Alfvén waves of a nonuniform plasma

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The spectral problem is analyzed for a fourth-order equation in which the coefficient of the highest derivative has a small parameter. This problem describes the Alfvén waves of a nonuniform plasma in cylindrical geometry. A series of Alfvén modes with frequencies $\gamma_s = i\nu + C_0\sigma^{-1/3}$ where $s = 1, 2, \ldots$, and $\sigma \gg 1$ is the plasma conductivity, is constructed. These are damped wave modes. The general method of the analysis is to join the solutions in the boundary layer near a singularity and outside this layer and to solve the equation for the boundary conditions in a complex plane.

1. INTRODUCTION

In this paper we construct a series of Alfvén wave modes of a nonuniform plasma. For definitiveness we consider the Alfvén waves in a cylindrical plasma in a strong longitudinal magnetic field. This model corresponds to the plasma configuration in a tokamak, but the present method is more general and can be used to study the Alfvén waves in a variety of plasma configurations. This method is based on an extremely detailed study of the asymptotic expansions of a fourth-degree equation in which the coefficient of the highest derivative contains a small parameter.

The Alfvén waves in a plasma in a strong longitudinal field are described by the linearized and simplified Kadowitz-Pogutse MHD equations. In dimensionless variables, the corresponding equations of Alfvén waves, $\psi(r) \exp(\ln \phi - \ln z + \gamma t)$, $\psi'(r) \exp(\ln \phi - \ln z + \gamma t)$, are

$$\gamma \Delta \psi + \left(\frac{m^2}{k_f^2}\right) \psi = 0,$$

$$\Delta \psi = \gamma \left(\frac{k_f^2}{k_t^2}\psi\right),$$

where

$$k_f^2 = \frac{q_s}{q} - \frac{1}{m^2} - \frac{1}{m^2},$$

$$k_t^2 = \frac{q_s}{q} - \frac{1}{m^2} - \frac{1}{m^2},$$

is the safety factor,

$$q^{-1}(r) = r^{-1} \int \frac{\psi}{\psi} \, r \, dr,$$

$|\psi(0)| < \infty$, $|\psi(0)| < \infty$,

$$\psi'(0) = 0, \quad \psi'(0) = \lambda \psi(0),$$

where $\lambda = -m(b_m + b_m)/b_m - b_m$. From (2) we find

$$q = \left(\frac{1}{c^3} \Delta \psi \right) \frac{k_f^2}{k_t^2}.$$

Substituting this relation into (1), and carrying out some simple manipulations, we find one fourth-order equation for $\psi$:

$$\frac{\gamma}{k_t^2} \Delta \gamma \Delta \psi + \frac{1}{k_t^2} \left(1 + \frac{\nu}{k_t^2}\right) \frac{d\psi}{dr} + D\psi = 0,$$

where

$$D = \frac{1}{k_t^2} \left(\frac{1}{k_f^2}\right) \frac{d^2}{dr^2} - \frac{m^2}{r} - \frac{m^2}{r}.$$
where the functions \( \Phi_1(r), \Phi_2(r), \) and \( \Phi_3(r) \) have no singularities at \( r = r_0; \) \( \Phi_1(r_0) = \Phi_2(r_0) = 1; \) \( \Phi_3(r_0) = 0; \) the superscripts \( \pm \) in \( C_{1}^{\pm} \) and \( C_{2}^{\pm} \) correspond to constants in the regions \( r > r_0 \) and \( r < r_0 \); \( C_{1}^{-} = C_{2}^{+}. \) Because of the logarithmic singularity, the kinetic energy \( T = \int \{ r \Phi(q(r)) + r \Phi'(q(r)) \} \, dr, \) where \( \Phi = -\Phi' \), of the oscillations described by a generalized eigenfunction becomes infinite.

