Finsler Manifolds with Nonpositive Flag Curvature and Constant S-curvature

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Abstract

The flag curvature is a natural extension of the sectional curvature in Riemannian geometry, and the S-curvature is a non-Riemannian quantity which vanishes for Riemannian metrics. There are (incomplete) non-Riemannian Finsler metrics on an open subset in $\mathbb{R}^n$ with negative flag curvature and constant S-curvature. In this paper, we are going to show a global rigidity theorem that every Finsler metric with negative flag curvature and constant S-curvature must be Riemannian if the manifold is compact. We also study the nonpositive flag curvature case.

1 Introduction

One of important problems in Finsler geometry is to understand the geometric meanings of various quantities and their impacts on the global geometric structures. Imaging a Finsler manifold as an Easter egg and a Riemannian manifold as a white egg, Finsler manifolds are not only curved, but also very “colorful”. The flag curvature $K$ tells us how curved is the Finsler manifold at a point. There are several non-Riemannian quantities which describe the “color” and its rate of change over the manifold, such as the mean Cartan torsion $I$, the mean Landsberg curvature $J$ and the S-curvature $S$ (see [17] or Section 2 below). These quantities interact with the flag curvature in a delicate way. The mean Landsberg curvature and the S-curvature reveal different non-Riemannian properties. For example, there is a family of Finsler metrics on $S^n$ with $K = 1$ and $S = 0$ [5]. However, every local Finsler metric with $K = 1$ and $J = 0$ must be Riemannian (Theorem 9.1.1 in [17]).

An $n$-dimensional Finsler metric is said to have constant $S$-curvature if $S = (n+1)cF$ for some constant $c$. It is known that every Randers metric of constant flag curvature has constant S-curvature [3], [4]. This is one of our motivations to consider Finsler metrics of constant S-curvature. In this paper, we are going to prove the following global metric rigidity theorem.

**Theorem 1.1** Let $(M, F)$ be an $n$-dimensional compact boundaryless Finsler manifold with constant $S$-curvature, i.e., $S = (n+1)cF$ for some constant $c$.
(a) If $F$ has negative flag curvature, $K < 0$, then it must be Riemannian;

(b) If $F$ has nonpositive flag curvature, $K \leq 0$, then the mean Landsberg curvature vanishes, $J = 0$, and the flag curvature $K(P, y) = 0$ for the flags $P = \text{span}\{y, I_y\} \subset T_xM$ whenever $I_y \neq 0$.

The compactness in Theorem 1.1 (a) can not be dropped. Consider the following family of Finsler metrics on the unit ball $B^m \subset \mathbb{R}^n$,

$$F = \sqrt{|y|^2 - (|x|^2 + \langle x, y \rangle)^2 + \langle x, y \rangle^2 \over 1 - |x|^2} + \langle y, x \rangle,$$

where $y \in T_xB^m \cong \mathbb{R}^n$ and $a \in \mathbb{R}^n$ is an arbitrary constant vector with $|a| < 1$. It is proved that $F$ has constant flag curvature $K = -\frac{1}{2}$ and constant S-curvature $S = \frac{1}{2}(n + 1)F$ (see [17][18]). Clearly, $F$ is not Riemannian.

The compactness in Theorem 1.1 (b) can not be dropped. Let $n \geq 2$ and

$$\mathcal{U} := \left\{ p = (s, t, \bar{y}) \in \mathbb{R}^2 \times \mathbb{R}^{n-2} \left| s^2 + t^2 < 1 \right. \right\}.
$$

Define

$$F := \sqrt{\left(-tu + sv\right)^2 + |y|^2 \left(1 - s^2 - t^2\right) - \left(-tu + sv\right)^2 \over 1 - s^2 - t^2},$$

where $y = (u, v, \bar{y}) \in T_p\mathcal{U} \cong \mathbb{R}^n$ and $p = (s, t, \bar{y}) \in \mathcal{U}$. $F$ is an incomplete Finsler metric on $\Omega$ with $K = 0$ and $S = 0$, but $J \neq 0$ [20].

The compactness condition in Theorem 1.1 can be replaced by a completeness condition together with certain growth condition on the mean Cartan torsion. See Theorems 4.1 and 4.2 below.

**Corollary 1.2** Let $(M, F)$ be a compact boundaryless Bervald manifold with nonpositive flag curvature. Then the following hold,

(a) If $F$ has negative flag curvature, $K < 0$, then it must be Riemannian;

(b) If $F$ has nonpositive flag curvature, $K \leq 0$, then $K(P, y) = 0$ for the flag $P = \text{span}\{y, I_y\}$ whenever $I_y \neq 0$.

In dimension two, we have the following

**Corollary 1.3** Let $(M, F)$ be a compact boundaryless Finsler surface. Suppose that $K \leq 0$ and $S = 3cF$ for some constant $c$, then $F$ is either locally Minkowskian or Riemannian.

