$ in $\mathcal{D}'(U)$.

For any $\epsilon > 0$, Proposition 1.13 gives that the regularizations satisfy
\[ h_n(z) \leq h_n * \psi_\epsilon(z) \quad \text{and} \quad h(z) \leq h * \psi_\epsilon(z). \]

Moreover, Proposition 1.10 gives that $h_n * \psi_\epsilon$ converges uniformly on compact sets to $h * \psi_\epsilon$.

Therefore,
\[ h_n(z) \leq h_n * \psi_\epsilon(z) \to h * \psi_\epsilon(z), \quad \text{implying} \quad \limsup_{n \to \infty} h_n(z) \leq h * \psi_\epsilon(z). \]

For any $\delta > 0$ and $n$ sufficiently large we have
\[ h * \psi_\epsilon(z) + \delta - h_n(z) > 0 \quad \text{and} \quad h * \psi_\epsilon(z) + \delta - h(z) > 0. \]

Let $\chi \geq 0 \in \mathcal{D}(U)$. Weak-$*$ convergence $h_n \to h$ implies that
\[ \int (h * \psi_\epsilon(z) + \delta - h_n(z)) \chi(z) \, d\text{Leb} \to \int (h * \psi_\epsilon(z) + \delta - h(z)) \chi(z) \, d\text{Leb}. \]
Moreover, since both integrands are positive
\[
\int |h - h_n| \chi \, d\text{Leb} = \int |(h * \psi_\epsilon(z) + \delta - h_n(z))\chi(z) - (h * \psi_\epsilon(z) + \delta - h(z))\chi(z)| \, d\text{Leb} \\
\leq \int |(h * \psi_\epsilon(z) + \delta - h_n(z))\chi(z) + |(h * \psi_\epsilon(z) + \delta - h(z))\chi(z)| \, d\text{Leb} \\
\rightarrow 2 \int (h * \psi_\epsilon(z) + \delta - h(z))\chi(z) \, d\text{Leb}.
\]
In brief,
\[
\lim \sup_{n \to \infty} \int |h - h_n| \chi \, d\text{Leb} \leq 2 \int (h * \psi_\epsilon(z) + \delta - h(z))\chi(z) \, d\text{Leb}.
\]
Since the right hand side goes to zero as \( \epsilon, \delta \to 0 \) and \( \chi \) is arbitrary, this gives that \( h_n \to h \) in \( L^1_{\text{loc}}(U) \).

\[\square\]

**Corollary 1.17.** If \( h_n \) is a sequence of subharmonic functions on \( U \) and \( h \) is subharmonic on \( U \) with \( h_n \to h \) in \( \mathcal{D}'(U) \), then \( h_n(z) \to h(z) \) for Lebesgue a.e. \( z \in U \).

The following compactness theorem (and its straightforward generalization to higher dimensions) plays a similar role in potential-theoretic proofs from one-variable dynamics to the role played by Montel’s Theorem in more classical one-variable dynamics.

**Theorem 1.18. (Compactness Criterion)** For any \( M \geq 0 \) and \( z_0 \in \mathbb{R}^2 \), the set
\[
\mathcal{S} \equiv \mathcal{S}_{M,z_0} := \{ h \text{ subharmonic on } \mathbb{R}^2 : h \leq M, h(z_0) \geq -M \}
\]
is compact in both the weak-* topology on \( \mathcal{D}'(\mathbb{R}^2) \) and the topology on \( L^1_{\text{loc}}(\mathbb{R}^2) \).

**Proof.** It suffices to show that for any sequence \( h_n \subset \mathcal{S} \) there is a subsequence \( h_{n_k} \) converging in each of the two senses (in \( \mathcal{D}' \) and in \( L^1_{\text{loc}} \)) to some \( h \in \mathcal{S} \). By Proposition 1.16, we need only prove weak-* convergence in \( \mathcal{D}'(U) \).

We will first show that there is a subsequence \( \{h_{n_k}\} \) that converges in \( \mathcal{D}'(\mathbb{R}^2) \) to some \( \mu \in \mathcal{D}'(\mathbb{R}^2) \). We will then use the Fundamental Equivalence (Proposition 1.15) to show that \( \mu \) is represented by a subharmonic function.

