Math 153 Test 3

At least you should spend 30 minutes on the problem before you read the solution.

1. (3.2) Find an equation of the circle with center \( C(-2, 6) \) and passing through \( P(-1, 2) \).

2. (3.2) Find the center and radius of the circle: \( 2x^2 + 2y^2 + 14x - 8y - 23 = 0 \).

3. (3.3) Given \( 2x = 15 - 3y \). Find the slope and the y-intercept and sketch the graph.

4. (3.4) Find the domain of \( f \) which is given by \( f(x) = \frac{\sqrt{x-3}}{x-5} \).

5. (3.4) If a linear function \( f \) satisfies the condition: \( f(-2) = 4 \) and \( f(3) = 6 \), find \( f(x) \).

6. (3.5) The graph of a function \( f \) with domain \([0, 4]\) is shown in the figure. Sketch the graph of the given equations.
   (a) \( y = f(x+1) \), (c) \( y = f(x) - 2 \), (e) \( y = -2f(x) \), (g) \( y = f(-x/2) \), (i) \( y = -f(x+1) + 2 \).

7. (3.6) The graph of a quadratic function \( f \) is a parabola with vertex \( V(-3, 2) \) and passing through the point \( P(2, 4) \). Find \( f(x) \).

8. (3.6) Given \( f(x) = 3x^2 - 9x + 1 \). (i) the standard form, (ii) find the maximum or minimum value of \( f(x) \), and (iii) sketch the graph.

9. (3.7) Given \( f(x) = \frac{x}{x-1} \) and \( g(x) = \frac{2x}{x+1} \). Find
   (a) \( (f+g)(x) \) and its domain.
   (b) \( (f/g)(x) \) and its domain.
   (c) \( f \circ g \) and its domain.

10. (3.8) Find the inverse function \( f^{-1} \) of \( f \) where \( f(x) = \frac{3x-2}{x+3} \).
MATH 153 Test 3

1. (3.2) Find an equation of the circle with center $C(-2,6)$ and passing through $P(-1,2)$.

**Key Formula:** The standard form of a circle equation is given by $(x-h)^2 + (y-k)^2 = r^2$, where $C(h,k)$ is the center and $r$ is the radius.

**Step 1:** The center is given by $C(-2,6)$. Thus

$$(x - (-2))^2 + (y - 6)^2 = r^2.$$ 

**Step 2:** Find the radius by plugging $x = -1$ and $y = 2$ into the equation.

$$r^2 = (-1 - (-2))^2 + (2 - 6)^2 = 9 + 16 = 25.$$ 

Thus $r = 5$.

**Step 3:** Write down the equation

$$(x + 2)^2 + (y - 6)^2 = 25.$$ 

2. (3.2) Find the center and radius of the circle: $2x^2 + 2y^2 + 14x - 8y - 23 = 0$.

**Key Formula:** $x^2 + kx + (k/2)^2 = (x + k/2)^2$.

**Step 1:** Dividing the equation by 2.

$$x^2 + y^2 + 7x - 4y - \frac{23}{2} = 0.$$ 

$$(x^2 + 7x + ?) + (y^2 - 4y + ?) = \frac{23}{2}$$

**Step 2:** Adding $(7/2)^2$ and $(-4/2)^2$ to both sides of the equation.

$$\left(x^2 + 7x + \frac{7}{2}\right) + \left(y^2 - 4y + \frac{-4}{2}\right) = \frac{23}{2} + \left(\frac{7}{2}\right)^2 + \left(\frac{-4}{2}\right)^2.$$ 

$$\left(x + \frac{7}{2}\right)^2 + \left(y - 2\right)^2 = \frac{111}{4}.$$ 

**Step 3:** Comparing the equation with the standard form $(x-h)^2 + (y-k)^2 = r^2$, we get

Center : $(h,k) = \left(-\frac{7}{2}, 2\right)$. Radius : $r = \frac{\sqrt{111}}{2}$.

3 (3.3) Given $2x = 15 - 3y$. Find the slope and the y-intercept and sketch the graph.

**Key Formula:** $y = mx + b$, where $m$ is the slope and $b$ is the y-intercept.

**Step 1:** Rewrite the equation as follows:

$$y = -\frac{2}{3}x + 5.$$ 

**Step 2:** Slope: $m = -\frac{2}{3}$ and the y-intercept: $b = 5$.

**Step 3:** Sketch the graph: $m = \frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta x}$. One may take $\Delta x = 3$ and $\Delta y = -2$. Starting from the point $(0,5)$, one can use the slope to find another point on the line. First walk to the right by $\Delta = 3$ units, then walk downward by $\Delta y = -2$ units, arriving at $(0 + 3, 5 - 2) = (3, 3)$. Then draw a line passing through $(0,5)$ and $(3,3)$.