The proof is simple. First, by Theorem 4.1 below, we know that $J = 0$, then the theorem follows from Theorem 7,3,2 in [2].
In dimension $n \geq 3$, we have some non-trivial examples satisfying the conditions and conclusions in Theorem 1.1 (b). Let $(N,h)$ be an arbitrary closed hyperbolic Riemannian manifold. For any $\epsilon \geq 0$, let

$$F_\epsilon := \sqrt{h^2(\overline{x}, \overline{y}) + w^2 + \epsilon \sqrt{h^2(\overline{x}, \overline{y}) + w^2}},$$

where $x = (\overline{x}, s) \in M$ and $y = \overline{y} \oplus w \overline{\nu} \in T_y M$. This family of Finsler metrics is constructed by Z.I. Szabó in his classification of Berwald metrics [21]. It is known that each $F_\epsilon$ is a Berwald metric. Thus $J = 0$ and $S = 0$ [17]. Further it can be shown that $F_\epsilon$ satisfies that $K \leq 0$ and $K(F, y) = 0$ for $P = \text{span}(y, I_y)$. The proof will be given in Section 5 below. A natural problem arises: Is the Finsler metric in Theorem 1.1 (b) a Berwald metric? This problem remains open.

Finally, we should point out that there are already several global rigidity results on the metric structure of Finsler manifolds with $K \leq 0$. For example, H. Akbar-Zadeh proves that every closed Finsler manifold with $K = -1$ must be Riemannian and every closed Finsler manifold with $K = 0$ must be locally Minkowskian [1]. Mo-Sheen prove that every closed Finsler manifold of scalar curvature with $K < 0$ must be of Randers type in dimension $\geq 3$ [15]. Here a Finsler metric $F$ is said to be of scalar curvature if the flag curvature $K = K(x, y)$ is independent of $P$ for any given direction $y \in T_x M$. Riemannian metrics of scalar curvature must have isotropic sectional curvature $K = K(x)$, hence they have constant sectional curvature in dimension $n \geq 3$ by the Schur Lemma. But there are lots of Finsler metrics of scalar curvature which have not been completely classified yet.

## 2 Preliminaries

In this section, we are going to give a brief description on the flag curvature and the above mentioned non-Riemannian quantities.

Let $M$ be an $n$-dimensional manifold and let $\pi : TM_0 := TM \setminus \{0\} \to M$ denote the slit tangent bundle. The pull-back tangent bundle is defined by $\pi^* TM := \{(x, y, v) \mid 0 \neq y, v \in T_x M\}$ and the pull-back cotangent bundle is defined by $\pi^* T^* M := \{\pi^* \theta \mid \theta \in T^* M\}$.

By definition, a Finsler metric $F$ on a manifold $M$ is a nonnegative function on $TM$ which is positively $y$-homogeneous of degree one with positive definite fundamental tensor $g := g_{ij} dx^i \otimes dx^j$ on $\pi^* TM$, where $g_{ij} := \frac{1}{2}[F^2]^i_j(x, y)$. A special class of Finsler metrics are Randers metrics in the form $F = \alpha + \beta$ where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form with $\|\beta\|_x := \sqrt{a^{ij}(x)b_i(x)b_j(x)} < 1$ for any $x \in M$.

For a Finsler metric $F$, the volume $dV = \sigma_F(x) dx^1 \cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\text{Vol}(B^n(1))}{\text{Vol}\{y^i \in \mathbb{R}^n \mid F(x, y^i, \frac{\partial}{\partial y^i}) < 1\}}.$$  

(2)
When $F = \sqrt{g_{ij}(x)y^iy^j}$ is Riemannian, then $\sigma_F (x) = \sqrt{\det(g_{ij}(x))}$. In general, the following quantity is not equal to zero, 

$$\tau(x, y) := \ln \left[ \frac{\sqrt{\det(g_{ij}(x), y))}}{\sigma_F (x)} \right].$$

$\tau = \tau(x, y)$ is a scalar function on $TM_\circ$, which is called the distortion [17]. The distortion is our primary non-Riemannian quantity. Let

$$I_i := \frac{\partial \tau}{\partial y^i}(x, y) = \frac{1}{2}g^{jk}(x, y)\frac{\partial g_{jk}}{\partial y^i}(x, y).$$

We have

$$I_i y^i = 0.$$  \hspace{1cm} (4)

The tensor $I := I_idx^i$ on $TM_\circ$ is called the mean Cartan tensor. According to Deicke's theorem [11], $F$ is Riemannian if and only if $I = 0$. Define the norm of $I$ at a point $x \in M$ by

$$\|I\|_x := \sup_{0 \neq y \in T_x M} \sqrt{I_i(x, y)g^{ij}(x, y)I_j(x, y)}.$$

For a point $p \in M$, let

$$I_p(r) := \sup_{\min(d(p, x), d(x, p)) < r} \|I\|_x.$$

The mean Cartan tensor $I$ is said to grow sub-linearly if for any point $p \in M$,

$$I_p(r) = o(r), \quad (r \to +\infty).$$

$I$ is said to grow sub-exponentially at rate of $k = 1$ if for any point $p \in M$,

$$I_p(r) = o(e^r), \quad (r \to +\infty).$$

It is known that for a Randers metric $F = \alpha + \beta$, $I$ is bounded, i.e.,

$$\|I\|_x \leq \frac{n + 1}{\sqrt{2}} \sqrt{1 - \sqrt{1 - \|\beta\|^2_2}} \leq \frac{n + 1}{\sqrt{2}}, \quad x \in M.$$

The bound in dimension two is suggested by B. Lackey: See Proposition 7.1.2 in [17] for a proof.

