Chapter 5

Geodesics on Metric Surfaces

5.1 Shortest Paths and Geodesics

Let \((S, F)\) be a Finsler surface. For a curve \(C\) on \(S\) issuing from \(p\) to \(q\), the length of \(C\) is given by

\[
L(C) := \int_a^b F(\dot{c}(t))dt,
\]

where \(c : [a, b] \to S\) is a coordinate map of \(C\) with \(c(a) = p\) and \(c(b) = q\). Assume that \(C\) is a shortest curve among those nearby \(C\) and issuing from \(p\) to \(q\). Take an arbitrary smooth family of curves \(C_s\), \(|s| < \varepsilon\), on \(S\) issuing from \(p\) to \(q\) such that \(C_0 = C\). Let

\[
L(s) := L(C_s).
\]

\(L(s)\) is the length of the curve \(C_s\). By assumption,

\[
L(s) \geq L(0).
\]

This implies that

\[
L'(0) = 0.
\]

Let \(c : [a, b] \to C\) be a constant speed coordinate map of \(C\). Assume that \(C\) is covered by a coordinate map \(\varphi : D \to S\). We can express \(c(t)\) by

\[
c(t) = \varphi(x(t), y(t)).
\]

Put

\[
L(x, y, u, v) := \frac{1}{2}F^2(y), \quad y = u\varphi_x + v\varphi_y.
\]
By the method in the calculus of variations, we know that the functions \(x(t)\) and \(y(t)\) satisfy the following second order ordinary equations:

\[
x''(t) + 2G \left( x(t), y(t), x'(t), y'(t) \right) = 0 \tag{5.1}
\]

\[
y''(t) + 2H \left( x(t), y(t), x'(t), y'(t) \right) = 0 \tag{5.2}
\]

where \(G = G(x, y, u, v)\) and \(H = H(x, y, u, v)\) are given by

\[
G : = \frac{(L_x L_{uv} - L_y L_{uv}) - (L_x - L_y L_u)L_v}{2 \left( L_{uu} L_{vv} - L_{uv}^2 \right)} \tag{5.3}
\]

\[
H : = \frac{(-L_x L_{uv} + L_y L_{uu}) + (L_x - L_y L_u)L_u}{2 \left( L_{uu} L_{vv} - L_{uv}^2 \right)} \tag{5.4}
\]

We call \(G\) and \(H\) the geodesic coefficients of \(F\).

**Definition 5.1.1** A geodesic on a Finsler surface \((S, F)\) is a regular map \(c : I \to S\) such that its coordinates \((x(t), y(t))\) in a coordinate system \((D, \varphi)\) satisfy (5.1) and (5.2). The image of a geodesic on \(S\) is called a path.

For any geodesic \(c(t)\) of a Finsler metric \(F\),

\[F(\dot{c}(t)) = \text{constant}.\]

The geodesic functions \(G\) and \(H\) have the following properties

(i) \(G\) and \(H\) are \(C^\infty\) on \(D \times (\mathbb{R}^2 - \{0\});\)

(ii) \(G\) and \(H\) are positively homogeneous of degree two in \((u, v)\), that is, for any \(\lambda > 0,\)

\[
G(x, y, \lambda u, \lambda v) = \lambda^2 G(x, y, u, v),
\]

\[
H(x, y, \lambda u, \lambda v) = \lambda^2 H(x, y, u, v).
\]

(iii) If \(\tilde{\varphi} : \tilde{D} \to S\) is another coordinate map with \(\tilde{\varphi}(\tilde{D}) \cap \varphi(D) \neq \emptyset\), then \((G, H)\) and \((\tilde{G}, \tilde{H})\) are related by

\[
2\tilde{G} = 2G \frac{\partial \tilde{x}}{\partial x} + 2H \frac{\partial \tilde{x}}{\partial y} - \frac{\partial^2 \tilde{x}}{\partial x^2} u^2 - 2 \frac{\partial^2 \tilde{x}}{\partial x \partial y} u v - \frac{\partial^2 \tilde{x}}{\partial y^2} v^2
\]

\[
2\tilde{H} = 2G \frac{\partial \tilde{y}}{\partial x} + 2H \frac{\partial \tilde{y}}{\partial y} - \frac{\partial^2 \tilde{y}}{\partial x^2} u^2 - 2 \frac{\partial^2 \tilde{y}}{\partial x \partial y} u v - \frac{\partial^2 \tilde{y}}{\partial y^2} v^2.
\]

