A GAUSS-BONNET-CHERN FORMULA FOR FINSLER MANIFOLDS

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Abstract. Let \( \pi : E \to M \) be an oriented fiber bundle with \( \text{dim } E_x = p \) and \( \mathcal{V}TE \) denote the vertical tangent bundle of \( E \). Given a projection \( p : TE \to \mathcal{V}TE \), a Riemann metric \( h \) on \( \mathcal{V}TE \) and a metric-compatible connection \( D \) on \( \mathcal{V}TE \), we construct a \(( p + 1 )\)-form \( \text{Pf} \) and a \( p \)-form \( \Pi \) such that \( d\Pi = \text{Pf} \) and \( \int_{E_x} \Pi = 1, \forall x \in M \). When \( E = S \) is the tangent sphere bundle of \( M \), we establish a Gauss-Bonnet-Chern formula for any triple \( \{ p, h, D \} \) over \( S \). Since every Finsler metric \( F \) on \( M \) naturally gives a triple \( \{ p, h, D \} \) on \( S \), we establish a Gauss-Bonnet-Chern formula for all Finsler manifolds.

0. Introduction

In 1944, S. S. Chern [Ch3] proves the Gauss-Bonnet theorem for Riemann manifolds. Let \( M \) be an \( n \)-dimensional oriented \( C^\infty \) manifold (\( n = \) even). The tangent sphere bundle \( \pi : S \to M \) consists of all rays \( [v] = \{ tv; t > 0 \} \). For any Riemann metric on \( M \), Chern constructs an \( n \)-form \( \text{Pf} \) on \( M \) from its curvature tensor and an \( (n - 1) \)-form \( \Pi \) on \( S \) such that \( d\Pi = \pi^* \text{Pf} \) and \( \int_S \Pi = 1 \). Then he obtains the following formula: \( \int_M \text{Pf} = \chi(M) \).

We shall call a formula of this type a Gauss-Bonnet-Chern (GBC) formula. Given a Finsler metric \( F \) on \( M \), one would like to establish an analogue of the GBC formula for \( F \). In this case, the problem becomes more difficult, since there is no canonical “metric-compatible” and “torsion-free” linear connection of \( F \) on \( TM \). Nevertheless, \( F \) naturally induces a Riemann metric \( g \) on \( \pi^* TM \). Here \( \pi^* TM \) denotes the pull-back tangent bundle over \( S \). There is a canonical section \( \ell \) of \( \pi^* TM \) given by \( \ell[v] = 1/F(v, v) \). Following [Ch3], one can construct an \( n \)-form \( \text{Pf} \) and an \( (n - 1) \)-form \( \Pi \) on \( S \) by

\[
\text{Pf} := (-1)^{\frac{n}{2} - 1} \frac{2}{n! \text{vol}(S^n)} \sum \epsilon^{i_1 \cdots i_n} \Omega_{i_1}^{i_2} \cdots \Omega_{i_{n-1}}^{i_n},
\]

and

\[
\Pi := \sum_{k=0}^{n} (-1)^k c_k \sum \epsilon^{i_1 \cdots i_n} \Omega^{i_2}_{i_1} \cdots \Omega^{i_{2k}}_{i_{2k-1}} \theta^{i_{2k+1}} \cdots \theta^{i_{n-1}} \ell^{i_n}.
\]

1991 Mathematics Subject Classification. Primary 53C60, Secondary 53C20, 53B40.
where $c_k$ are determined by $c_k = \frac{(n-2k-1)}{2k} c_{k-1}$, $c_0 = \frac{1}{(n-1)\text{Vol}(S^{n-1})}$. When $F$ is Riemannian, according to [Ch3], one has

\begin{align}
(0.3) \quad & d\Pi = Pf, \\
(0.4) \quad & \int_{S_x} \Pi = 1, \quad \forall x \in M.
\end{align}

Then one can prove the Gauss-Bonnet theorem by the Hopf theorem. However, (0.3)(0.4) does not hold simultaneously for a general Finsler metric $F$, no matter which connection we take.

Take the Cartan connection on $\pi^*TM$, which is metric-compatible with $g$. Lichnerowicz [L] first verifies that $d\Pi = Pf$. He also notices that $\int_{S_x} \Pi \neq 1$. Therefore he restricts himself to the class of Finsler metrics with $(S_x, h_x) = \mathbb{S}^{n-1}, \forall x \in M$ (hence $\int_{S_x} \Pi = 1$), where $h_x$ denotes the induced Riemann metric on $S_x$. However, this is a very strong restriction. According to a theorem of Brickell [Br], Finsler metrics with $(S_x, h_x) = \mathbb{S}^{n-1}, \forall x \in M$ must be Riemannian, provided that $n \geq 3$ and $F(-\nu) = F(\nu)$.

It is Bao and Chern [BC2] who first make the following non-trivial observation. Bao and Chern show that the $(n-1)$-form $\Pi$ of the Chern connection (or any torsion-free connection) has very nice properties. First, $d\Pi = Pf + \mathfrak{F}$. The additional term $\mathfrak{F}$ occurs, because a torsion-free connection is not metric-compatible. Second, the the restriction of $\Pi$ to $S_x$ is a multiple of the volume form of $(S_x, h_x)$. This observation leads to a GBC formula for Finsler manifolds with $\text{vol}(S_x, h_x) = \text{constant}$.

