Estimates of Green Functions of Difference Operators in Arbitrary Domains and Their Applications*

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§1. Introduction

Let $M_n$ be the set of all $n \times n$ complex matrices and $\mathcal{A} = l_1(\mathbb{Z}'', M_n)$ the set of functions $a: \mathbb{Z}' \to M_n$ such that

$$\|a\|_1 = \sum_{x \in \mathbb{Z}'} |a_x(x)| < \infty,$$

where $|a(x)|$ is the norm of the matrix $a(x)$ as an operator from $\mathbb{C}^n$ to $\mathbb{C}^n$. The set $\mathcal{A}$ is a Banach algebra with respect to convolution:

$$(a \ast b)(x) = \sum_{y \in \mathbb{Z}'} a(x - y)b(y).$$

The element $e = e(x) = \delta_{x,0}E$, where $E$ is the unit matrix, is the unit in $\mathcal{A}$. Let $H = l_1(\mathbb{Z}''', \mathbb{C}')$ be the Hilbert space of functions $f: \mathbb{Z}'' \to \mathbb{C}'$ with the scalar product

$$(f, g) = \sum_{x \in \mathbb{Z}'''} f(x)\overline{g(x)}.$$  

An element $a \in \mathcal{A}$ defines a linear operator $A: H \to H$ by the formula

$$(Af)(x) = \sum_{y \in \mathbb{Z}'''} a(x - y)f(y) = (a \ast f)(x);$$

$a \ast b$ defines the operator $AB$, and $\|A\| \leq \|a\|_1$. Assume that $a \in \mathcal{A}$ is invertible, $a \ast g = g \ast a = e$ and $(Gf)(x) = \sum_{y \in \mathbb{Z}'''} g(x - y)f(y)$. Let $\Lambda \subset \mathbb{Z}'', H_\Lambda = l_2(\Lambda, \mathbb{C}'')$, $A: H_\Lambda \to H_\Lambda$ be defined by the formula

$$(A_\Lambda f)(x) = \sum_{y \in \Lambda} a(x - y)f(y).$$

Suppose that \( A_\Lambda \) is invertible, i.e. there exists \( T_\Lambda \) such that \( T_\Lambda A_\Lambda = A_\Lambda T_\Lambda = I \); then
\[
(T_\Lambda f)(x) = \sum_{y \in \Lambda} t_\Lambda(x, y) f(y).
\]

The main goal of this paper is to derive estimates of \( |t_\Lambda(x, y)| \) and \( |t_\Lambda(x, y) - g(x - y)| \) for arbitrary subsets \( \Lambda \subset Z^r \). The necessity of estimates of this kind arises in various problems of probability theory and statistical physics. They are of special interest in connection with the recent study of Gaussian random fields (see [1]) and their perturbations (see [2], [7]). In addition they are interesting in the problem of interpolation of random fields and in the general theory of difference schemes [3], [6]. In connection with the latter, note that the proof of the estimates in §2 might be carried out by making use of a priori estimates based on the maximum principle more familiar in the theory of difference schemes. However, our method seems somewhat simpler. Note also that the estimates from §3 directly concern difference operators with constant coefficients but can be carried over without much complication to operators of divergent form with variable coefficients.

The paper consists of three parts. In the first part we obtain quite precise estimates of \( |t_\Lambda(x, y)| \) and \( |t_\Lambda(x, y) - g(x - y)| \) under one of two assumptions: either
\[
a = e - b, \quad \|b\|_1 < 1
\]  
(1.1)

or
\[
g = e - h, \quad \|h\|_1 < 1.
\]  
(1.2)

The proof of these estimates is based on some perturbation-theoretic developments in powers of \( b \) or \( h \).

In the second part we establish an estimate of \( |t_\Lambda(x, y)| \) when instead of (1.1) and (1.2) we assume the strict positivity of \( A \). This estimate generalizes the result of Dobrushin and Zagradnik [2], where the fact that \( a(x) \) has compact support was essential. At the same time the deduction of our estimate is based on an important idea from [2].

