Project 4. Guessing Limits Numerically

Objective
To guess the limit of a function at a point numerically.

Narrative
If you have not already done so, read Section 2.2 in the text.
Prior to having theorems on limits at our disposal, there are two major issues surrounding the limit of a function at a point: The first is guessing what the limit is, if it even exists; this issue can often be approached either graphically or numerically. The second issue involves proving that the guess is correct; this issue involves using the formal definition of limit.

Having alluded to guessing limits graphically in the last project, we address the issue of guessing limits numerically in this project. In Project 5 we address the issue of proving a guess to be correct. In this project we also illustrate how to perform repeated computations in Maple using a “do loop”.

Task

a) Type the command lines in the left-hand column below into Maple in the order in which they are listed. These commands will help you estimate \( \lim_{x \to 1} \frac{x^3 - 1}{x - 1} \) numerically, if it exists. (Note: It’s OK to type the entire first for n from 1 to 6 do ... od: on one line, and the second for n from 1 to 6 do ... od: on one line.)

```maple
> # Project 4. Guessing Limits Numerically
> restart;
> f := x -> (x^3-1)/(x-1);
> plot(f(x),x=0..2,y=0..6);
> a := 1.0;
> f(a);
> for n from 1 to 6 do
x := a-1/10^n:
print(evalf(x), evalf(f(x)));
od:
> for n from 1 to 6 do
x := a+1/10^n:
print(evalf(x), evalf(f(x)));
od:
```

b) On the basis of this data, do you think \( \lim_{x \to 1} \frac{x^3 - 1}{x - 1} \) exists? If so, what do you think it is (to 4 decimal places of accuracy)? Justify your answer.

Your lab report will be a hard copy of your typed input and Maple’s responses (including both text and graphics), together with your written response.
Comments

1. You can *guess* whether or not \( \lim_{x \to 1} \frac{x^3 - 1}{x - 1} \) exists, and if it does exist, what its value is, on the basis of numerical "evidence" (as provided by Maple in this project), but it is impossible to say *for sure* that you’re ever correct: you can never perform more than a finite number of computations, and however close \( x \) is to 1, you may *miss* some critical behavior of \( f(x) = \frac{x^3 - 1}{x - 1} \) that might otherwise affect your guess. It is because of this that we *must* turn to the formal concept of the limit.

2. Different rates of convergence can be achieved by replacing \( 1/10^n \) by \( 1/2^n \) (this is slower), \( 1/n^2 \) (this is slower yet), and \( 1/n^n \) (this is faster).

3. The physical limitations of your computer may limit the accuracy of your computations.

4. Maple has a built-in command \( \text{limit}(f(x), x=a) \) that allows you to compute some limits automatically. (Variations on this command include \( \text{limit}(f(x), x=\text{infinity}) \) and \( \text{limit}(f(x), x=-\text{infinity}) \) for computing limits at \( \pm \infty \), and \( \text{limit}(f(x), x=a, \text{left}) \) and \( \text{limit}(f(x), x=a, \text{right}) \) for computing left- and right-hand limits.) Since we are interested not just in what limits are, but how they are computed, we intentionally avoided this command in this project.

5. At the end of the do loops in the above code, Maple will think that \( n = 6 \) and \( x = a \pm 1/10^n \). (You can check this by entering the commands \( n; \) and \( x; \) after each loop.) This is important to know since if, subsequent to the appropriate do loop, you wanted to reuse \( n \) or \( x \) as a variable then you would have to redefine it as a variable using the command \( n = \text{'n'} \) or the command \( x = \text{'x'} \).