After [BC2], the author [S] also establishes several GBC formulas for certain class of Finsler manifolds, by analysing the geometric data of $\Pi$ on $S_x$ for the $(n-1)$-form $\Pi$ of the Cartan connection. Other attempts (in lower dimensions) can be found in [B] [R] [Ch4] [M][BSC2], etc.

Our goal in this paper is to establish a GBC formula for all Finsler manifolds. Recall that a Finsler metric $F$ on $M$ naturally induces a Riemannian metric $g$ on $\pi^*TM$. Let $\nabla$ denote the Cartan connection on $\pi^*TM$. The Cartan connection has the following property. $\nabla: TS \to \pi^*TM$ is a bundle map of rank $n-1$ such that $\nabla: VTS \to \ell^\perp$ is an isomorphism. Let $\{e_i\}_{i=1}^n$ be an orthonormal frame for $\pi^*TM$ with $e_n = \ell$. Put $\nabla e_i = \theta^\alpha \otimes e_\alpha$ and $\nabla e_j = \theta^j_\alpha \otimes e_i$ (hence $\theta^\alpha = \theta_n^\alpha$). Let $\{f_\alpha\}_{\alpha=1}^{n-1}$ be the basis for $VTS$ determined by $\theta^\alpha(f_\beta) = \delta^\alpha_\beta$. Then we get a triple $\{p, h, D\}$ over $S$ by

\begin{equation}
(0.5) \quad p = \theta^\alpha \otimes f_\alpha, \quad h = \theta^\alpha \otimes \theta^\alpha|_{VTS}, \quad D f_\beta = \theta_\beta^\alpha \otimes f_\alpha.
\end{equation}

Here $p: TS \to VTS$ is a projection, $h$ is a Riemann metric on $VTS$ and $D$ is a metric-compatible connection on $(VTS, h)$.

The curvature form $(\Omega^i_j)$ of $\nabla$ is defined by by

\begin{equation}
(0.6) \quad \Omega^i_j := d\theta^i_j - \theta^i_j \wedge \theta_k^i.
\end{equation}

The torsion form $(\Theta^\alpha)$ and the curvature form $(\Theta_\beta^\alpha)$ of $D$ are defined by

\begin{align}
(0.7) \quad & \Theta^\alpha := d\theta^\alpha - \theta^2 \wedge \theta_\beta^\alpha, \\
(0.8) \quad & \Theta_\beta^\alpha := d\theta_\beta^\alpha - \theta_\beta^\tau \wedge \theta_\tau^\alpha.
\end{align}
\{\Omega^i_j\} \text{ and } \{\Theta^\alpha, \Theta^\beta_\alpha\} \text{ are related by}

\begin{align*}
\Theta^\alpha &= \Omega^n_\alpha, \\
\Theta^\alpha_\beta &= \Omega^\alpha_\beta - \theta^\beta \wedge \theta^\alpha. 
\end{align*}

In order to establish a GBC formula for a Finsler metric, it suffices to establish a GBC formula for an arbitrary triple \{p, h, D\} over \mathcal{S}.

Let \(M\) be a closed oriented manifold of dimension \(n = p + 1\). Let \{p, h, D\} be an arbitrary triple over \(\mathcal{S}\); namely, \(p : TS \to VTS\) be a projection map, \(h\) is a Riemann metric on \(VTS\) and \(D\) is a metric-compatible connection on \((VTS, h)\). Let \(\{f_\alpha\}\) be a positive orthonormal frame for \(VTS\). Let \((\theta^\alpha), (\theta^\beta_\alpha), (\Theta^\alpha)\) and \((\Theta^\beta_\alpha)\) be given by (0.5) and (0.8), respectively. The vertical parts \(Q^\alpha\) and \(Q^\beta_\alpha\) of \(\Theta^\alpha\) and \(\Theta^\beta_\alpha\) are determined by

\begin{align*}
Q^\alpha &\equiv \Theta^\alpha, \quad Q^\beta_\alpha \equiv \Theta^\beta_\alpha, \quad \text{over } VTS.
\end{align*}

Put

\begin{align*}
\Omega^\beta_\alpha &= dQ^\beta_\alpha + Q^\mu_\beta \wedge \theta^\alpha \wedge \theta^\mu \wedge Q^\alpha_\mu.
\end{align*}

The following forms are well-defined on \(\mathcal{S}\).

\begin{align*}
\Phi_k &= \sum \epsilon^{\alpha_1 \cdots \alpha_p} \Theta_{\alpha_1}^{\alpha_2} \cdots \Theta_{\alpha_{2k-1}}^{\alpha_{2k}} \theta^{\alpha_{2k+1}} \cdots \theta^{\alpha_p}, \\
\Psi_k &= \sum \epsilon^{\alpha_1 \cdots \alpha_p} \Theta_{\alpha_1}^{\alpha_2} \cdots \Theta_{\alpha_{2k-1}}^{\alpha_{2k}} \Theta_{\alpha_{2k+1}}^{\alpha_{2k+2}} \cdots \theta^{\alpha_p}, \\
\Phi_k^{(v)} &= \sum \epsilon^{\alpha_1 \cdots \alpha_p} Q_{\alpha_1}^{\alpha_2} \cdots Q_{\alpha_{2k-1}}^{\alpha_{2k}} \theta^{\alpha_{2k+1}} \cdots \theta^{\alpha_p}, \\
\Psi_k^{(v)} &= \sum \epsilon^{\alpha_1 \cdots \alpha_p} Q_{\alpha_1}^{\alpha_2} \cdots Q_{\alpha_{2k-1}}^{\alpha_{2k}} \Theta_{\alpha_{2k+1}}^{\alpha_{2k+2}} \cdots \theta^{\alpha_p}, \\
F_k^{(v)} &= \sum \epsilon^{\alpha_1 \cdots \alpha_p} \Theta_{\alpha_1}^{\alpha_2} \cdots Q_{\alpha_{2k-1}}^{\alpha_{2k}} \theta^{\alpha_{2k+1}} \cdots \theta^{\alpha_p}.
\end{align*}

Here we put \(F_0^{(v)} = 0\) and \(\Psi_k = \Psi_k^{(v)} = 0\) if \(2k = p\).