We introduce

\[
\gamma^- = \min_{0 < \gamma < 1} k^2(r)/\rho(r), \quad \gamma^+ = \max_{0 < \gamma < 1} k^2(r)/\rho(r)
\]

The continuous spectrum fills the segments \([i\gamma^-, i\gamma^+]\) and \([i\gamma^-, -i\gamma^+]\) along the imaginary axis. In addition to the continuous spectrum, Eq. (7) (with the appropriate boundary conditions) may have points of a discrete spectrum. The discrete spectrum for the case of a parabolic current profile \( j(r) = j_0 (1 - r^2) \) was studied numerically in Ref. 9, and it was found that the discrete spectrum (discrete in \( \gamma \)) consists of a single point. Figure 1, from Ref. 9 for the \( m = 2 \) mode, shows the \( q_a \) dependence of the continuous spectrum (the hatched region) and the discrete spectrum (the solid line). The density is assumed to have a profile \( \rho(r) \in 1. \) Interestingly, for \( 2 < q_a < 2.6 \) the discrete point in the spectrum disappears (the dashed line). By analogy with problems in quantum mechanics, this point moves onto the "nonsense" sheet of the analytic continuation of the resolvent (it goes to a resonance).

When we transform from the simplified equation (7) to the complete equation (6) we find that the continuous spectrum breaks up into a discrete spectrum. This spectrum was studied in Refs. 8–13 and in other papers (see Ref. 13 for a bibliography). A plane model was studied in Ref. 8, and regions were found in which there were definitely no points of the spectrum of the complete equation. In particular, the following, somewhat unexpected, result was found: With \( \gamma \) denoting a point of the continuous spectrum, the corresponding singularity of the simplified equation lies a distance \( \approx \sigma^{-1/2} \) from the plasma boundary. There are then no points of the spectrum of the complete equation near \( \gamma; \) in other words, the typical points \( \gamma \) of the continuous spectrum do not (as might be expected) give rise to points of the discrete spectrum of the complete system in their vicinity. Then where do the points of the discrete spectrum of the complete equation appear? We seek the answer to this question below.

We introduce \( \gamma = |k_0| (1/ \sqrt{|\rho(1)|}) \), so that at \( \gamma = \pm 1 \) the singularity \( r_0 \) lies at the plasma boundary: \( r_0 = 1. \) We will show that a series of eigenvalues \( \gamma_s (s = 1, 2, \ldots) \) moves away from the point \( 1 \gamma \) at an angle of \( \pi/6 \) from the imaginary axis (Fig. 2). Our calculations yield an asymptotic formula for the points of the spectrum:

\[
\gamma_s \approx \frac{2n_1}{3} \text{ if } (k^2(p_0)^{(1)}) > 0,
\]

\[
\gamma_s \approx \frac{2n_1}{3} \text{ if } (k^2(p_0)^{(1)}) < 0,
\]

where \( C = (1/2) (1/ \sqrt{|\rho(1)|}) (k_0^2)^{(1)} \), \( \sigma = \sigma(1) \gg 1, \) and \( x_8 > 0 \) is the negative of the \( 8 \)-th zero of the Airy function \( Ai(x). \) The first few values of \( x_8 \) are 2.34, 4.09, 5.52, 6.79, 7.94, and 9.02 (Ref. 14). The following asymptotic formula holds:

\[
x_s = \left( \frac{3n_1}{2} - (s - 4/3) \right), \quad s \gg 1.
\]

Expression (10) is valid for \( s \ll \sigma^{-1/2}. \) Our method shows that at \( |\gamma - 1\gamma| \ll 1 \) there are no points of the spectrum which differ from \( \gamma \). Since \( E \) is real, a complex-conjugate series of points \( \gamma_s (s = 1, 2, \ldots) \) moves away from the point \( -1\gamma \).

In the derivation of (10) we assume

\[
(k^2(p_0)^{(1)}) \neq 0, \quad k_0(p_0) \neq 0
\]

(11)

(the condition that the plasma be nonuniform and the condition that there be no resonance at the plasma boundary), and for \( \gamma = 1\gamma \) we assume that the solution of simplified equation (7), with the condition \( \Phi(0) \ll \infty, \) has a logarithmic singularity at the extreme point \( r = 1. \) This assumption is the condition for the general case of this problem, and it continues to hold for the case of small perturbations of the functions \( j, \rho, \) and \( \sigma. \) We note that these assumptions are important, and if they are not correct expression (10) becomes incorrect.

