1 Let $g = g_{ij}(x)dx^i \otimes dx^j$ be a Riemannian metric on a domain $\Omega \subset \mathbb{R}^n$ and let $F := \sqrt{g_{ij}(x)y^i y^j}$. Verify that
\[
\frac{g_{ij}}{4} \left\{ [F^2]_{x^i y^j} y^k - [F^2]_{x^i} y^k \right\} = \frac{F y^k}{2F} y^i + \frac{g_{ij}}{2} \left\{ F_{x^i y^j} y^k - F_{x^i} \right\}.
\]

2 Let $g = (1 + \mu |x|^2)^{-2} \{ (1 + \mu |x|^2) \delta_{ij} - \mu x^i x^j \} dx^i \otimes dx^j$ and $F = \sqrt{g_{ij}(x)y^i y^j}$, where $\mu$ is an arbitrary constant. Show that
\[
F := \sqrt{1 + \mu |x|^2} |y|^2 - \mu \langle x, y \rangle^2
\]
where $|x| = \sqrt{\sum_i (x_i)^2}$, $|y| = \sqrt{\sum_i (y_i)^2}$ and $\langle x, y \rangle = \sum_i x_i y_i$. Verify that
\[
F_{x^i y^j} y^k - F_{x^i} = 0.
\]

3 Let $B^n$ be the unit ball in $\mathbb{R}^n$. For $x \in B^n$ and $0 \neq y \in \mathbb{R}^n$, define $\Theta = \Theta(x, y) > 0$ to be the number such that
\[
x + \frac{y}{\Theta} \in \partial B^n.
\]
Find an explicit formula for the following function
\[
F := \frac{\Theta(x, y) + \Theta(x, -y)}{2}
\]
and compare it with the function in (1). This metric is called the Klein metric on $B^n$.

4 Let $\rho = \rho(p)$ be a function on a Riemannian manifold $(M, g)$. In a local coordinate system $\varphi = (x^i)$ in $M$, let $G^i = \frac{1}{2} \Gamma^i_{jk} (x) y^j y^k$ denote the geodesic coefficients of $g$. Consider a new Riemannian metric defined by
\[
\tilde{g} := e^{\rho} g.
\]
Express the geodesic coefficients $\tilde{G}^i = \frac{1}{2} \tilde{\Gamma}^i_{jk} (x) y^j y^k$ of $\tilde{g}$ in terms of $G^i$ and the partial derivatives of $\rho$ as a function $\rho(x) := \rho \circ \varphi^{-1}(x)$ defined on $\mathbb{R}^n$.

5 Let $g := \frac{4 \delta_{ij}}{(1 - |x|^2)^2} dx^i \otimes dx^j$ be the Poincaré metric on the unit ball $B^n \subset \mathbb{R}^n$. Find the geodesic coefficients $G^i = \frac{1}{2} \Gamma^i_{jk} (x) y^j y^k$. 