Let \(h_x = h_{\mathcal{S}_x}\). The function \(V(x) := \text{vol}(\mathcal{S}_x, h_x)\) is called the volume function on \(M\). In general, \(V(x) \neq \text{constant}\).

For arbitrary constants \(c_k\), define

\begin{align*}
Pf &= \frac{1}{p! V(x)} \left\{ \sum_{k=0}^{[\frac{p}{2}]} c_k [(p - 2k)(\Psi_k - \Psi_k^{(v)}) - kF_k^{(v)}] \\
&\quad - \pi^* d(\log V) \wedge (\Phi_k - \Phi_k^{(v)}) + [p \Psi_0 - \pi^* d(\log V) \wedge \Phi_0] \right\}.
\end{align*}

The following is the main theorem.
**Theorem 0.1.** Let $M$ be an oriented closed manifold of dimension $n = p + 1$. Given a projection $p : TS \to VTS$, a Riemann metric $h$ on $VTS$, and a metric-compatible connection on $VTS$. For any vector field $X$ on $M$ with isolated zeros, the $n$-form $Pf$ in (0.18) satisfies

\[
\int_M [X]^* Pf = \chi(M),
\]

where $[X] : M \setminus \{\text{zeros}\} \to S$ denotes the section defined by $X$.

The proof of Theorem 0.1 will be given in §1-§3. Note that when $V(x) = \text{constant}$, (0.18) reduces to

\[
Pf = \frac{1}{p! V(x)} \sum_{k=0}^{\left[ \frac{p}{2} \right]} c_k ((p - 2k)(\Psi_k - \Psi_k^{(v)}) - kF_k^{(v)} + p\Psi_0).
\]

Applying (0.19) to the special case when the triple $\{p, h, D\}$ over $S$ is given by a Finsler metric $F$, one obtains a GBC formula for $F$ (Theorem 4.2). In §5, we shall derive the Gauss-Bonnet-Chern formula for Riemann manifolds from Theorem 4.2, by choosing a suitable set of constants $\{c_k\}$ in (0.18).

**Acknowledgements.** The author would like to thank David Bao and S.S.Chern for many valuable discussions.

1. **Exact forms on a fiber bundle**

In §0, we briefly describe how to get the triple $\{p, h, D\}$ over $S$ from a Finsler metric $F$ on a smooth manifold $M$ of dimension $n = p + 1$. Then we define an $n$-forms $Pf$ (0.18) and state Theorem 0.1. In this section we shall study a general triple $\{p, h, D\}$ over an arbitrary fiber bundle $\pi : E \to M$ with $\dim E_x = p$, where $p : TE \to VTE$ is a projection, $g$ is a Riemann metric on $VTE$ and $D$ is a metric-compatible connection on $(VTE, h)$.

Let $\{f_\alpha\}_{\alpha=1}^p$ be a positive orthonormal frame for $(VTE, h)$. Let $\{\theta^\alpha\}$ and $\{\theta_\beta^\alpha\}$ be given by

\[p = \theta^\alpha \otimes f_\alpha, \quad Df_\beta = \theta_\beta^\alpha \otimes f_\alpha.\]

Let $(\Theta^\alpha), (\Theta_\beta^\alpha), \Phi_k$ and $\Psi_k$ be given by (0.7), (0.8), (0.13) and (0.14), respectively. We have

**Lemma 1.1.**

\[
d\Phi_k = (p - 2k)\Psi_k, \quad 0 \leq k \leq \left[ \frac{p}{2} \right]
\]

Here we put $\Psi_k = 0$ when $2k = p$.

**Proof.** We have the following Bianchi identities

\[
d\Theta^\alpha = -\Theta^\beta \wedge \theta_\beta^\alpha + \theta^\beta \wedge \Theta_\beta^\alpha
\]

\[
d\Theta_\beta^\alpha = -\Theta_\beta^\mu \wedge \theta_\mu^\alpha + \theta_\beta^\mu \wedge \Theta_\mu^\alpha.
\]
We first prove (1.1) for \( k = 0 \).

\[
d\Phi_0 = \sum \varepsilon_{\alpha_1 \cdots \alpha_p} \sum_{i=1}^{p} (-1)^{i-1} \theta^{\alpha_1} \cdots (\sum_{\beta} \theta^{\beta}) \wedge \theta^{\alpha_i} \cdots \theta^{\alpha_p} + \sum \varepsilon_{\alpha_1 \cdots \alpha_p} \sum_{i=1}^{p} (-1)^{i-1} \theta^{\alpha_1} \cdots \Theta^{\alpha_i} \cdots \theta^{\alpha_p}
\]

\[
= \sum \varepsilon_{\alpha_1 \cdots \alpha_p} \theta^{\alpha_1} \cdots (\sum_{i=1}^{p} \Theta^{\alpha_i}) \cdots \theta^{\alpha_p}
\]

\[
= p\Psi_0.
\]