The eigenfunctions \( \psi_s(r) \) corresponding to the eigenvalues \( \gamma_s \) are complex. Figure 3 shows sketches of the functions \( \text{Re} \psi_s(r) \) and \( \text{Im} \psi_s(r). \)

2. CONSTRUCTION OF THE EIGENFUNCTIONS

Equation (6) has the two slowly varying solutions \( \psi_1(r) \) and \( \psi_2(r), \) which become the solutions of the simplified equation (7) in the limit \( \sigma \to \infty, \) and it also has two rapidly varying solutions, a decreasing solution \( \psi_3(r) \) and an increasing solution \( \psi_4(r) \) (Ref. 15). We assume that a singularity \( r_0 \) of the simplified equation lies near the plasma boundary; i.e., we assume \( r_0 \ll 1 \) and \( r_0 \to 1 \) in the limit \( \sigma \to \infty. \) Since the eigenvalue \( \gamma \) is generally complex, the point \( r_0 \) lies in the complex plane. In the
limit \( \sigma \to \infty \), \( \psi_1(\tau) \) becomes a smooth solution of the simplified equation, while \( \psi_2(\tau) \) becomes a solution with a logarithmic singularity at the point \( \tau = 1 \).

The general solution of the complete equation (6) can be written in the form 
\[ \psi(\tau) = \sum_{i=1}^{n} C_i \psi_i(\tau). \]
Near the plasma boundary \( \tau = 1 \), only the slowly varying solutions \( \psi_1(\tau) \) and \( \psi_2(\tau) \) and the rapidly growing solution \( \psi_3(\tau) \) are important, and \( \psi(\tau) \) can be written in the form 
\[ \psi(\tau) = C_1 \psi_1(\tau) + C_2 \psi_2(\tau) + C_3 \psi_3(\tau) \]
where \( C_2 \psi_2(\tau) \) is negligible near the plasma boundary \( \tau = 1 \). The constants \( C_1 \), \( C_2 \), and \( C_3 \) are defined within a factor. Outside the neighborhood of the boundary, \( \psi(\tau) \) becomes the solution of the simplified equation 
\[ \psi(\tau) = C_1 \psi_1(\tau) + C_2 \psi_2(\tau). \]
The ratio \( C_1/C_2 \) is determined from the solution of the simplified equation. Specifically, to find \( C_1/C_2 \), we solve simplified equation (9) with the condition \( \psi(0) = 0 \) at \( \tau = 0 \), and then approximate \( \psi(\tau) \) near a singularity.

In the region \( \sigma^{-1/3} \ll |\tau - r_0| \ll 1 \) on the complex \( \tau \)-plane, the asymptotic behavior of the solutions \( \psi_1(\tau) \), \( \psi_2(\tau) \), and \( \psi_3(\tau) \) can be found by the method of Ref. 10:
\[ \psi_1(\tau) \sim 1, \quad \psi_2(\tau) \sim 0, \quad \psi_3(\tau) \sim |\tau - r_0|^{\gamma}, \]
where \( x = (\tau - r_0)^{1/3} \) and \( B = (1/\gamma) (k^{1/\rho}) (r_0)^{1/3} \). These expressions, however, are not adequate for our purpose, since we wish to know the asymptotic behavior of the solutions at \( |\tau - r_0| \ll 1 \).

It is shown in the appendix that for \( |\tau - r_0| \ll 1 \) the following asymptotic formulas hold:
\[ \psi_1(\tau) \sim 1, \quad \psi_2(\tau) \sim 0, \quad \psi_3(\tau) \sim (|\tau - r_0|)^{\gamma}, \]
where \( x = (\tau - r_0)^{1/3} \) and \( B = (1/\gamma) (k^{1/\rho}) (r_0)^{1/3} \). These equations determine the singular points \( r_0 \) at which the boundary conditions \( \psi(1) = \psi(1) = 0 \) hold. The singular point \( r_0 \) is related to the eigenvalue \( \gamma \) by the equation 
\[ (k^{1/\rho}) (r_0)^{1/3} + \gamma^3 = 0. \]

For this equation we use the same estimates as above, and we find Eq. (18).