To find a convergent subsequence, we will use “the usual weak-* compactness” given by Alaoglu’s Theorem; see [4, Thm. 5.58]. However, Alaoglu’s Theorem does not directly apply to \( \mathcal{D}'(\mathbb{R}^2) \) since \( \mathcal{D}(\mathbb{R}^2) \) is not a normed space. Instead, we will apply over each in a sequence of closed discs of increasing radii and then take a diagonal subsequence.

Without loss of generality, we can suppose that \( h_n \leq 0 \) and \( h_n(z_0) \geq -M \) by subtracting \( M \) from every element of the sequence and then dividing by 2.

Let \( \overline{D}_R \equiv \overline{D}_R(z_0) \) be the disc of radius \( R > 0 \) centered at \( z_0 \). For every \( n \)
\[
-M \leq h_n(z_0) \leq \int_{\overline{D}_R} h_n(z) \, d\text{Leb} \leq 0
\]
with the second inequality following from the SMVP. Since \( h_n(z) \) has a constant sign, this gives \( \| h_n \|_{L^1} \leq M \).

Hölder’s Inequality gives
\[
\int_{\overline{D}_R} |h_n| \, d\text{Leb} \leq \| h_n \|_{L^1} \cdot \| \eta \|_{L^\infty} \leq M \cdot \| \eta \|_{L^\infty},
\]
where \( \eta \) is represented by a subharmonic function and the last inequality follows from the fact that \( \eta \) is constant.

Hence, \( h_n \) is compact in \( L^1_{\text{loc}}(\overline{D}_R) \) for each \( R > 0 \). Since \( \mathcal{S} \) is closed in \( L^1_{\text{loc}}(\mathbb{R}^2) \), a diagonal argument gives a subsequence \( h_{n_k} \to h \) in the weak-* topology.
for any $\eta \in C^0(\mathbb{D}_R)$. In other words, the operator norm of each $h_n$ is bounded by $M$. Since $C^0(\mathbb{D}_R)$ is a separable normed space, Alaoglu’s Theorem gives existence of subsequence $h_{n_k}$ so that

$$\int_{\mathbb{D}_R} \eta h_{n_k} \, d\text{Leb}$$

converges for every $\eta \in C^0(\mathbb{D}_R)$.

Let us now consider the countable sequence of discs $\mathbb{D}_R(z_0)$ for $R = 1, 2, \ldots$, exhausting all of $\mathbb{R}^2$. We inductively form a sequence of subsequences $\{h_{n_k}^R\}$ of $\{h_n\}$ so that $\{h_{n_k}^R\}$ is chosen as a subsequence of $\{h_{n_k}^{R-1}\}$ so that (1.20) converges for every $\eta \in C^0(\mathbb{D}_R)$.

Consider the diagonal subsequence $\{h_{n_k}^k\}$ of $\{h_{n_k}\}$. For any test function $\phi \in \mathcal{D}(\mathbb{R}^2)$ there is some $R \in \mathbb{N}$ so that $\text{supp} \phi \subset \mathbb{D}_R(z_0)$. Since $\{h_{n_k}^k\}$ is eventually a subsequence of $\{h_{n_k}\}$ we have that

$$\int_{\mathbb{R}^2} \phi h_{n_k}^k \, d\text{Leb}$$

converges. For the remainder of the proof we denote the subsequence $\{h_{n_k}^k\}$ by just $\{h_{n_k}\}$.

Let $\mu$ be defined by

$$\langle \mu, \phi \rangle := \lim_{k \to \infty} \int_{\mathbb{R}^2} \phi h_{n_k} \, d\text{Leb}$$

for any $\phi \in \mathcal{D}(\mathbb{R}^2)$. In order for $\mu$ to be a distribution, it must act linearly and continuously on $\mathcal{D}(\mathbb{R}^2)$. Linearity follows immediately from linearity of the integrals. If $\phi_i \to \phi$ in $\mathcal{D}(\mathbb{R}^2)$, then there exists some $R > 0$ so that $\text{supp} \phi_i \subset \mathbb{D}_R$ for every $i$. By (1.19),

$$\left| \int_{\mathbb{D}_R} (\phi_i - \phi) h_{n_k} \, d\text{Leb} \right| \leq M \cdot \|\phi_i - \phi\|_\infty$$

for each $k$. Therefore, $|\langle \mu, \phi_i - \phi \rangle| \leq M \cdot \|\phi_i - \phi\|_\infty$, giving $\lim_{i \to \infty} \langle \mu, \phi_i \rangle = \langle \mu, \phi \rangle$.