Then we deal with the general case, \( 1 \leq k \leq \left\lceil \frac{p}{2} \right\rceil \). Observe that

\[
d\Phi_k = -2k \sum \varepsilon_{\alpha_1 \cdots \alpha_p} \left( \sum_{t=1}^{2k} \Theta^{\alpha_t} \wedge \Theta^{\alpha_{t+2}} \right) \Theta^{\alpha_4} \cdots \Theta^{\alpha_{2k-1}} \theta^{2k+1} \cdots \theta^{\alpha_p}
\]

\[
+ (p - 2k) \sum \varepsilon_{\alpha_1 \cdots \alpha_p} \Theta^{\alpha_1} \cdots \Theta^{\alpha_{2k}} \left( \sum_{t=1}^{p} \Theta^{\beta} \wedge \theta^{\alpha_{t+2k+1}} \right) \theta^{\alpha_{2k + 2}} \cdots \theta^{\alpha_p}
\]

\[
= -2k \sum \varepsilon_{\alpha_1 \cdots \alpha_p} \left( \sum_{t=1}^{2k} \Theta^{\alpha_t} \wedge \Theta^{\alpha_{t+2}} \right) \Theta^{\alpha_4} \cdots \Theta^{\alpha_{2k-1}} \theta^{2k+1} \cdots \theta^{\alpha_p}
\]

\[
- 2k \sum \varepsilon_{\alpha_1 \cdots \alpha_p} \left( \sum_{t=1}^{p} \Theta^{\alpha_t} \wedge \Theta^{\alpha_{t+2}} \right) \Theta^{\alpha_4} \cdots \Theta^{\alpha_{2k-1}} \theta^{2k+1} \cdots \theta^{\alpha_p}
\]

\[
+ (p - 2k) \sum \varepsilon_{\alpha_1 \cdots \alpha_p} \Theta^{\alpha_1} \cdots \Theta^{\alpha_{2k}} \left( \sum_{t=1}^{p} \Theta^{\beta} \wedge \theta^{\alpha_{t+2k+1}} \right) \theta^{\alpha_{2k + 2}} \cdots \theta^{\alpha_p}
\]

\[
+ (p - 2k) \Psi_k
\]

\[
= -2k A - 2k B + (p - 2k) C + (p - 2k) \Psi_k.
\]

We assert that \( A = 0 \).

\[
A = \sum \varepsilon_{\alpha_1 \cdots \alpha_p} \left( \sum_{t=1}^{2k} \Theta^{\alpha_t} \wedge \Theta^{\alpha_{t+2}} \right) \Theta^{\alpha_4} \cdots \Theta^{\alpha_{2k-1}} \theta^{2k+1} \cdots \theta^{\alpha_p}
\]

\[
= 2k \sum \varepsilon_{\alpha_1 \cdots \alpha_4 \cdots \alpha_p} \Theta^{\alpha_3} \Theta^{\alpha_3} \Theta^{\alpha_4} \cdots \Theta^{\alpha_{2k-1}} \theta^{2k+1} \cdots \theta^{\alpha_p}
\]

\[
= 2k \sum \varepsilon_{\alpha_4 \cdots \alpha_1 \cdots \alpha_p} \Theta^{\alpha_3} \Theta^{\alpha_3} \Theta^{\alpha_1} \cdots \Theta^{\alpha_{2k-1}} \theta^{2k+1} \cdots \theta^{\alpha_p}
\]

\[
= -2k \sum \varepsilon_{\alpha_1 \cdots \alpha_4 \cdots \alpha_p} \Theta^{\alpha_3} \Theta^{\alpha_3} \Theta^{\alpha_1} \cdots \Theta^{\alpha_{2k-1}} \theta^{2k+1} \cdots \theta^{\alpha_p}
\]

\[
= -A.
\]

Thus \( A = 0 \). It is not difficult to verify that

\[
B = (p - 2k) \sum \varepsilon_{\alpha_1 \cdots \alpha_p} \Theta^{\alpha_1} \cdots \Theta^{\alpha_{2k}} \theta^{2k} \theta^{\alpha_{2k+1}} \theta^{\alpha_{2k+2}} \cdots \theta^{\alpha_p}.
\]

\[
C = 2k \sum \varepsilon_{\alpha_1 \cdots \alpha_p} \Theta^{\alpha_1} \cdots \Theta^{\alpha_{2k}} \theta^{2k} \theta^{\alpha_{2k+1}} \theta^{\alpha_{2k+2}} \cdots \theta^{\alpha_p}.
\]
Remark: If \( p = 2m \), then
\[
\Phi_m := \sum \varepsilon^{\alpha_1 \cdots \alpha_p} \Theta_{\alpha_1}^{\alpha_2} \cdots \Theta_{\alpha_p}^{\alpha_{p-1}}
\]
is a closed form representing the Euler class of \( VE \).

Let \( \{ c_k \} \) be an arbitrary set of constants. Define
\[
\tilde{\Pi} := \sum_{k=0}^{\lfloor \frac{p}{2} \rfloor} c_k \Phi_k,
\]
(1.4)
\[
\tilde{\Psi} := \sum_{i=0}^{\lfloor \frac{p}{2} \rfloor} (p - 2i) c_k \Psi_k.
\]
(1.5)

It follows from Lemma 1.1 that \( d \tilde{\Pi} = \tilde{\Psi} \). However, \( \int_{E_x} \tilde{\Pi} \neq \text{constant} \). In \( \S 2 \), we shall modify \( \tilde{\Pi} \) as well as \( \tilde{\Psi} \) to get the desired \( \Pi \) and \( \Psi \) satisfying (0.3)(0.4).