Let \(c(t) = \varphi \left( x(t), y(t) \right)\) be a geodesic with \(x'(t) > 0\). Then \(x(t)\) is an increasing function so that the inverse function \(t = t(x)\) exists and \(y = y(t) =\)
\( y(t(x)) \) becomes a function of \( x \). By the chain rule, we have

\[
\frac{dy}{dx} = \frac{y'(t)}{x'(t)},
\]

(5.5)

\[
\frac{d^2y}{dx^2} = \frac{d}{dt} \left( \frac{y''(t)}{x'(t)} \right) = \frac{y''(t)x'(t) - x''(t)y'(t)}{x'(t)^3}.
\]

(5.6)

By (5.1) and (5.2), we obtain

\[
\frac{d^2y}{dx^2} = -2H \left( x(t), y(t), x'(t), y'(t) \right) x'(t) + 2G \left( x(t), y(t), x'(t), y'(t) \right) y'(t).
\]

Let

\[
\Phi := 2 \frac{v^y}{u} G(x, y, u, v) - 2H(x, y, u, v).
\]

\( \Phi \) satisfies

\[
\Phi(x, y, \lambda u, \lambda v) = \lambda^2 \Phi(x, y, u, v), \quad \lambda > 0.
\]

Now two ODEs (5.1) and (5.2) are combined into the following ODE after we eliminate the parameter \( t \),

\[
\frac{d^2y}{dx^2} = \Phi \left( x, y, 1, \frac{dy}{dx} \right).
\]

(5.7)

If \( x'(t) < 0 \), we can still view \( y(t) \) as a function of \( x \). In this case, two ODEs (5.1) and (5.2) are combined into the following ODE:

\[
\frac{d^2y}{dx^2} = \Phi \left( x, y, -1, -\frac{dy}{dx} \right).
\]

(5.8)

**Proposition 5.1.1** Let \((S, F)\) be a Finsler surface and \( \varphi : D \to S \) a coordinate map such that the geodesic coefficients \( G \) and \( H \) are in the following forms

\[
G = P(x, y, u, v) u, \quad H = P(x, y, u, v) v.
\]

Then any path on the surface must be the image of a straight line under \( \varphi \).

**Proof:** Let \( c(t) = \varphi \left( x(t), y(t) \right) \) be a geodesic. Since

\[
\Phi = 2 \frac{v^y}{u} P u - 2P v = 0,
\]

\( y = y(x) \) satisfies

\[
\frac{d^2y}{dx^2} = 0.
\]

The general solution of the above ODE is that \( y = mx + b \). Q.E.D.
Example 5.1.1 Let $F$ be the Funk metric on a strongly convex domain $\mathcal{U} \subset \mathbb{R}^2$. Hence it satisfies
\[ F_x = FF_u, \quad F_y = FF_v. \] (5.9)
A direct computation yields
\[ G = \frac{1}{2} F u, \quad H = \frac{1}{2} F v. \] (5.10)
Thus paths are straight lines in $\mathcal{U}$.

A Finsler surface $(S, F)$ is said to be \textit{positively complete} if every geodesic $c : (a, b) \to S$ can be extended to a geodesic defined on $(a, \infty)$.

Let $(S, F)$ be a positively complete Finsler surface. For a point $p \in S$, define
\[ \exp_p : T_p S \to S \]
by
\[ \exp_p(y) = c_y(1), \]
where $c_y : [0, \infty) \to S$ denotes the geodesic with $\dot{c}_y(0) = y$. We have
\[ \exp_p(ty) = c_y(t), \quad t \geq 0. \]
We call $\exp_p$ the \textit{exponential map} at $p$.

Theorem 5.1.1 Let $F$ be a positively complete Finsler metric on a surface. For any point $p \in S$,
\[ \exp_p : T_p S \to S \]
is an onto map.

I. Geodesics on Riemann Surfaces:
Consider a Riemannian metric $F$ on a surface $S$,
\[ F(y) = \sqrt{au^2 + 2buv + cv^2}, \quad y = \varphi_x + \varphi_y \in T_p S, \]
where $a = a(x, y), b = b(x, y)$ and $c = c(x, y)$ are $C^\infty$ functions on $D$ satisfying
\[ a > 0, \quad c > 0, \quad ac - b^2 > 0. \]
The geodesic coefficients $G$ and $H$ are given by
\[ G = \frac{(ca_y + ba_y - 2bb_x)u^2 + 2(ca_x - bc_x)uv + (2cb_y - cc_x - bc_y)v^2}{4(ac - b^2)}, \]
\[ H = \frac{(2ab_x - ba_x - aa_y)u^2 + 2(ac_x - ba_y)uv + (bc_x + ac_y - 2bb_y)v^2}{4(ac - b^2)}. \]
Thus $G$ and $H$ are quadratic functions of $(u, v) \in \mathbb{R}^2$.  