It is relevant to note here that the invertibility of \( A \) does not guarantee the invertibility of \( A_\Lambda \), much less any estimates of \( |t_\Lambda(x, y)| \). Even in the 1-dimensional case, for the simplest situation, when \( \Lambda \) is a half line, the question of the invertibility of \( A_\Lambda \) is, as we know, a rather difficult problem; see [4]. Therefore it is necessary to impose extra restrictions on \( a(x) \). Our conditions (1.1) and (1.2) or the positivity condition seem to be rather natural. This seems especially true for the positivity condition, since in most of the applications \( A \) serves as the operator of a positive definite quadratic form and might be supposed to be positive. In both the first and the question of what can be \( |t_\Lambda(x, y) - g(x - y)| \) if it is better estimate. In the last part of the estimates.

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§2. Perturbation Theory

First suppose that (1.1) is a perturbation theory \( g = a^+ \) and \( \|g\|_1 \leq 1 + \|b\|_1 + \|h\|_1 \) for the formula \( \delta(x) = |b(x)| \). Let
\[
g \in I_1(Z^r, R^1)
\]
and \( \tilde{g} \in I_1(Z^r, R^1) \). For \( \Omega \subset Z^r \)

**Theorem 2.1.** If (1.1) holds
\[
|t_\Lambda(x, y)| \leq \sum_{z_1, \ldots, z_r \in \Omega} |b(x, y)|.
\]

where
\[
L = \|a\|_1, \quad d = \max\{|a(x)| : x \in \Omega\}
\]

**Proof.** Let \( b_\Lambda'(x, y) \) be the
\[
|b_\Lambda'(x, y)| = \sum_{z_1, \ldots, z_r \in \Omega} \sum_{e \in E} e \left| \sum_{z \in \Lambda} \phi(z, e) t_\Lambda(x, z) \right|.
\]

(2.1)

It is easy to calculate
\[
L = L_0 + \sum_{z_1, \ldots, z_r \in \Omega} \sum_{e \in E} e \left| \sum_{z \in \Lambda} \phi(z, e) t_\Lambda(x, z) \right|.
\]

(2.2)

It is also easy to calculate
\[
\sum_{z_1, \ldots, z_r \in \Omega} \sum_{e \in E} e \left| \sum_{z \in \Lambda} \phi(z, e) t_\Lambda(x, z) \right|.
\]

(2.3)

Finally, we can calculate
\[
|b_\Lambda'(x, y)| = \sum_{z_1, \ldots, z_r \in \Omega} \sum_{e \in E} e \left| \sum_{z \in \Lambda} \phi(z, e) t_\Lambda(x, z) \right|.
\]

(2.4)

Therefore, we see that (2.1) is a perturbation theory for the formula (2.2).

In the second part we establish an estimate of \( |t_\Lambda(x, y)| \) when instead of (1.1) and (1.2) we assume the strict positivity of \( A \). This estimate generalizes the result of Dobrushin and Zagradnik [2], where the fact that \( a(x) \) has compact support was essential. At the same time the deduction of our estimate is based on an important idea from [2].

It is relevant to note here that the invertibility of \( A \) does not guarantee the invertibility of \( A_\Lambda \), much less any estimates of \( |t_\Lambda(x, y)| \). Even in the 1-dimensional case, for the simplest situation, when \( \Lambda \) is a half line, the question of the invertibility of \( A_\Lambda \) is, as we know, a rather difficult problem; see [4]. Therefore it is necessary to impose extra restrictions on \( a(x) \). Our conditions (1.1) and (1.2) or the positivity condition seem to be rather natural. This seems especially true for the positivity condition, since in most of the applications \( A \) serves as the operator of a positive definite quadratic form and might be supposed to be positive.
such that $T \Lambda_A = A \Lambda T = 1$.