2. The construction of \( \Psi \) and \( \Pi \) on a fiber bundle

Let \( \pi : E \rightarrow M \) be an oriented fiber bundle with \( \dim E_x = p \). Given a triple \( \{ p, h, D \} \) over \( E \). In this section we shall construct a \( (p+1) \)-form \( \Psi \) and a \( p \)-form \( \Pi \), satisfying (0.3)(0.4) on \( E \).

Let \( \{ f_\alpha \} \) be a positive orthonormal basis for \( \langle \mathcal{V}TE, h \rangle \). Put
\[
p := \theta^\alpha \otimes f_\alpha, \quad Df_\beta = \Theta_\beta^\alpha \otimes f_\alpha.
\]

Let \( \{ \omega^i \}_{i=1}^q \) be a positive co-frame for \( \pi^* T^* E \subset T^* E \). Then \( T^* E \) has the following direct decomposition
\[
T^* E = \text{span}\{ \omega^i \} \oplus \text{span}\{ \theta^\alpha \}.
\]

Let \( (\Theta^\alpha) \) and \( (\Theta_\beta^\alpha) \) be given by (0.7)(0.8). \( (\Theta^\alpha) \) and \( (\Theta_\beta^\alpha) \) can be expressed as follows
\[
\Theta^\alpha = \frac{1}{2} R^\alpha_{ij} \omega^i \wedge \omega^j + P^\alpha_{i\mu} \omega^i \wedge \theta^\mu + \frac{1}{2} Q^\alpha_{\lambda\mu} \theta^\lambda \wedge \theta^\mu
\]
(2.1)
\[
\Theta_\beta^\alpha = \frac{1}{2} R^\alpha_{i\beta} \omega^i \wedge \omega^j + P^\alpha_{i\mu} \omega^i \wedge \theta^\mu + \frac{1}{2} Q^\alpha_{\lambda\mu} \lambda^\mu \wedge \theta^\mu
\]
(2.2)

Let \( (Q^\alpha) \) and \( (Q_\beta^\alpha) \) be given by (0.11). we have
\[
Q^\alpha = \frac{1}{2} Q^\alpha_{\lambda\mu} \theta^\lambda \wedge \theta^\mu
\]
(2.3)
\[
Q_\beta^\alpha = \frac{1}{2} Q_\beta^\alpha \lambda^\mu \theta^\lambda \wedge \theta^\mu.
\]
(2.4)

Let \( h_x = (i_x)^* h \), where \( i_x : E_x \rightarrow E \) denotes the natural embedding. The volume function \( V \) is defined by \( V(x) = \text{vol}(E_x, h_x) \). Let \( \{ \dot{f}_\alpha \} \) be the local frame for \( TE_x \) such that \( (i_x)_*(\dot{f}_\alpha) = f_\alpha \). The induced linear connection \( \dot{D} \) on \( TE_x \) is given by
\[
\dot{D} \dot{f}_\beta = (i_x)^* \theta_\beta^\alpha \otimes \dot{f}_\alpha.
\]
(2.5)
Since \((i_x)^*\omega^i = 0\),

(2.6) \((i_x)^*\Theta^\alpha = (i_x)^*Q^\alpha\) is the torsion form of \(\hat{D}\), and

(2.7) \((i_x)^*\Theta^\beta_* = (i_x)^*Q^\alpha_\beta\) is the curvature form of \(\hat{D}\).

From (2.6)(2.7), one can see that if \(Q^\alpha = 0\), then \((i_x)^*Q^\alpha_\beta\) is the Riemann curvature form of \(h_x\) w.r.t. \(\lbrace f_\alpha \rbrace\).

Let \(Q^\alpha_k, \Phi_k, \Psi_k, \Phi^{(v)}_k, \Psi^{(v)}_k\) and \(F^{(v)}_k\) be given by (0.12)-(0.17). By the same argument as for Lemma 1.1, we get the following

**Lemma 2.1.** For \(0 \leq k \leq \left\lfloor \frac{p}{2} \right\rfloor\), the following hold

\[
(2.8) \quad d\Phi^{(v)}_k = (p - 2k)\Psi^{(v)}_k + kF^{(v)}_k.
\]

Let \(c_k\) be arbitrary constants. Define \(P_f\) as in (0.18) and \(\Pi\) by

(2.9) \[
\Pi = \frac{1}{p!V(x)} \left\{ \sum_{k=0}^{\left\lfloor \frac{p}{2} \right\rfloor} c_k (\Phi_k - \Phi_k^{(v)}) + \Phi_0 \right\}
\]

**Proposition 2.2.** Let \(P_f\) and \(\Pi\) be constructed by (0.18) and (2.9). Then

\[
(2.10) \quad d\Pi = P_f
\]

and \(\Pi\) satisfies

(2.11) \[
\int_{E_x} (i_x)^*\Pi = 1, \quad \forall x \in M.
\]

**Proof.** Define \(dV\) on \(E\) by

\[
dV = \theta^1 \cdots \theta^p.
\]

Clearly, \(dV_x := (i_x)^*dV\) is the Riemann volume form of \((E_x, h_x)\). By Lemmas 1.1 and 2.1, one can easily verify (2.10). It follows from (2.6)(2.7) that

\[
(2.12) \quad (i_x)^*\Phi_k = (i_x)^*\Phi_k^{(v)}, \quad (i_x)^*\Phi_0 = p!(i_x)^*dV.
\]

Thus

\[
\int_{E_x} (i_x)^*\Pi = \frac{1}{V(x)} \int_{E_x} (i_x)^*dV = 1.
\]

\[\square\]

A natural question arises: Under what curvature condition \(V(x) = \text{constant}\)? We the following

**Proposition 2.5.** If \(P^\alpha_{k\beta} + P^\beta_{k\alpha} = 0\), then all fibers \((E_x, h_x)\) are isometric to each other. If \(P^\alpha_{k\alpha} = 0\), then \(V(x) = \text{constant}\).