The geodesics in a Finsler manifold are characterized by a system of second order ordinary differential equations

$$\ddot{\sigma}^i + 2G^i(\sigma, \dot{\sigma}) = 0,$$

where $G^i = G^i(x, y)$ are positively $y$-homogeneous functions of degree two. When $F$ is Riemannian, $G^i = \frac{1}{2}F^i_{jk}(x)y^jy^k$ are quadratic in $y \in T_x M$. A Finsler metric with such a property is called a Berwald metric. There are many non-Riemannian Berwald manifolds (see Section 5 below).
For a non-zero vector \( y \in T_xM \), set
\[
S(x, y) := \frac{d}{dt} \left[ \tau \left( \sigma(t), \dot{\sigma}(t) \right) \right]_{t=0},
\]
where \( \sigma = \sigma(t) \) is the geodesic with \( \sigma(0) = x \) and \( \dot{\sigma}(0) = y \). \( S = S(x, y) \) is a scalar function on \( TM_o \) which is called the \( S \)-curvature [16][17]. Let \( dV = \sigma_F(x)dx^1 \cdots dx^n \) be the volume form on \( M \). The \( S \)-curvature can be expressed by
\[
S = \frac{\partial G^m}{\partial y^m}(x, y) - y^m \frac{\partial}{\partial x^m} \left[ \ln \sigma_F(x) \right].
\]
(5)

It is proved that \( S = 0 \) for Berwald metrics [16][17]. An \( n \)-dimensional Finsler metric \( F \) is said to have constant \( S \)-curvature if there is a constant \( c \) such that \( S = (n + 1)cF \). It is known that all Randers metrics of constant flag curvature must have constant \( S \)-curvature [3] (see [4] for the classification of such metrics).

There is a distinguished linear connection \( \nabla \) on \( \pi^*TM \) which is called the Chern connection [10]. Let \( \{e_i\} \) be a local frame for \( \pi^*TM \) and \( \{\omega^i\} \) the dual local frame for \( \pi^*T^*M \), \( \nabla \) can be expressed by
\[
\nabla V = \left\{ dV^j + V^j \omega^j \right\} \otimes e_i,
\]
where \( V = V^i e_i \in C^\infty(\pi^*TM) \). The Chern connection can be viewed as a generalization of the Levi-Civita connection in Riemannian geometry. Let
\[
\omega^{n+i} := dy^i + y^j \omega^j_i,
\]
where \( y^i \) are local functions on \( TM_o \) defined by the canonical section \( Y = y^i e_i \) of \( \pi^*TM \). We obtain a local coframe \( \{\omega^i, \omega^{n+i}\} \) for \( T^*(TM_o) \).

Let
\[
\Omega^i := d\omega^{n+i} - \omega^{n+j} \wedge \omega^j_i.
\]
\( \Omega^i \) can be expressed as follows,
\[
\Omega^i = \frac{1}{2} R^k_{ijl} \omega^k \wedge \omega^l - L^k_{ijl} \omega^k \wedge \omega^{n+l},
\]
where \( R^k_{ijl} + R^k_{lji} = 0 \) and \( L^k_{ijl} = L^l_{ijk} \). The anti-symmetric tensor \( R = R^k_{ijl} e_i \otimes \omega^k \otimes \omega^j \) is called the Riemann tensor and the symmetric tensor \( L = L^k_{ijl} e_i \otimes \omega^k \otimes \omega^j \) is called the Landsberg tensor.

Let
\[
R^i_{jk} := R^i_{kji}, \quad R_{jk} := g_{ij}R^i_{jk}.
\]
We have
\[
R^i_{jk} y^k = 0, \quad R_{jk} = R_{kj}.
\]
(6)

See [17] for details. The tensor \( R := R^k_{ijl} e_i \otimes \omega^k \) is still called the Riemann tensor. The notion of Riemann (curvature) tensor for general Finsler metrics is introduced by L. Berwald using the Berwald connection [6][7]. Let
\[
J_k := L^m_{km}.
\]
The tensor $\mathbf{J} := J_k \omega^k$ is called the mean Landsberg tensor. For a Berwald metric, $\mathbf{J} = 0$ [17].

For a scalar function on $TM_o$, say $\tau$, we define its covariant derivatives by

$$d\tau = \tau_k \omega^k + \tau_i \omega^{n+k}.$$  

From (3), we have

$$\tau_i = \frac{\partial \tau}{\partial y^i} = I_i.$$  

We have

$$S := \tau |_{m} y^m.$$  

For a tensor, say, $\mathbf{T} = I_i \omega^i$, the covariant derivatives are defined in a canonical way by

$$dI_i - I_i \omega^i_{-k} = I_i \omega_k + I_i \omega^{n+k}.$$  

We have

$$J_i = I_i |_{m} y^m$$  

Hence

$$J_i y^i = 0.$$  

See [17] for details.

Now we interpret the above geometric quantities in a different way.

Let $F$ be a Finsler metric on an $n$-dimensional manifold $M$. For a non-zero tangent vector $y = y^j \frac{\partial}{\partial x^j} |_x \in T_x M$, define

$$g_y(u,v) := g_{ij}(x,y)u^i v^j, \quad u = u^i \frac{\partial}{\partial x^i} |_x, v = v^j \frac{\partial}{\partial x^j} |_x \in T_x M,$$

where $g_{ij}(x,y) = \frac{1}{2} [F^2] y^i y^j(x,y)$. Each $g_y$ is an inner product on the tangent space $T_x M$.