43
Example 5.1.2 Consider the following Riemannian metric on a domain \( \mathcal{U} \) in \( \mathbb{R}^2 \),

\[
F = e^{\eta/2} \sqrt{u^2 + v^2},
\]
(5.11)

where \( \eta = \eta(x,y) \) is a \( C^\infty \) function on \( \mathcal{U} \). The geodesic coefficients \( G \) and \( H \) are given by

\[
G = (u^2 - v^2) \eta_x + 2uv \eta_y \\
H = 2uv \eta_x - (u^2 - v^2) \eta_y.
\]

Example 5.1.3 Let

\[
F = \sqrt{\frac{u^2 + v^2 + (xv - yu)^2}{1 + x^2 + y^2}}.
\]

\( F \) is a Riemannian metric on \( \mathbb{R}^2 \). The geodesic coefficients \( G \) and \( H \) are given by

\[
G = -\frac{1}{2} F^2 x \\
H = -\frac{1}{2} F^2 y.
\]

Thus any geodesic \( c(t) = (x(t), y(t)) \) with \( \lambda = F(\dot{c}(t)) > 0 \) must be given by

\[
x(t) = a \cosh(\lambda t) + b \sinh(\lambda t) \\
y(t) = c \cosh(\lambda t) + d \sinh(\lambda t).
\]

Example 5.1.4 Consider the following Riemannian metric on \( \mathbb{R}^2 \),

\[
F = \sqrt{\frac{u^2 + v^2 + (xv - yu)^2}{1 + x^2 + y^2}}.
\]

The geodesic coefficients \( G \) and \( H \) are given by

\[
G = P u, \quad H = P v,
\]

where

\[
P = -\frac{xu + yv}{1 + x^2 + y^2}.
\]

Thus paths are straight lines.
Example 5.1.5 Let $S$ be the graph in $\mathbb{R}^3$ which is defined by

$$z = f(x, y), \quad (x, y) \in D.$$  

The standard coordinate map of $S$ is given by

$$\varphi(x, y) = \left(x, y, f(x, y)\right).$$

The induced Riemannian metric is given by

$$F(y) := \sqrt{\left(1 + |f_x|^2\right)u^2 + 2f_x f_y uw + \left(1 + |f_y|^2\right)v^2},$$

where $y = u\varphi_x + v\varphi_y \in T_pS$. The geodesic coefficients $G$ and $H$ are given by

$$G = \frac{f_x}{2(1 + |f_x|^2 + |f_y|^2)} \left(f_{xx}u^2 + 2f_{xy}uw + f_{yy}v^2\right), \quad (5.12)$$

$$H = \frac{f_y}{2(1 + |f_x|^2 + |f_y|^2)} \left(f_{xx}u^2 + 2f_{xy}uw + f_{yy}v^2\right). \quad (5.13)$$

If

$$z = f(x, y) = a(y) + b(y)x,$$

where $a(y)$ and $b(y)$ are functions of $y$, then

$$H(x, y, u, 0) = 0.$$  

Thus the straight lines $C_m$ on the surface $S$,

$$C_m := \left\{ \left(x, m, a(m) + b(m)x\right) \right\},$$

are paths.

II. Geodesics on Randers Surfaces: Consider a Randers metric $F = \alpha + \beta$ on a surface $S$, where

$$\alpha(y) : = \sqrt{a(x, y)u^2 + 2b(x, y)uw + c(x, y)v^2}$$

$$\beta(y) : = \lambda(x, y)u + \mu(x, y)v$$

where $y = u\varphi_x + v\varphi_y \in T_pS$.