Since the proof is quite simple (compare [S]), so is omitted here. Let

(2.13) \[
\mathcal{P} := P^\alpha_{k\alpha}\omega^k.
\]

We have, in general

(2.14) \[
d(\log V)(u) = \frac{1}{V(x)} \int_{E_x} (i_x)^*\mathcal{P}(X_u)dV, \quad \forall u \in TM
\]

where \(X_u\) denote the horizontal lift of \(u\). Thus if \(P^\alpha_{k\alpha} = 0\), then \(V(x) = \text{constant}\) (compare [BS]).
3. Proof of Theorem 0.1

Let \( M \) be as in Theorem 0.1. Let \( X \) be an arbitrary vector field with isolated singularities \( \{ x_i \}^q_{i=1} \). It follows from (2.10) that

\[
\int_{M \setminus (\cup_{i=1}^q B_r(x_i))} [X]^* Pf = \int_{M \setminus (\cup_{i=1}^q B_r(x_i))} d[X]^* \Pi = \sum_{i=1}^q \int_{\partial B_r(x_i)} [X]^* \Pi.
\]

Here \( B_r(x) := \varphi^{-1}(B^n_r) \) for some coordinate system \( \varphi : U \to \mathbb{R}^n \) with \( \varphi(x) = 0 \). Using (2.11), we can easily get

\[
\lim_{\epsilon \to 0^+} \int_{\partial B_r(x_i)} [X]^* \Pi = \text{ind}_{x_i}(X),
\]

where \( \text{ind}_{x_i}(X) \) denotes the index of \( X \) at \( x_i \). Thus

\[
\int_{M} [X]^* Pf = \sum_{i=1}^q \text{ind}_{x_i}(X).
\]

Theorem 0.1 follows from the Hopf theorem that the Euler number

\[
\chi(M) = \sum_{i=1}^q \text{ind}_{x_i}(X).
\]

4. Finsler manifolds

In this section, we shall apply Theorem 0.1 to Finsler manifolds.

Let \( M \) be a \( n \)-dimensional manifold \( (n = p + 1) \). Let \( \pi : S \to M \) denote the tangent sphere bundle of \( M \) and \( \pi^* TM \) denote the pull-back tangent bundle over \( S \). The vectors in \( \pi^* TM \) are denoted by \((v, w)\), where \([v] \in S_x, w \in T_x M\). There is a canonical bundle map \( \rho : TS \to \pi^* TM \) defined by

\[
\rho(\tilde{X}) = ([v], \pi_*(\tilde{X})), \quad \forall \tilde{X} \in T[v]S.
\]

Let \( (x^i) \) be a local coordinate system in \( M \) and \((x^i, y^j)\) the standard coordinate system in \( TM \). Denote by \( \partial_i|_v = ([v], \frac{\partial}{\partial x^i}|_x) \) the natural local basis for \( \pi^* TM \) at \([v] \in S_x\). Let \( F \) be a Finsler metric on \( M \) and write \( F(x, y) = F(y^i \frac{\partial}{\partial x^i}|_x) \). The induced Riemann metric \( g : \pi^* TM \otimes \pi^* TM \to \mathbb{R} \) and the Cartan tensor \( A : \pi^* TM \otimes \pi^* TM \otimes \pi^* TM \to \mathbb{R} \) are defined by

\[
g(\partial_i, \partial_i|_v) = \frac{1}{2} \frac{\partial^2 [F^2]}{\partial y^i \partial y^j} (x, y)
\]

\[
A(\partial_i, \partial_j, \partial_k|_v) = \frac{1}{4} F \frac{\partial^3 [F^2]}{\partial y^i \partial y^j \partial y^k} (x, y).
\]

Here \( v = y^i \frac{\partial}{\partial x^i}|_x \). The canonical section \( \ell \) of \( \pi^* TM \) is defined by

\[
\ell := ([v], \frac{v}{F(v)}).
\]
Let \( \{b_i\}_{i=1}^n \) be an arbitrary local frame for \( \pi^*TM \). Write \( g_{ij} = h(b_i, b_j), A_{ijk} = A(b_i, b_j, b_k) \), and \( \ell = \ell^i b_i \). We have

\[
(4.1) \quad g_{ij} \ell^i \ell^j = 1, \quad A_{ijk} \ell^i = 0.
\]

Let \( \{\omega^i\}_{i=1}^n \) be defined by \( \rho = \omega^i \otimes b_i \).