The Riemann tensor can be viewed as a family of endomorphisms on tangent spaces.

$$R_y(u) := R^i_k (x,y) u^k \frac{\partial}{\partial x^i} |_x,$$

where $u = u^i \frac{\partial}{\partial x^i} |_x \in T_x M$. The coefficients $R^i_k = R^i_k (x,y)$ are given by

$$R^i_k = 2 \frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2 G^j \frac{\partial^2 G^i}{\partial y^i \partial y^j} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.$$  

It follows from (6) that

$$R_y(y) = 0, \quad g_y(R_y(u),v) = g_y(u,R_y(v)),$$

where $u,v \in T_x M$. The family $\mathbf{R} := \{R_y | y \in T_x M \setminus \{0\}\}$ is called the Riemann curvature.
Using the Chern connection $\nabla$ on $\pi^*TM$, one can define the covariant derivative of a vector field $X = X^i(t) \frac{\partial}{\partial x^i}|_{c(t)}$ along a curve $c$ by

$$D_c X(t) := \left\{ \frac{dX^i}{dt}(t) + X^j(t) \Gamma^i_{jk}(c(t), \dot{c}(t)) \frac{\partial}{\partial x^j}|_{c(t)} \right\} \frac{\partial}{\partial x^i}|_{c(t)}.$$ 

If $H = H(s, t)$ is a family of geodesics, i.e., for each $s$, $\sigma_s(t) := H(s, t)$ is a geodesic, the variation field $V_s(t) := \frac{\partial H}{\partial s}(s, t)$ satisfies the following Jacobi field along $\sigma_s$,

$$D_{\sigma_s} D_{\sigma_s} V_s(t) + R_{\sigma_s(t)}(V(t)) = 0.$$ 

For a tangent plane $P \subset T_xM$ and a vector $0 \neq y \in P$, let

$$K(P, y) := \frac{g_y(R_y(u), u)}{g_y(y, y)g_y(u, u) - [g_y(y, u)]^2},$$

where $P = \text{span}\{y, u\}$. By (10), one can see that $K(P, y)$ is well-defined, namely, independent of the choice of a particular $u \in T_xM$.

The mean Cartan tensor and the mean Landsberg tensor can be viewed as families of vectors on the manifold, i.e.,

$$I_y = I^i(x, y) \frac{\partial}{\partial x^i}|_x, \quad J_y = J^i(x, y) \frac{\partial}{\partial x^i}|_x,$$

where $I^i := g^{ij}I_j$ and $J^i := g^{ij}J_j$. It follows from (4) and (8) that

$$g_y(I_y, y) = 0 = g_y(J_y, y).$$

Thus $I_y$ and $J_y$ are perpendicular to $y$ with respect to $g_y$. We call $I := \{I_y \mid y \in TM \setminus \{0\}\}$ and $J := \{J_y \mid y \in TM \setminus \{0\}\}$ the mean Cartan torsion and the Landsberg curvature, respectively.

If $F$ is a Berwald metric, then $J = 0$ and $S = 0$. The converse is true too in dimension two, but it is not clear in higher dimensions (cf. [17]).

### 3 Finsler metrics with constant S-curvature

The following lemma is crucial for the proof of Theorem 1.1.

**Lemma 3.1** Let $(M, F)$ be an $n$-dimensional Finsler manifold. Suppose that there is a constant $c$ and a closed 1-form $\gamma$ such that

$$S(x, y) = (n + 1)cF(x, y) + \gamma_x(y), \quad y \in T_xM,$$

then along any geodesic $\sigma = \sigma(t)$, the vector field $I(t) := I^i(\sigma(t), \dot{\sigma}(t)) \frac{\partial}{\partial x^i}|_{\sigma(t)}$ satisfies the following equation:

$$D_{\sigma} D_{\sigma} I(t) + R_{\dot{\sigma}(t)}(I(t)) = 0.$$  \hfill (11)
Proof: It is known that the Landsberg tensor satisfies the following equation [13] [15]:

$$J_k|m y^m + I_m R^m_{k m} = -\frac{1}{3}\left\{2 R^m_{k m} + R^m_{m k}\right\}$$

(12)

and the S-curvature satisfies the following equation [8] [14]:

$$S_k|m y^m - S_k = -\frac{1}{3}\left\{2 R^m_{k m} + R^m_{m k}\right\}.$$  

(13)

It follows from (12) and (13) that

$$J_k|m y^m + I_m R^m_{k m} = S_k|m y^m - S_k.$$

By (7), we can rewrite the above equation as follows

$$I^i_{\mid k\mid q} y^q y^g + R^i_{m k} I^m = g^{j k}\left\{S_k|m y^m - S_k\right\}.$$ 

(14)

Note that $F = \sqrt{g_{ij} y^i y^j}$ satisfies

$$F_{\mid m} = \frac{g_{\mid k m} y^i y^j}{2F} = 0, \quad F_{\mid k m} = \frac{g_{k m} y^i}{F} = 0.$$

Since $\gamma = \gamma_i dx^i$ is closed, it satisfies

$$\gamma_k|m y^m - \gamma_k = \left\{\frac{\partial \gamma_k}{\partial x^m} - \frac{\partial \gamma_m}{\partial x^k}\right\} y^m = 0.$$

We have

$$S_k|m y^m - S_k = (n + 1)\left\{F_{\mid k m} y^m - F_{\mid k}\right\} + \gamma_k|m y^m - \gamma_k = 0.$$

Then (14) is reduced to

$$I^i_{\mid k\mid q} y^q y^g + R^i_{m k} I^m = 0.$$ 

(15)

Since $\sigma$ is a geodesic, we have

$$D_{\hat{\sigma}} D_{\hat{\sigma}} I(t) = I^i_{\mid k\mid \sigma(t), \hat{\sigma}(t)} \hat{\sigma}^p(t) \hat{\sigma}^q(t) \frac{\partial}{\partial x^i} \mid_{\sigma(t)}.$$

Then (15) restricted to $\sigma(t)$ gives rise to (11).