Let

$$A = \lambda_x - \frac{ca_x + ba_y - 2bb_x}{2(ac - \beta^2)}\lambda - \frac{2ab_x - ba_x - ac_y}{2(ac - \beta^2)}\mu$$

$$B = \frac{1}{2} \left(\lambda_y + \mu_x\right) - \frac{ca_y - bc_x}{2(ac - \beta^2)}\lambda - \frac{ac_x - ba_y}{2(ac - \beta^2)}\mu$$

$$C = \mu_y - \frac{2cb_y - cc_x - bc_y}{2(ac - \beta^2)}\lambda - \frac{bc_x + ac_y - 2bb_y}{2(ac - \beta^2)}\mu$$

$$D = \frac{1}{2} \left(\lambda_y - \mu_x\right).$$
Let \( G, H \) denote the geodesic functions of the Riemannian metric \( \alpha \) and \( \tilde{G}, \tilde{H} \) denote the geodesic functions of the Randers metric \( F = \alpha + \beta \). Then

\[
\tilde{G} = G + \frac{u}{2F} \left( A u^2 + 2B u v + C v^2 \right) + \frac{D a^2}{(a c - b^2) F} \left( b u + c v + a \mu \right),
\]

\[
\tilde{H} = H + \frac{\nu}{2F} \left( A u^2 + 2B u v + C v^2 \right) - \frac{D a^2}{(a c - b^2) F} \left( a u + b v + a \lambda \right).
\]

Let

\[
\tilde{\Phi} := 2\frac{\nu}{u} \tilde{G}(x, y, u, v) - 2\tilde{H}(x, y, u, v).
\]

\[
\Phi := 2\frac{\nu}{u} G(x, y, u, v) - 2H(x, y, u, v).
\]

Observe that

\[
\tilde{\Phi} = \Phi + \frac{2Da^3}{(ac - b^2) u}, \tag{5.14}
\]

Then the paths \( y = y(x) \) of \( F \) satisfies

\[
\begin{align*}
\frac{d^2 y}{d x^2} &= \Phi \left( x, y, \pm 1, \pm \frac{dy}{dx} \right) \\
&\pm \frac{\lambda_y(x, y) - \mu_x(x, y)}{a(x, y)c(x, y) - b(x, y)^2} \left( a(x, y) + 2b(x, y) \frac{dy}{dx} + c(x, y) \left( \frac{dy}{dx} \right)^2 \right)^{3/2}.
\end{align*}
\]

The sign in the above equation is equal to that of \( x'(t) \). We conclude that if

\[
\lambda_y = \mu_x,
\]

then the paths of \( F = \alpha + \beta \) coincide with that of \( \alpha \).

**Example 5.1.6** Let \( | \cdot |_\kappa \) be the Randers norm on \( \mathbb{R}^3 \) given by

\[
| (u, v, w) |_\kappa = \sqrt{u^2 + v^2 + w^2 + \kappa w}.
\]

Let \( S \) be a graph in \((\mathbb{R}^3, | \cdot |_\kappa)\) given by

\[
\varphi(x, y) = \left( x, y, f(x, y) \right).
\]

The induced Finsler metric \( F_\kappa \) is given by

\[
F_\kappa = \sqrt{a \ u^2 + 2b \ u v + c \ v^2 + \lambda \ u + \mu \ v},
\]

where

\[
a = 1 + |f_x|^2, \quad b = f_x f_y, \quad c = 1 + |f_y|^2 \quad \lambda = \kappa f_x, \quad \mu = \kappa f_y.
\]

Note that

\[
\lambda_y = \mu_x.
\]

Thus the paths of \( F_\kappa \) remains unchanged when \( \kappa \) changes.
Example 5.1.7 Consider the following special Randers metric on $S$

$$F(y) = \sqrt{u^2 + \phi^2(x) v^2} + \psi(x)u, \quad y = u\varphi_x + v\varphi_y \in T_p S.$$ 

We have

$$G = -\frac{1}{2} \phi(x)\phi'(x)u^2 + \frac{\psi'(x)u^2 + \psi(x)\phi(x)\phi'(x)v}{2(\sqrt{u^2 + \phi^2(x)v^2} + \psi(x)u)}u$$

$$H = \frac{\phi'(x)}{\phi(x)} uv + \frac{\psi'(x)u^2 + \psi(x)\phi(x)\phi'(x)v}{2(\sqrt{u^2 + \phi^2(x)v^2} + \psi(x)u)}v.$$
5.2 Exercises