S. S. Chern proves the following theorem ([Ch1][Ch2][BC1]).

**Theorem 4.1 (Chern).** There is a unique set of local 1-forms \( \{\omega^i_j\} \) on \( S \) satisfying

\[
(4.2) \quad d\omega^i = \omega^j \wedge \omega^i_j,
\]

\[
(4.3) \quad dh_{ij} = h_{kj}\omega^i_k + h_{ik}\omega^j_k + 2A_{ijk}\theta^k,
\]

where

\[
(4.4) \quad \theta^i := d\ell^i + \ell^j \omega^i_j.
\]

Define a set of 1-forms \( \{\theta^i_j\} \) by

\[
\theta^i_j := \omega^i_j + A^i_{jk}\theta^k,
\]

where \( A^i_{jk} = g^{il}A_{jkl} \). It is easy to verify that

\[
(4.5) \quad d\omega^i = \omega^j \wedge \theta^i_j - A^i_{jk}\omega^j \wedge \theta^k,
\]

\[
(4.6) \quad dh_{ij} = h_{kj}\theta^i_k + h_{ik}\theta^j_k.
\]

By (4.1) we also have

\[
(4.7) \quad \theta^i = d\ell^i + \ell^j \theta^i_j.
\]

The Cartan connection \( \nabla \) and the Chern connection \( \nabla' \) on \( \pi^*TM \) are given by

\[
\nabla b_j = \theta^i_j \otimes b_i, \quad \nabla' b_j = \omega^i_j \otimes b_i.
\]

(4.2) means that \( \nabla' \) is torsion-free and (4.6) means that \( \nabla \) is metric-compatible.

Define a bundle map \( \mu : TS \to \pi^*TM \) by

\[
\mu := \nabla \ell = \theta^i \otimes b_i.
\]

It is easy to check that \( \text{rank} \mu = n - 1 \) and

\[
\mu|_{VTS} : VTS \to \ell^\perp
\]

is a bundle isomorphism.
From now on, we always let $\{e_i\}_{i=1}^n$ be an orthonormal frame for $\pi^*T\mathcal{M}$ such that $e_n = \ell$. Let $\{f_\alpha\}_{\alpha=1}^n$ be the orthonormal basis for $\mathcal{V}T\mathcal{S}$ such that $\mu(f_\alpha) = e_\alpha$.

Put $\rho = \omega^i \otimes e_i$ and $\mu = \theta^\alpha \otimes e_\alpha$. We have a direct decomposition for $T^*\mathcal{S}$

\[
T^*\mathcal{S} = \text{span}\{\omega^i\} \oplus \text{span}\{\theta^\alpha\}.
\]

Put $\nabla e_j = \theta^i_j \otimes e_i$. Hence $\theta^\alpha = \theta^\alpha_n$. Then we get a triple $\{p, h, D\}$ given by (0.5), that is,

\[
p = \theta^\alpha \otimes f_\alpha, \quad h = \theta^\alpha \otimes \theta^\alpha|\mathcal{V}T\mathcal{S}, \quad Df_\beta = \theta^\alpha_\beta \otimes f_\alpha.
\]

The curvature form $(\Omega^i_j)$ of $\nabla$ is given by (0.6). It can be expressed by

\[
\Omega^i_j = \frac{1}{2} \tilde{R}^i_{jk} \omega^k \land \omega^l + \tilde{P}^i_{jk} \omega^j \land \theta^\alpha + \frac{1}{2} \tilde{Q}^i_{jk} \theta^\alpha \land \theta^\beta.
\]

Let $(\Theta^\alpha)$ and $(\Theta^\alpha_\beta)$ be given by (0.7)(0.8). From the definition, it is easy to show that

\[
\Theta^\alpha = \Omega^\alpha_n, \quad \Theta^\alpha_\beta = \Omega^\alpha_\beta - \theta^\beta \land \theta^\alpha.
\]

\[
Q^\alpha = \tilde{Q}^\alpha_n, \quad Q^\alpha_\beta = \tilde{Q}^\alpha_\beta - \theta^\beta \land \theta^\alpha.
\]

Let $(\Theta^\alpha)$ and $(\Theta^\alpha_\beta)$ be expressed by (2.1)(2.2). It follows from (4.10)-(4.13) that

\[
R^\alpha_{ij} = \tilde{R}^\alpha_{ij}, \quad P^\alpha_{i\mu} = \tilde{P}^\alpha_{i\mu}, \quad Q^\alpha_{\lambda\mu} = \tilde{Q}^\alpha_{\lambda\mu}.
\]

\[
R^\alpha_{i\beta} = \tilde{R}^\alpha_{i\beta}, \quad P^\alpha_{i\mu} = \tilde{P}^\alpha_{i\mu}, \quad Q^\alpha_{\beta\lambda\mu} = \tilde{Q}^\alpha_{\beta\lambda\mu} + \delta^\alpha_\lambda \delta^\beta_\mu - \delta^\alpha_\mu \delta^\beta_\lambda.
\]

By the well-known fact (see e.g. [BSc1]), we have

\[
\tilde{Q}^i_{j\alpha\beta} = A^i_{j\alpha A^i_{j\beta} - A^i_{j\beta} A^i_{j\alpha}}.
\]

\[
Q^\alpha_{\lambda\mu} = \tilde{Q}^\alpha_{\lambda\mu} = 0.
\]

Remark. It follows from (4.17) that $Q^\alpha = 0$. By (2.5)-(2.7), one can see that the induced $\tilde{D}$ on $\mathcal{S}_x$ is the Christoffel (Levi-Civita) connection of $h_x$. Further, $(i_x)^*Q^\alpha_\beta$ is the Riemann curvature of $h_x$.

Define a form $\text{Pf}$ as in (0.18) in terms of $\Phi_k, \Psi_k, \Phi_k^{(v)}, \Psi_k^{(v)}, F_k^{(v)}$, which are related to $\theta^i$ and $\Omega^i_j$ by (0.11)-(0.17) and (4.10)-(4.13). The following is just a corollary of Theorem 0.1.