We have thus shown that the solutions \( x \) of Eq. (17) lie near the zeros \( -\sigma x = 1, 2, \ldots \) of the Airy function \( A(3) \).

For this reason, we conclude that the boundary conditions \( \psi(1) = \psi(1) = 0 \) hold.
The solutions of the equation

$$A(\gamma) = \sum_{l=0}^{\infty} a_l (1-r_0)^l$$

where $A(\gamma) = \sum_{l=0}^{\infty} a_l (1-r_0)^l$ and $r_0$ is expressed in terms of $\gamma$ by the equation $1 + 2\alpha^2 \rho/k_0^2 (r_0) = 0$.

We have calculated $\gamma_0$ in second-order approximation theory in $x_0 \sigma^{-1/3}$ for the case $\rho(r) = 1$; we found

$$\gamma_0 = \gamma^0 + C_1 x_0 \sigma^{-\alpha} \exp \left( \pm \frac{n_i}{3} \right) + C_2 (x_0 \sigma^{-\alpha}) \exp \left( \pm \frac{n_i}{6} \right),$$

where

$$C_1 = - (1/2) \{ \langle k_0^2 \rangle \langle 1 \rangle \langle \sigma^2 \rangle \}, \quad C_2 = - (1/24) \{ \langle k_0^2 \rangle \langle 1 \rangle \langle \sigma^2 \rangle \} \times \langle k_0^2 \rangle \langle 1 \rangle.$$

This expression is a refinement of (10) for $\rho(r) = 1$ and $\sigma(r) = \text{const} \gg 1$.

**APPENDIX**

We seek the asymptotic form of the solutions of a fourth-order equation near a singular point.

We denote by $r_0$ a singular point of the simplified equation. Near $r_0$, the complete equation has solutions of the boundary-layer type. The thickness of the boundary layer is of the order of $\sigma^{-1/3}$. Inside the boundary layer we introduce the coordinate $x = (r - r_0) \sigma^{-1/3}$, where $x = [\gamma (k_0^2/\rho) (r_0) (\sigma^2/\rho) (r_0)]^{1/3} = [\gamma (k_0^2/\rho) (r_0)]^{1/3}$. Substituting $\psi(x) = V(x)$ into the complete equation (6), and retaining the leading terms in $\sigma$, we find the model equation

$$\frac{dV}{dx} = \frac{d^2V}{dx^2}$$

whose solutions determine the asymptotic behavior of $\psi(r)$ at $r \approx r_0$. For $W = V^1$, this equation reduces to the homogeneous Airy equation

$$W'' = z W + \text{const.}$$

The solutions of this equation are the Airy function $A(z)$ and the orthogonality condition $W(x) = \int \exp(iax + i\alpha^2/3) du$.

Correspondingly, Eq. (A.1) has the following fundamental system of solutions: $V_1(x) = 1$, $V_2(x) = \int W(x) dx$, $V_3(x) = \int \exp(3ax + 3\alpha^2/3) dx$.

The functions $V_2(x)$ and $V_4(x)$ are determined within constant terms. We choose these terms in such a manner that we have $V_2(x) = -\ln x - \ln(\sigma/x^3) - \alpha x$ in the sector $-\pi/3 < arg x < \pi$ and $V_4(x) = -\ln x - \ln(\sigma/x^3) - \alpha x$ in the sector $\pi/3 < arg x < \pi/3$. This choice guarantees the asymptotic behavior $\psi_2 \approx \psi_1(\rho)$ in the sector $1 > r_0 - r > \sigma^{-1/3}$. Equation (A.1) has the property that if $V(x)$ is a solution of this equation, then $V(ax)$, where $a = \exp(2\sigma/3)$ or $\exp(4\sigma/3)$, is also a solution. Where necessary, we can thus replace the function $V_1(x)$ by $V_4(x)$ and thereby rotate the sector in which this function decreases or slowly grows.
Since \( A_l^n = xA_l \) and \( W_l^n = xW_l - 1 \), we have \( W^nA_l = xW^nA_l - xW^nA_l + A_l = A_l \) or \( V_2^nV_4^n - V_2^nV_4' = V_4' \).


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