We have estimated $|t_\Lambda(x, y)|$. The necessity of estimates of $|t_\Lambda(x, y)|$ and $|t_\Lambda(x, y) - g(x - y)|$ is discussed in the recent study of Gaussian fields [2, 7]. In addition, they are frequent fields and in the general case, the latter, note that the fact by making use of a priori estimates familiar in the theory of somewhat simpler. Note also that constant operators with constant complication to operators of

In this part we obtain quite precise estimates under one of two assumptions:

\begin{equation}
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where \( 
abla^j \phi(x) = \nabla * \cdots * \nabla \phi(x) \) \((j\text{ times})\). Thus
\[
|t_\Lambda(x, y)| = \left| \sum_{j=0}^{\infty} \nabla^j \phi(x, y) \right| \leq \sum_{j=0}^{\infty} \nabla^j \phi(x - y) = \tilde{g}(x - y).
\]
Relation (2.2) is proved.

To prove (2.3) assume that \( d(y, \Lambda') \geq d(x, \Lambda') \). The opposite case may be considered by using, for example, dual operators.

Since \( T_\Lambda A_\Lambda = 1 \), the identity \( T_\Lambda - G_\Lambda = -T_\Lambda (A_\Lambda G_\Lambda - 1) \) holds. Since \( AG = 1 \), the kernel of \( A_\Lambda G_\Lambda - 1 \) can be transformed into the form
\[
\sum_{z \in \Lambda} a(x - z)g(z - y) - \delta(x - y) = \left( \sum_{z \in \Lambda} a(x - z)g(z - y) - \delta(x - y) \right)
- \sum_{z \in \Lambda} a(x - z)g(z - y)
= - \sum_{z \in \Lambda} a(x - z)g(z - y).
\]

Therefore
\[
t_\Lambda(x - y) - g(x - y) = \sum_{z \in \Lambda} \sum_{z' \in \Lambda'} t_\Lambda(x, z) a(z' - z') g(z' - y).
\]

It is clear that
\[
|g(z' - y)| \leq \max_{|z' - y| \geq d(y, \Lambda')} |g(z' - y)| = \eta(d);
\]

hence
\[
|t_\Lambda(x, y) - g(x - y)| \leq \sum_{z \in \Lambda} \sum_{z' \in \Lambda'} |t_\Lambda(x, z)| \cdot |a(z' - z')| \cdot \eta(d)
\leq \eta(d) \sum_{z \in \Lambda} \tilde{g}(x - z) \sum_{z' \in \Lambda'} |a(z' - z')|
\leq \eta(d) \sum_{z \in \Lambda} \tilde{g}(x - z) \sum_{z' \in \Lambda'} |a(z' - z')| = \eta(d) \| \tilde{g} \|_1 \cdot \| a \|_1.
\]

Theorem 2.1 is proved.

Now suppose that instead of (1.1), condition (1.2) holds. Put
\[
\hat{\Lambda}(x) = |\hat{h}(x)|, \quad \hat{a} = e + \hat{h} = \hat{h} * \cdots = (e - \hat{h})^{-1}, \quad \| g \|_1 = \| g \|.
\]

Theorem 2.2. When (1.2) holds,
\[
|t_\Lambda(x, y)| \leq \| t_\Lambda \|_1 \cdot \| a \|_1 \cdot \| g \|_1, \quad \text{and}
\]
where \( L_0 = \| \hat{a} \|_1 \cdot \| g \|_1, \) and \( \tilde{g} \) is a nonnegative measure.

Proof. Let \( q_\Lambda(x, y) \) be the kernel of \( A_\Lambda - 1 \).