Q.E.D.

4 Proof of Theorem 1.1

In this section, we are going to prove a slightly more general version of Theorem 1.1.
Theorem 4.1 Let \((M, F)\) be an \(n\)-dimensional complete Finsler manifold with nonpositive flag curvature \(K \leq 0\) and almost constant \(S\)-curvature \(S = (n + 1)\epsilon F + \gamma\) \((\epsilon = \text{constant and } \gamma \text{ is a closed 1-form})\). Suppose that the mean Cartan torsion grows sub-linearly. Then \(J = 0\) and \(K(F, y) = 0\) for the flag \(P = \text{span}\{1, y\}\) whenever \(1, y \neq 0\). Moreover \(F\) is Riemannian at points where \(K < 0\).

Proof: Let \(y \in T_x M\) be an arbitrary non-zero vector and let \(\sigma = \sigma(t)\) be the geodesic with \(\sigma(0) = x\) and \(\dot{\sigma}(0) = y\). Since the Finsler metric is complete, one may assume that \(\sigma\) is defined on \((-\infty, \infty)\). \(I\) and \(J\) restricted to \(\sigma\) are vector fields along \(\sigma\),

\[
I(t) := I' \left( \sigma(t), \dot{\sigma}(t) \right) \frac{\partial}{\partial x^i} |_{\sigma(t)}, \quad J(t) := J' \left( \sigma(t), \dot{\sigma}(t) \right) \frac{\partial}{\partial x^i} |_{\sigma(t)}.
\]

It follows from (7) that

\[
D_{\sigma}I(t) = I \left|_{m} \left( \sigma(t), \dot{\sigma}(t) \right) \sigma^m(t) \frac{\partial}{\partial x^i} |_{\sigma(t)} \right) = J(t).
\]

If \(I(t) \equiv 0\), then by (16), \(J_y = D_{\sigma}I(0) = 0\). From now on, we assume that \(I(t) \neq 0\). Let

\[
\varphi(t) := \sqrt{g_{\sigma(t)} (I(t), I(t))}.
\]

Let \(I = (a, b) \neq 0\) be a maximal interval on which \(\varphi(t) > 0\). We have

\[
\varphi \varphi' = g_{\sigma} \left( I, D_{\sigma}I \right) \leq \sqrt{g_{\sigma} \left( I, I \right)} \sqrt{g_{\sigma} \left( D_{\sigma}I, D_{\sigma}I \right)} = \varphi \sqrt{g_{\sigma} \left( D_{\sigma}I, D_{\sigma}I \right)}.
\]

This is,

\[
\varphi' \leq \sqrt{g_{\sigma} \left( D_{\sigma}I, D_{\sigma}I \right)}.
\]

By assumption \(K \leq 0\) and (18), we have

\[
\frac{1}{2} \left[ \varphi^2 \right]' = g_{\sigma} \left( D_{\sigma}D_{\sigma}I, I \right) + g_{\sigma} \left( D_{\sigma}I, D_{\sigma}I \right)
\]

\[
= -g_{\sigma} \left( R_{\sigma}(1, I) \right) + g_{\sigma} \left( D_{\sigma}I, D_{\sigma}I \right)
\]

\[
\geq g_{\sigma} \left( D_{\sigma}I, D_{\sigma}I \right) \geq \varphi^2.
\]

We obtain that \(\varphi''(t) \geq 0\).

We claim that \(\varphi'(t) \equiv 0\). Suppose that \(\varphi'(t_o) \neq 0\) for some \(t_o \in I\). If \(\varphi'(t_o) > 0\), then

\[
\varphi(t) \geq \varphi'(t_o)(t - t_o) + \varphi(t_o), \quad t > t_o.
\]

Thus \(b = +\infty\). If \(\varphi'(t_o) < 0\), then

\[
\varphi(t) \geq \varphi'(t_o)(t - t_o) + \varphi(t_o) > \varphi(t_o) > 0, \quad t < t_o.
\]

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Thus \( a = -\infty \). In either case, \( \varphi(t) \) grows at least linearly. Note that for \( p = \sigma(t_0), I_p[t-t_0] \geq \varphi(t) \). We see that \( I \) grows at least linearly. This is impossible. Thus \( \varphi'(t) \equiv 0 \) and \( \varphi(t) = \text{constant} > 0 \). In this case, \( I = (-\infty, \infty) \).

It follows from (19) that

\[
g_\varphi \left( \mathbf{R}_\varphi(\mathbf{I}), \mathbf{I} \right) = 0, \quad D_\varphi \mathbf{I} = 0.
\]

By (16), we get \( \mathbf{J}_y = D_\varphi \mathbf{I}(0) = 0 \). Since \( \mathbf{I}_y \) is orthogonal to \( y \) with respect to \( g_y \), \( \mathbf{K}(P, y) = 0 \) for \( P = \text{span}\{\mathbf{I}_y, y\} \) whenever \( \mathbf{I}_y \neq 0 \).