One can also find another point by solving the equation. Let $x = 3$ in the equation. Then $2(3) = 15 - 3y$. One gets $y = (15 - 6)/3 = 3$. Namely, $(3,3)$ is another point on the line.

4 (3.4) Find the domain of $f$ which is given by $f(x) = \frac{\sqrt{2x^2}}{x^2}$.

**Step 1:** Look at the numerator and the denominator. The numerator is defined only when $x - 3 \geq 0$ and the fraction is defined only when $x - 5 \neq 0$. We get two conditions:

$$x - 3 \geq 0, \quad x - 5 \neq 0.$$ 

**Step 2:** Solve the above inequalities:

$$x \geq 3, \quad x \neq 5.$$ 

**Step 3:** Express the solution in the form of intervals: $[3,5) \cup (5,\infty)$.

5. (3.4) If a linear function $f$ satisfies the condition: $f(-2) = 4$ and $f(3) = 6$, find $f(x)$. 

2
Method 1: $f$ is given by $f(x) = mx + b$. By assumption,

\[
\begin{align*}
    f(-2) &= m(-2) + b = -2m + b = 4 \\
    f(3) &= m(3) + b = 3m + b = 6.
\end{align*}
\]

(2) - (1) yields

\[
5m = 2, \quad m = \frac{2}{5}
\]

By (2), we obtain

\[
b = 6 - 3m = 6 - 3\left(\frac{2}{5}\right) = \frac{24}{5}.
\]

Thus

\[
f(x) = \frac{2}{5}x + \frac{24}{5}.
\]

Method 2: The graph of $f$ is a line and its equation is

\[
y = f(x) = mx + b,
\]

where $m$ is the slope and $b$ is the $y$-intercept. The given condition can be translated to the following statement: the line passes through $A(-2, 4)$ and $B(3, 6)$. First, we find the slope.

\[
m = \frac{6 - 4}{3 - (-2)} = \frac{2}{5}.
\]

The point-slope form of the equation is $y - y_1 = m(x - x_1)$. We get

\[
y - 6 = \frac{2}{5}(x - 3),
\]

which can be written as

\[
y = \frac{2}{5}x + \frac{24}{5}.
\]

We get

\[
f(x) = \frac{2}{5}x + \frac{24}{5}.
\]

6. (3.5) The graph of a function $f$ with domain $[0, 4]$ is shown in the figure. Sketch the graph of the given equations.

(a) $y = f(x + 1)$, (c) $y = f(x) - 2$, (e) $y = -2f(x)$, (g) $y = f(-x/2)$, (i) $y = -f(x + 1) + 2$. 

![Graphs of the given equations](image-url)
7 (3.6) The graph of a quadratic function \( f \) is a parabola with vertex \( V(-3, 2) \) and passing through the point \( P(2, 4) \). Find \( f(x) \).

**Key Formula:** If \( f(x) = a(x-h)^2 + k \), then the graph of \( f \) is a parabola with vertex \( V(h, k) \).

**Step 1.** From the given information, we get

\[
f(x) = a(x - (-3))^2 + 2 = a(x + 3)^2 + 2.
\]

**Step 2.** Determine the value of \( a \) using the point \( P(2, 4) \):

\[
4 = a(2 + 3)^2 + 2 = 25a + 2.
\]

\[
a = \frac{2}{25}
\]

**Step 3.** Write down the formula \( f(x) \):

\[
f(x) = \frac{2}{25}(x + 3)^2 + 2.
\]

8 (3.6) Given \( f(x) = 3x^2 - 9x + 1 \). (i) the standard form, (ii) find the maximum or minimum value of \( f(x) \), and (iii) sketch the graph.