The main point in the estimate for \( q_\Lambda(x, y) \). To obtain this, we can consider \( q_\Lambda(x, y) \) as \( 2 \times 2 \) matrix operators, namely
\[
A = \begin{pmatrix} A_\Lambda & V \\ \Lambda & \Lambda' \end{pmatrix}
\]

Since \( AG = 1 \), we have
\[
A_\Lambda G_\Lambda + V \sigma = 1,
\]

implying that \( G_\Lambda = T_\Lambda V_\sigma W_1 \)
\[
T_\Lambda = A_\Lambda^{-1} = 0,
\]
or
\[
t_\Lambda(x, y) = g(x - y).
\]

Now it follows from (2.6) that
\[
|t_\Lambda(x, y) - g(x - y)| \leq \| t_\Lambda \|_1 \cdot \| \tilde{g} \|_1 \cdot \| a \|_1 \cdot \| g \|_1.
\]

proving (2.4).

To prove the estimates (2.5),
\[
|g(z' - y)| \leq \| g \|_1 \cdot \| a \|_1 \cdot \| g \|_1
\]
Theorem 2.2. When (1.2) holds then

\[ |t_\Lambda(x, y)| \leq |g(x - y)| + |g| \cdot \hat{a} \cdot |g|(x - y), \]
\[ |t_\Lambda(x, y) - g(x - y)| \leq L_0 \eta(d), \]

where \( L_0 = \|\hat{a}\|_1 \cdot \|g\|_1 \), and \( \eta(t) \) and \( d \) are defined in Theorem 2.1.

Proof. Let \( q_\Lambda(x, y) \) be the kernel of \( Q_\Lambda = G_\Lambda^{-1} \). By Theorem 2.1,

\[ |q_\Lambda(x, y)| \leq \hat{a}(x - y). \]

The main point in the estimation of \( t_\Lambda(x, y) \) is a formula which relates \( t_\Lambda(x, y) \) and \( q_\Lambda(x, y) \). To obtain this, let us follow [7] and write the operators \( A \) and \( G \) as \( 2 \times 2 \) matrix operators, making use of the decomposition \( Z' = \Lambda \cup \Lambda' \):

\[ A = \begin{pmatrix} A_\Lambda & V_0 \\ V_1 & A_\Lambda' \end{pmatrix}, \quad G = \begin{pmatrix} G_\Lambda & W_0 \\ W_1 & G_\Lambda' \end{pmatrix}. \]

Since \( AG = 1 \), we have

\[ A_\Lambda G_\Lambda + V_0 W_1 = 1, \quad A_\Lambda W_0 + V_0 G_\Lambda = 0, \]

implying that \( G_\Lambda + T_\Lambda V_0 W_1 = T_\Lambda \) and \( W_0 + T_\Lambda V_0 G_\Lambda = 0 \); therefore

\[ T_\Lambda = A_\Lambda^{-1} = G_\Lambda - W_0 G_\Lambda^{-1} W_1 = G_\Lambda - W_0 Q_\Lambda W_1, \]

or

\[ t_\Lambda(x, y) = g(x - y) - \sum_{z \in \Lambda'} \sum_{z' \in \Lambda'} g(x - z) q_\Lambda(z, z') g(z' - y). \]

Now it follows from (2.6) that

\[ |t_\Lambda(x, y) - g(x - y)| \leq \sum_{z \in \Lambda'} \sum_{z' \in \Lambda'} |g(x - z)| \cdot \hat{a}(z - z') \cdot |g(z' - y)| \]

\[ \leq \sum_{z \in \Lambda'} \sum_{z' \in \Lambda'} |g(x - z)| \cdot \hat{a}(z - z') \cdot |g(z' - y)| \]

\[ = |g| \cdot \hat{a} \cdot |g|(x - y), \]

proving (2.4).