Assume that \( \mathbf{K} < 0 \) at a point \( x \in M \). It follows from \( g_y(\mathbf{R}_y(\mathbf{I}_y), \mathbf{I}_y) = 0 \) that \( \mathbf{I}_y = 0 \) for all \( y \in T_xM \setminus \{0\} \). By Deicke’s theorem [11], \( F \) is Riemannian.

Q.E.D.

Two natural problems arise:

(a) Is there any complete non-Landsberg metric on \( \mathbb{R}^n \) with \( \mathbf{K} \leq 0 \), \( \mathbf{S} = (n+1)cF \) and \( I_p(r) \sim Cr \) (as \( r \to +\infty \))?  

(b) What is the metric structure of a complete Finsler metric on \( \mathbb{R}^n \) \((n \geq 3)\) satisfying \( \mathbf{K} = 0 \), \( \mathbf{J} = 0 \) and \( \mathbf{S} = 0 \)?

If the flag curvature is strictly negative, we have the following

**Theorem 4.2** Let \((M, F)\) be an \( n \)-dimensional complete Finsler manifold with \( \mathbf{K} \leq -1 \) and almost constant \( S \)-curvature. Suppose that the mean Cartan torsion \( I \) grows sub-exponentially at a rate of \( k = 1 \). Then \( F \) is Riemannian.

**Proof:** The proof is similar. Assume that \( \mathbf{I}_y \neq 0 \) for some non-zero vector \( y \in T_xM \). Let \( y \in T_xM \) be an arbitrary vector and \( \sigma = \sigma(t) \) be the geodesic with \( \sigma(0) = x \) and \( \sigma(0) = y \). Let \( \varphi(t) \) be defined by (17). Let \( I = (a, b) \neq 0 \) be the maximal interval on which \( \varphi(t) > 0 \) and \( 0 \in I \). By assumption \( \mathbf{K} \leq -1 \) and (18), we obtain

\[
\frac{1}{2}[\varphi'^2]' = g_\varphi \left( D_\varphi D_\varphi \mathbf{I}, \mathbf{I} \right) + g_\varphi \left( D_\varphi \mathbf{I}, D_\varphi \mathbf{I} \right) \\
= -g_\varphi \left( \mathbf{R}_\varphi(\mathbf{I}), \mathbf{I} \right) + g_\varphi \left( D_\varphi \mathbf{I}, D_\varphi \mathbf{I} \right) \\
\geq \varphi'^2 + \varphi'^2.
\]

This gives rise to the following inequality

\[
\varphi'' - \varphi \geq 0.
\]

We claim that \( \varphi''(t) \equiv 0 \). Suppose that \( \varphi''(t_0) \neq 0 \) for some \( t_0 \in I \). Let

\[
\varphi_0(t) := \varphi(t_0) \cosh(t - t_0) + \varphi'(t_0) \sinh(t - t_0).
\]

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Let $h(t) := \varphi'(t)/\varphi(t)$ and $h_o(t) := \varphi'_o(t)/\varphi_o(t)$.

$$
\chi(t) := e^{\int_{t_o}^t (h(r)+h_o(r))dr} [h(t) - h_o(t)].
$$

It is easy to verify that $\chi(t) \geq 0$ and $\chi(t_o) = 0$. Thus $\chi(t) \geq 0$ for $t > t_o$ and
$\chi(t) \leq 0$ for $t < t_o$. This implies that $h(t) \geq h_o(t)$ for $t > t_o$ and $h(t) \leq h_o(t)$ for $t < t_o$. Note that

$$
h(t) - h_o(t) = \frac{\varphi'(t)}{\varphi(t)} - \frac{\varphi'_o(t)}{\varphi_o(t)} = \frac{d}{dt} \left[ \ln \frac{\varphi(t)}{\varphi_o(t)} \right].
$$

Thus $[\varphi/\varphi_o]'(t) \geq 0$ for $t > t_o$ and $[\varphi/\varphi_o]'(t) \leq 0$ for $t < t_o$. We conclude that

$$
\varphi(t) \geq \varphi_o(t), \quad a < t < b.
$$

If $\varphi'(t_o) > 0$, then

$$
\varphi(t) \geq \varphi_o(t) > 0, \quad t > t_o.
$$

Thus $b = +\infty$ and

$$
\liminf_{t \to +\infty} \frac{\varphi(t)}{\varphi(t_o)} \geq \frac{\varphi(t_o) + \varphi'(t_o)}{2} > 0.
$$

If $\varphi'(t_o) < 0$, then

$$
\varphi(t) \geq \varphi_o(t) > 0, \quad t < t_o.
$$

Thus $a = -\infty$ and

$$
\liminf_{t \to -\infty} \frac{\varphi(t)}{\varphi(t_o)} \geq \frac{\varphi(t_o) - \varphi'(t_o)}{2} > 0.
$$

Note that $I_p(|t - t_o|) \geq \varphi(t)$ for $p = \sigma(t_o)$. Thus $I$ grows at least exponentially at rate of $k = 1$. This contradicts the assumption. Therefore, $\varphi'(t) \equiv 0$.

