1 Given an affine connection $D$ on a manifold $M$, let $R = R_{ijkl} dx^i \otimes \frac{\partial}{\partial x^i} \otimes dx^k \otimes dx^l$ denote the curvature tensor. For a vector $y \in T_pM$, define $R_y : T_pM \rightarrow T_pM$ by

$$R_y(u) := R_{ik}^j u^k \frac{\partial}{\partial x^i}|_p = y^j R_{ik}^j y^l \frac{\partial}{\partial x^i}|_p.$$ 

$R_y$ is called the Riemann curvature.

Let $G_i := \frac{1}{2} \Gamma_i^{jk}(x) y^j y^k$. Suppose that $G_i = P y^i$, where $P = P_i(x) y^i$. Show that $R_y$ is in the following form

$$R_y(u) = \Xi(y) u - \theta_y(u) y,$$

where $\theta_y$ is a linear functional on $T_pM$.

2 Let $g = g_{ij} dx^i \otimes dx^j$ be a Riemannian metric defined on $\mathbb{R}^n$ by

$$g_{ij} = \frac{(1 + |x|^2) \delta_{ij} - x^i x^j}{(1 + |x|^2)^2},$$

where $|x| := \sqrt{\sum_{i=1}^n (x^i)^2}$. Find a formula for $G_i = \frac{1}{2} \Gamma_i^{jk}(x) y^j y^k$ and describe the geodesics of $g$ in $\mathbb{R}^n$.

3 The Riemann curvature tensor $R = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l$ satisfies

$$R_{ijkl} + R_{jikl} = 0, \quad R_{ijkl} + R_{klij} + R_{lijk} = 0, \quad R_{ijkl} + R_{ijkl} = 0.$$ 

Show that

$$R_{ijkl} = R_{klij}. $$

Hint: Let $T_{ijkl} := R_{ijkl} + R_{jikl}$ and simplify $T_{ijkl} - T_{klij} + T_{ijlk} + T_{ijkl} - T_{jikl} + T_{klij}$ using the above identities.

4 Let $R = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l$ be the Riemannian curvature tensor of a Riemannian metric $g$. Using the fact $R_y(y) = 0$, show that for any tangent plane $P \subset T_pM$ with $y \in P$, the following ratio

$$K(P) := \frac{g(R_y(u), u)}{g(y, y) g(u, u) - g(y, u)^2}$$

is independent of $u \in P$ with $P = \text{span}\{y, u\}$.

5 Recall that the Riemann curvature $R_y : T_pM \rightarrow T_pM$ is self-adjoint,

$$g(R_y(u), v) = g(u, R_y(v)),$$

and satisfies $R_y(y) = 0$. Suppose that $R_y$ is in the following form

$$R_y(u) = \Xi(y) u - \theta_y(u) y,$$

where $\Xi(y)$ is a quadratic functional on $T_pM$ and $\theta_y$ is a linear functional on $T_pM$. Show that

$$R_y(u) = \frac{\Xi(y)}{g(y, y)} \{g(y, y) u - g(y, u) y\}.$$