**Key idea:** Rewrite \( f(x) \) into the form \( f(x) = a(x-h)^2 + k \) by completing the square.

**Step 1:** Set

\[
y = 3x^2 - 9x + 1.
\]

\[
y - 1 = 3x^2 - 9x.
\]

\[
y - 1 = 3x^2 - 9x + \left(\frac{3}{2}\right)^2 = x^2 - 3x + \left(\frac{-3}{2}\right)^2 = \left(x - \frac{3}{2}\right)^2.
\]

\[
\frac{y - 1}{3} = \left(x - \frac{3}{2}\right)^2 - \frac{9}{4}.
\]

\[
y - 1 = 3\left(x - \frac{3}{2}\right)^2 - \frac{27}{4}.
\]

\[
y = 3\left(x - \frac{3}{2}\right)^2 - \frac{23}{4}.
\]

**Step 2:** Observe that

\[
f(x) = 3\left(x - \frac{3}{2}\right)^2 - \frac{23}{4} \geq -\frac{23}{4},
\]

and \( f(x) = -\frac{23}{4} \) when \( x = \frac{3}{2} \). Thus \( f \) has minimum value \( f(3/2) = 23/4 \).

**Step 3:** First find the vertex \( V(3/2, -23/4) \). Then find the x-intercepts and the y-intercept.

The x-intercept:

\[
3\left(x - \frac{3}{2}\right)^2 - \frac{23}{4} = 0.
\]

\[
x = \frac{3}{2} \pm \sqrt{\frac{23}{12}} \approx 2.88, 0.12.
\]

The y-intercept:

\[
y = 1.
\]

**Step 4:** Based on the above information, sketch the graph (omitted).

9. (3.7) Given \( f(x) = \frac{x}{x+1} \) and \( g(x) = \frac{2x}{x+1} \). Find

(a) \( (f + g)(x) \) and its domain.
\[(f + g)(x) = \frac{x}{x - 1} + \frac{2x}{x + 1} = \frac{3x^2 - x}{x^2 - 1}.
\]

The domain of \(f + g\) is the set of \(x\) for which \(f(x)\) AND \(g(x)\) both are defined. Namely, the set of \(x\) such that
\[
x - 1 \neq 0, \quad x + 1 \neq 0.
\]
\[
x \neq 1, -1.
\]

The domain is given by
\[
D = \{x \mid x \neq 1, -1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty).
\]

(b) \((f/g)(x)\) and its domain.
\[
(f/g)(x) = \frac{f(x)}{g(x)} = \frac{x}{x - 1} = \frac{x + 1}{2x} = \frac{x + 1}{2(x - 1)}
\]

The domain of \(f/g\) is the set of \(x\) for which \(f(x)\) and \(g(x)\) both are defined and \(g(x) \neq 0\) (because that \(g(x)\) is in the denominator). Namely,
\[
x - 1 \neq 0, \quad x + 1 \neq 0, \quad 2x \neq 0.
\]
We get
\[
x \neq -1, 1, 0.
\]

The domain is given by
\[
D = \{x \mid x \neq 1, -1\} = (-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (1, \infty).
\]

Warning: Do not look at the simplified formula \((f/g)(x)\). The same caution should be paid to the following one.

(c) \(f \circ g\) and its domain.
\[
f \circ g(x) = f(g(x)) = \frac{g(x)}{g(x) - 1} = \frac{2x}{2x + 1 - 1} = \frac{2x}{x - 1}
\]

The domain of \(f \circ g\) is the set of \(x\) for which \(g(x)\) is defined and \(g(x)\) is in the domain of \(f\) (because that \(g(x)\) is inside \(f\)). Namely,
\[
x + 1 \neq 0(such \ that \ g(x) \ is \ defined), \quad g(x) - 1 \neq 0(such \ that \ g(x) \ is \ in \ the \ domain \ of \ f).
\]

Let us look at the equation \(g(x) - 1 = 0\).
\[
\frac{2x}{x + 1} = 1, \quad 2x = x + 1, \quad x = 1.
\]

Thus \(g(x) \neq 1\) if and only if \(x \neq 1\). We get
\[
D = \{x \mid x \neq -1, 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty).
\]

10 (3.8) Find the inverse function \(f^{-1}\) of \(f\) where \(f(x) = \frac{3x - 3}{2x - 3}\).

**Key idea:** set \(y = f(x)\), solve for \(x\) in terms of \(y\) and get \(x = g(y)\). Then the inverse \(f^{-1}\) is given by \(f^{-1}(x) = g(x)\).

**Step 1:** Set
\[
y = f(x) = \frac{3x - 2}{2x - 5},
\]
\[
(2x - 5)y = 3x - 2.
\]
\[
2yx - 5y = 3x - 2.
\]
\[
2yx - 3x = 5y - 2.
\]
\[
(2y - 3)x = 5y - 2.
\]
\[
x = \frac{5y - 2}{2y - 3} = g(y).
\]

**Step 2:** \(f^{-1}\) is given by
\[
f^{-1}(x) = \frac{5x - 2}{2x - 3}.
\]