Exercise 5.2.1 Let $F$ be a Finsler metric on a domain $D \subset \mathbb{R}^2$. Express a Finsler metric $F$ in the form

$$ F = u \, f\left( x, y, \frac{v}{u} \right), \quad u > 0, \quad (5.15) $$

where $f = f(x, y, \xi)$ is a function of $(x, y, \xi) \in D \times \mathbb{R}$. Express the geodesic coefficients $G$ and $H$ of $F$ in the form

$$ G = \frac{1}{2} \Theta \left( x, y, \frac{v}{u} \right) \, u^2 $$

$$ H = \frac{1}{2} \Phi \left( x, y, \frac{v}{u} \right) \, uv - \frac{1}{2} \Phi \left( x, y, \frac{v}{u} \right) \, u^2, $$

where $\Theta = \Theta(x, y, \xi)$ and $\Phi = \Phi(x, y, \xi)$ are functions of $(x, y, \xi)$. Show that

$$ \Phi = \frac{f_y - f_x - \xi f_{y\xi}}{f_{\xi\xi}}, \quad (5.16) $$

$$ \Theta = \frac{f_y - f_x - \xi f_{y\xi}}{f_{\xi\xi}} \cdot \frac{f_{\xi\xi}}{f} + \frac{f_x + \xi f_y}{f}. \quad (5.17) $$

Exercise 5.2.2 Consider the following Riemannian metric on the right half plane of $\mathbb{R}^2$.

$$ F = \frac{1}{x} \sqrt{u^2 + v^2}. \quad (5.18) $$

(a) Let $c(t) = (x(t), y(t))$ be a geodesic of $F$. Show that $x(t)$ and $y(t)$ satisfy

$$ x''(t) - \frac{1}{x(t)} \left( x'(t)^2 - y'(t)^2 \right) = 0 $$

$$ y''(t) - \frac{1}{x(t)} x'(t) y'(t) = 0. $$

(b) View a geodesic $c(t) = \left( x(t), y(t) \right)$ as a graph of $y = f(x)$. Show that it satisfies

$$ \frac{d^2 y}{dx^2} = \frac{1}{x} \left( \frac{dy}{dx} + \left( \frac{dy}{dx} \right)^3 \right). $$

(c) Verify that semi-circles

$$ x^2 + (y - b)^2 = a^2, \quad x > 0 $$

are paths.

Exercise 5.2.3 Let $S$ be a graph in $\mathbb{R}^4$ given by

$$ \varphi(x, y) = \left( x, \ y, \ x^2 + y^2, \ x^2 - y^2 \right). $$

Find $G$ and $H$ of the induced Riemannian metric $F$. 

48
Exercise 5.2.4 Consider the following Randers metric \( F = \alpha + \beta \) on a domain \( \Omega = \{(x,y), \ x^2 + y^2 < 1 \} \) in \( \mathbb{R}^2 \),
\[
F = \frac{\sqrt{u^2 + v^2 - (xv - yu)^2} + xu + yv}{1 - x^2 - y^2}.
\]
Find the geodesic coefficients \( G \) and \( H \) of \( F \). Describe the paths of \( F \) in \( \Omega \)?

Exercise 5.2.5 Let \( |\kappa| < 1 \). Consider the following Randers metric on a domain \( \mathcal{U} \) in \( \mathbb{R}^2 \).
\[
F = \sqrt{\frac{1 + \kappa^2 x^2}{1 - (1 - \kappa^2)x^2} u^2 + x^2 v^2 + \frac{\kappa x u}{\sqrt{1 - (1 - \kappa^2)x^2}}}. 
\]
Show that
\[
H(x, y, u, 0) = 0.
\]
Explain why any horizontal line (after an appropriate parametrization) is a path of \( F \).

Exercise 5.2.6 Let
\[
F = \sqrt{\frac{u^2 + v^2 - (xv - yu)^2}{1 - x^2 - y^2}}.
\]
\( F \) is a Riemannian metric on the unit disk \( \mathbb{D}^2 \subset \mathbb{R}^2 \).

(a) Verify that the geodesics of \( F \) are given by
\[
G = \frac{1}{2} F^2 x, \quad H = \frac{1}{2} F^2 y.
\]

(b) Show that any geodesic \( c(t) = (x(t), y(t)) \) with \( \lambda = F(\dot{c}(t)) > 0 \) is given by
\[
x(t) = a \cos(\lambda t) + b \sin(\lambda t), \\
y(t) = c \cos(\lambda t) + d \sin(\lambda t),
\]
where \( a, b, c \) and \( d \) are constants. Explain why the paths of \( F \) are either elliptic curves with center at the origin or straight lines passing through the origin.