Since $\varphi'(t) \equiv 0$, we conclude that $\varphi(t) \equiv 0$ by (20). In particular, $I_y = \varphi(0) = 0$. This contradicts the assumption at the beginning of the argument.

Therefore $I \equiv 0$ and $F$ is Riemannian by Deicke’s theorem [11], Q.E.D.

A natural problem arises: Is there any non-Riemannian complete Finsler metric on $\mathbb{R}^n$ satisfying $K \leq -1$, $S = (n+1)cF$, but $I_p(r) \sim Ce^r$ (as $r \to +\infty$)? This problem remains open.

**Example 4.3** Let $\phi = \phi(y)$ be a Minkowski norm on $\mathbb{R}^n$ and $\mathcal{U} := \{ y \in \mathbb{R}^n | \phi(y) < 1 \}$. Let $\Theta = \Theta(x, y)$ be a function on $T\mathcal{U} \cong \mathcal{U} \times \mathbb{R}^n$ defined by

$$
\Theta(x, y) = \phi \left( y - \Theta(x, y)x \right).
$$

$\Theta$ is a Finsler metric on $\mathcal{U}$ which is called the Funk metric [12]. The Funk metric satisfies the following important equation

$$
\Theta_{\mathcal{U}}(x, y) = \Theta(x, y)\Theta_{\mathcal{U}}(x, y).
$$

(21)
Let $a \in \mathbb{R}^n$ be an arbitrary constant vector $a \in \mathbb{R}^n$ with $|a| < 1$. Let
\[ F := \Theta(x, y) + \frac{\langle a, y \rangle}{1 + \langle a, x \rangle}, \quad y \in T\mathcal{U} \equiv \mathcal{U} \times \mathbb{R}^n. \]
Clearly, $F$ is a Finsler metric near the origin. By (21), one sees that the spray coefficients of $F$ are given by $G^i = Py^i$, where
\[ P := \frac{1}{2} \left( \Theta(x, y) - \frac{\langle a, y \rangle}{1 + \langle a, x \rangle} \right). \]
Then using the above formula for $G^i$ and (9), one can easily show that $F$ has constant flag curvature $K = -\frac{1}{4}$ (see Example 5.3 in [19]). Now let us compute the S-curvature of $F$. A direct computation gives
\[ \frac{\partial G^m}{\partial y^m} = (n + 1)P. \]
Let $dV = \sigma_F(x) dx^1 \cdots dx^n$ be the Finsler volume form on $M$. Using (5), we obtain
\[ S = (n + 1)P(x, y) - y^m \frac{\partial}{\partial x^m} \left( \ln \sigma_F(x) \right) \]
\[ = \frac{n + 1}{2} F(x, y) - (n + 1) \frac{\langle a, y \rangle}{1 + \langle a, x \rangle} - y^m \frac{\partial}{\partial x^m} \left( \ln \sigma_F(x) \right) \]
\[ = \frac{1}{2} (n + 1) F(x, y) + d\varphi(y), \]
where
\[ \varphi := -\ln \left[ (1 + \langle a, x \rangle)^{n+1} \sigma_F(x) \right]. \tag{22} \]
Thus $F$ has almost constant S-curvature. Note that $F$ is not Riemannian in general.

Example 4.3 shows that the completeness in Theorem 4.1 can not be replaced by the positive completeness.

5 An Example

The local/global structures of Berwald metrics have been completely determined by Z.I. Szabo [21], but their curvature properties have not been discussed thoroughly. Here we are going to compute the Riemann curvature and the mean Cartan torsion for a special class of Berwald manifolds constructed from a pair of Riemannian manifolds. Then we show that these metrics satisfy the conditions and the conclusions in Theorem 1.1 (b).

Let $(M_i, \alpha_i), i = 1, 2,$ be arbitrary Riemannian manifolds and $M = M_1 \times M_2$. Let $f : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ be an arbitrary $C^\infty$ function satisfying
\[ f(\lambda s, \lambda t) = \lambda f(s, t), \quad (\lambda > 0) \quad \text{and} \quad f(s, t) \neq 0 \text{ if } (s, t) \neq 0. \]
Define

\[ F := \sqrt{f \left( [\alpha_1(x_1, y_1)]^2, [\alpha_2(x_2, y_2)]^2 \right)}, \tag{23} \]

where \( x = (x_1, x_2) \in M \) and \( y = y_1 \oplus y_2 \in T_{(x_1, x_2)}(M_1 \times M_2) \cong T_{x_1} M_1 \oplus T_{x_2} M_2 \). Clearly, \( F \) has the following properties:

(a) \( F(x, y) \geq 0 \) with equality holds if and only if \( y = 0 \);

(b) \( F(x, \lambda y) = \lambda F(x, y), \ \lambda > 0 \);

(c) \( F(x, y) \) is \( C^\infty \) on \( TM \setminus \{0\} \).