We now check that $\mu$ is represented by a subharmonic function $h$, so that $h_{n_k} \to h$ in $\mathcal{D}'(\mathbb{R}^2)$. By the Fundamental Equivalence (Proposition 1.15), $\Delta h_{n_k} \geq 0$ for each $k$. Therefore,

$$\Delta h = \Delta \lim h_{n_k} = \lim \Delta h_{n_k} \geq 0,$$

since differential operators act continuously on $\mathcal{D}'(\mathbb{R}^2)$. Using the Fundamental Equivalence again, $\Delta \mu \geq 0$ implies that the distribution $\mu$ is represented by a subharmonic function $h$. \hfill \Box

**Corollary 1.19.** (Converting a.e. convergence to genuine convergence):
Suppose that $\{h_n\} \in \mathcal{S}_{M,z_0}$ for some $M$ and $z_0$ and that $\lim_{n \to \infty} h_n(z)$ exists for Lebesgue a.e. $z \in \mathbb{R}^2$. Then, there is a subharmonic function $h \in \mathcal{S}_M$ so that all three modes of convergence apply:

1. $\lim_{n \to \infty} h_n(z) = h(z)$ for a.e. $z \in \mathbb{R}^2$.
2. $h_n \to h$ in $\mathcal{D}'(\mathbb{R}^2)$,
3. $h_n \to h$ in $L^1_{\text{loc}}(\mathbb{R}^2)$, and
Proof. Theorem 1.18 gives a subsequence $h_{n_k}$ converging in the $L^1_{\text{loc}}$ topology to some subharmonic function $h$. In particular, $\lim_{k \to \infty} h_{n_k}(z) = h(z)$ for a.e. $z \in \mathbb{R}^2$. The hypothesis that $\lim h_n(z)$ exists for a.e. $z \in \mathbb{R}^2$ then forces that $$\lim h_n(z) = \lim h_{n_k}(z) = h(z)$$ for a.e. $z$, giving (1).

By Proposition 1.16, (2) will imply (3). Let $\phi$ be any test function from $\mathcal{D}(\mathbb{R}^2)$ and $\{h_{n_k}\}$ be a subsequence so that $$\limsup_{n \to \infty} \int \phi \, h_n \, d\text{Leb} = \lim_{k \to \infty} \int \phi \, h_{n_k} \, d\text{Leb}$$ Since $\{h_{n_k}\} \subset \mathcal{S}_M$, we can extract a further subsequence (which we will also call $\{h_{n_k}\}$) converging in the weak-* topology to some subharmonic $\tilde{h}$ and satisfying $\lim_{k \to \infty} h_{n_k}(z) = \tilde{h}(z)$ for a.e. $z$. We have

$$\limsup_{n \to \infty} \int \phi \, h_n \, d\text{Leb} = \lim_{k \to \infty} \int \phi \, h_{n_k} \, d\text{Leb} = \int \phi \, \tilde{h} \, d\text{Leb} = \int \phi \, h \, d\text{Leb}$$

(1.22) with the second inequality following from convergence of $\{h_{n_k}\}$ to $\tilde{h}$ in the weak-* topology. The last equality follows because $\lim_{k \to \infty} h_{n_k}(z) = \tilde{h}(z)$ for a.e. $z$, together with (1) gives that $h(z)$ and $\tilde{h}$ are equal almost everywhere.

Essentially the same proof gives

$$\liminf_{n \to \infty} \int \phi \, h_n \, d\text{Leb} = \int \phi \, h \, d\text{Leb}$$

(1.23) Since (1.22) and (1.23) both hold for any $\phi \in \mathcal{D}(\mathbb{R}^2)$, we have that $h_n$ converges to $h$ in the weak-* topology. \qed

References