**Theorem 4.2.** Let $(M, F)$ be an oriented closed Finsler manifold of dimension $n = p + 1$. For any vector field $X$ with isolated zeros on $M$,

\[
\int_M [X]^*\text{Pf} = \chi(M).
\]
5. RIEMANNIAN MANIFOLDS

In this section we shall briefly derive the Gauss-Bonnet-Chern formula for Riemann manifolds from Theorem 4.2, by choosing a suitable set of constants \( \{c_k\} \) in (0.18).

Let \((M, \bar{g})\) be an oriented Riemannian manifold of dimension \( n = p + 1 \). Thus \( F(v) := \sqrt{\bar{g}(v,v)} \) is a special Finsler metric. We shall continue to use the notations in §4. Since \( A = 0 \), it follows from (4.15) (4.16) that

\[
Q_{\beta}^\alpha = -\theta^\beta \wedge \theta^\alpha.
\]

By (2.7), we know that \((S_x, h_x)\) has constant curvature = 1 (if \( n \geq 3 \)). Thus all \((S_x, h_x)\) are naturally isometric to the standard unit sphere \( S^{n-1} \) in \( \mathbb{R}^n \), in particular, \( V(x) = \text{vol}(S^{n-1}) \). It follows from (5.1) that

\[
O_{\beta}^\alpha = -\Theta^\beta \wedge \theta^\alpha + \theta^\beta \wedge \Theta^\alpha.
\]

Substituting (5.1) and (5.2) into \( \Psi_k^{(v)} \) and \( F_k^{(v)} \) yields

\[
\Psi_k^{(v)} = (-1)^k \Psi_0, \quad F_k^{(v)} = (-1)^k 2 \Psi_0.
\]

Suppose that constants \( c_k \) satisfy

\[
\sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^i c_i = 1.
\]

For such set of \( c_k \), define Pf as in (0.18). It follows from (5.3)(5.4) that

\[
Pf = \frac{1}{(n-1)! \text{vol}(S^{n-1})} \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (n - 1 - 2i)c_i \Psi_i.
\]

Define

\[
Pf_o = -n \sum c^{\alpha_1 \cdots \alpha_{n-1}} \Omega_{\alpha_1}^{\alpha_2} \cdots \Omega_{\alpha_{n-1}}.
\]

Without much difficulty, one can find constants \( c_i \) satisfying (5.4), for which, the \( n \)-form \( Pf \) in (5.5) has the following form

\[
Pf = (-1)^{\frac{n}{2}} \frac{2}{n! \text{vol}(S^n)} Pf_o.
\]

Let \( \{\bar{e}_i\}_{i=1}^n \) be an arbitrary orthonormal frame for \((TM, \bar{g})\), and \( \{e_i = ([v]; \bar{e}_i)\}_{i=1}^n \) be the corresponding frame for \( \pi^*TM \). The induced Riemann metric \( g \) on \( \pi^*TM \) is given by \( g(e_i, e_j) = \bar{g}(\bar{e}_i, \bar{e}_j) \). Let \( \{\bar{\omega}^i\}_{i=1}^n \) be the dual co-frame frame for \( T^*M \). Then \( \{\omega^i\} \) defined by \( \rho \) (see §4) satisfy

\[
\omega^i = \pi^* \bar{\omega}^i.
\]
Let \((\tilde{\omega}_j^i)\) be the Levi-Civita connection form on \(TM\) and \((\bar{\Omega}_j^i)\) denote the curvature form. Then the Chern/Cartan connection form \((\omega_j^i) = (\theta_j^i)\) satisfies
\[
\omega_j^i = \theta_j^i = \pi^* \omega_j^i.
\]
The curvature form \((\Omega_j^i)\) has the following form
\[
(5.8) \quad \Omega_j^i = \pi^* \bar{\Omega}_j^i.
\]
Then \(Pf_o\) in (5.6) can also expressed by
\[
(5.10) \quad Pf_o = \sum \epsilon_{i_1 \cdots i_n} \Omega_{i_2}^{i_1} \cdots \Omega_{i_{n-1}}^{i_n} = \pi^* \sum \epsilon_{i_1 \cdots i_n} \bar{\Omega}_{i_2}^{i_1} \cdots \bar{\Omega}_{i_{n-1}}^{i_n}.
\]
It follows from (5.10) that the \(n\)-form \(Pf\) in (5.7) has the following form
\[
Pf = (-1)^{\frac{n}{2}} \frac{2}{n! \text{vol}(S^n)} \pi^* \sum \epsilon_{i_1 \cdots i_n} \bar{\Omega}_{i_2}^{i_1} \cdots \bar{\Omega}_{i_{n-1}}^{i_n}.
\]
Thus for any vector filed with isolated zeros on \(M\),
\[
[X]^* Pf = (-1)^{\frac{n}{2}} \frac{2}{n! \text{vol}(S^n)} \sum \epsilon_{i_1 \cdots i_n} \bar{\Omega}_{i_2}^{i_1} \cdots \bar{\Omega}_{i_{n-1}}^{i_n}.
\]
It follows from (4.18) that
\[
(5.11) \quad (-1)^{\frac{n}{2}} \frac{2}{n! \text{vol}(S^n)} \int_M \sum \epsilon_{i_1 \cdots i_n} \bar{\Omega}_{i_2}^{i_1} \cdots \bar{\Omega}_{i_{n-1}}^{i_n} = \chi(M).
\]
This is just the Gauss-Bonnet-Chern theorem proved by S. S. Chern in [Ch3].

**References**


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