Now we are going to find additional condition on \( f = f(s, t) \) under which the matrix \( g_{ij} := \frac{1}{2} [F^2]_{y^iy^j} \) is positive definite. Take standard local coordinate systems \((x^a, y^a)\) in \( TM_1 \) and \((x^a, y^a)\) in \( TM_2 \). Then \((x^i, y^j) := (x^a, x^a, y^a, y^a)\) is a standard local coordinate system in \( TM \). Express

\[ \alpha_1(x_1, y_1) = \sqrt{\tilde{g}_{ab}(x_1) y^a y^b}, \quad \alpha_2(x_2, y_2) = \sqrt{\tilde{g}_{ab}(x_2) y^a y^b}, \]

where \( y_1 = y^a \frac{\partial}{\partial x^a} \) and \( y_2 = y^a \frac{\partial}{\partial x^a} \). We obtain

\[ \left( g_{ij} \right) = \left( \begin{array}{cc} 2f_s \tilde{g}_{ab} + f_t \tilde{g}_{ab} & 2f_s \tilde{g}_{ab} + f_t \tilde{g}_{ab} \\ 2f_s \tilde{g}_{ab} + f_t \tilde{g}_{ab} & 2f_t \tilde{g}_{ab} + f_s \tilde{g}_{ab} \end{array} \right), \tag{24} \]

where \( \tilde{g}_a := \tilde{g}_{ab} y^b \) and \( \tilde{g}_a := \tilde{g}_{ab} y^b \). By an elementary argument, one can show that \( \left( g_{ij} \right) \) is positive definite if and only if \( f \) satisfies the following conditions:

\[ f_s > 0, \quad f_t > 0, \quad f_s + 2sf_{st} > 0, \quad f_t + 2tf_{st} > 0, \]

and

\[ f_s f_t - 2f_f s t > 0. \]

In this case,

\[ \det \left( g_{ij} \right) = h \left( [\alpha_1]^2, [\alpha_2]^2 \right) \det \left( \tilde{g}_{ab} \right) \det \left( \tilde{g}_{ab} \right), \tag{25} \]

where

\[ h := (f_s)^{n_1 - 1} (f_t)^{n_2 - 1} \left( f_s f_t - 2f_f s t \right), \]

where \( n_1 := \dim M_1 \) and \( n_2 := \dim M_2 \).

By a direct computation, one knows that the spray coefficients of \( F \) are split into the direct sum of the spray coefficients of \( \alpha_1 \) and \( \alpha_2 \), that is,

\[ G^a(x, y) = \tilde{G}^a(x_1, y_1), \quad G^a(x, y) = \tilde{G}^a(x_1, y_1), \tag{26} \]

where \( \tilde{G}^a \) are the spray coefficients of \( \alpha_1 \) and \( \alpha_2 \). From (26), one can see that the spray of \( F \) is independent of the choice of a particular function \( f \). In particular, \( G^a \) is quadratic in \( y \in T_x M \). Thus \( F \) is a Berwald
metric. This fact is claimed in [21]. Since $F$ is a Berwald metric, $J = 0$ and $S = 0$ [17].

The Riemann tensor of $F$ is given by

\[
\left( \tilde{R}^i_j \right) = \left( \tilde{R}^a_j \right) = \begin{pmatrix} \tilde{R}^a_b & 0 \\ 0 & \tilde{R}^a_\beta \end{pmatrix},
\]

where $\tilde{R}^a_b$ and $\tilde{R}^a_\beta$ are the coefficients of the Riemann tensor of $\alpha_1$ and $\alpha_2$ respectively. Let $\tilde{R}_{ij} := g_{ik} \tilde{R}^k_j$, $\tilde{R}_{ab} := \tilde{g}_{ac} \tilde{R}^c_b$ and $\tilde{R}_{\alpha\beta} := \tilde{g}_{\alpha\gamma} \tilde{R}^\gamma_\beta$. Using (24), one obtains

\[
\left( R_{ij} \right) = \begin{pmatrix} f_s \tilde{R}_{ab} & 0 \\ 0 & f_t \tilde{R}_{\alpha\beta} \end{pmatrix}.
\]

For any vector $v = v^a \frac{\partial}{\partial x^a} |_x \in T_x M$,

\[
g_y \left( R_y(v), v \right) = f_s \tilde{R}_{ab} v^a v^b + f_t \tilde{R}_{\alpha\beta} v^\alpha v^\beta. \tag{27}
\]

It follows from (27) that if $\alpha_1$ and $\alpha_2$ both have nonpositive sectional curvature, then $F$ has nonpositive flag curvature.

Using (25), one can compute the mean Cartan torsion. First, observe that

\[
I_i = \frac{\partial}{\partial y^i} \left[ \ln \sqrt{\det(g_{jk})} \right] = \frac{\partial}{\partial y^i} \left[ \ln \sqrt{h([\alpha_1]^2, [\alpha_2]^2)} \right].
\]

One obtains

\[
I_a = \frac{h_s}{h} \tilde{g}_a, \quad I_\alpha = \frac{h_t}{h} \tilde{g}_\alpha,
\]

where $\tilde{g}_a := \tilde{g}_{ab} y^b$ and $\tilde{g}_\alpha := \tilde{g}_{\alpha\beta} y^\beta$. Since $\tilde{g}_a \tilde{R}^a_b = 0$ and $\tilde{g}_\alpha \tilde{R}^\alpha_\beta = 0$, one obtains

\[
g_y \left( R_y(I_y), I_y \right) = I_i \tilde{R}_j I^j = \frac{h_s}{h} \tilde{g}_a \tilde{R}^a_b t^b + \frac{h_t}{h} \tilde{g}_\alpha \tilde{R}^\alpha_\beta t^\beta = 0.
\]

Therefore $F$ satisfies the conditions and conclusions in Theorem 4.1.

References


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