To prove the estimates (2.5), suppose that \( d(y, \Lambda') \geq d(x, \Lambda') \) and notice that

\[ |g(z' - y)| \leq \max_{|z' - y| \geq d(y, \Lambda')} |g(z' - y)| = \eta(d) \]
for $z' \in \Lambda'$. Hence
\[
|t_\Lambda(x, y) - g(x - y)| \leq \eta(d) \sum_{z \in \Lambda'} \sum_{z' \in \Lambda'} |g(x - z)| \cdot \dot{a}(z - z')
\leq \eta(d) \sum_{z \in \Lambda'} \sum_{z' \in \Lambda'} |g(x - z)| \cdot \dot{a}(z - z') = \eta(d) \cdot \|g\|_1 \cdot \|\dot{a}\|_1.
\]

Theorem 2.2 is proved.

Now let us discuss what we gain from Theorem 2.1 in two cases that are interesting in applications: (1) $a(x)$ decreases exponentially (in particular, has compact support); and (2) $a(x)$ admits a power estimate. Similar deductions can be made from Theorem 2.2 by considering different rates of decrease of $g(x)$.

First suppose that
\[
|a(x)| < C_0 \exp(-\gamma|x|)
\]
for some $\gamma > 0$ and that (1.1) holds. Then the Fourier series
\[
\tilde{a}(\tau) = \sum_{x \in \Lambda'} \exp(i\tau x) a(x)
\]
of $a(x) = \delta_{x,0} - |b(x)|$ defines a periodic analytic function with analytic continuation to the domain $|\text{Im} \tau| < \gamma$,
\[
\text{Im} \tau = (\text{Im} \tau_1, \ldots, \text{Im} \tau_n).
\]
For real $\tau$ we have
\[
|\tilde{a}(\tau)| \geq 1 - |\tilde{b}(\tau)| = 1 - \left| \sum_{x \in \Lambda'} \exp(i\tau x) b(x) \right| \geq 1 - \sum_{x \in \Lambda'} |b(x)| > 0,
\]
i.e. $\tilde{a}(\tau)$ does not vanish. Since $\tilde{a}(\tau)$ is periodic, by continuity it also does not vanish for small $|\text{Im} \tau|$. Let $\tau_0$ be a root of $\tilde{a}(\tau)$ with minimal $|\text{Im} \tau|$; and if $\tilde{a}(\tau)$ has no zeros in the domain $|\text{Im} \tau| < \gamma$, put $|\text{Im} \tau_0| = \infty$. Let $0 < \gamma' < \min\{\gamma, |\text{Im} \tau_0|\}$. Then $\tilde{a}^{-1}(\tau)$ is analytic in the domain $|\text{Im} \tau| \leq \gamma'$ and by the Paley–Wiener theorem the function
\[
\hat{g}(x) = (2\pi)^{-n} \int_{-\gamma}^{\gamma} \cdots \int_{-\gamma}^{\gamma} \exp(-i\tau x) \tilde{a}^{-1}(\tau) d\tau
\]
decreases exponentially, i.e. $\hat{g}(x) < C_1 \exp(-\gamma'|x|)$. By Theorem 2.1 we have
\[
|t_\Lambda(x, y)| \leq C_2 \exp(-\gamma' |x - y|),
\]
where $C_2 = C_1 L$, $d = \max\{d(x, \Lambda'), d(y, \Lambda')\}$.

Now suppose that condition
\[
|a(x)| < C_0(1 + |x|)^{-r},
\]
where $A_0$ is chosen so large that
\[
\beta(x) = \frac{A_0}{|x|^r}
\]
This is possible since
\[
\sum_{x \in \Lambda'} |b(x)| = \tilde{b}(x) \geq 0,
\]
\[
\tilde{b}(x) = 1 + \tilde{b}(x) + (\tilde{b} \ast \tilde{b})
\]
therefore it suffices to estimate
\[
\beta(x)
\]
of $\tilde{b}(x)$ possesses the following property:

1. $\beta(x)$ is periodic,
2. $1 - \beta(x) \neq 0$ for real $\tau$,
3. $\beta(x)$ is infinitely differentiable,
4. for any $N > 0$ the function
\[
\beta(x) = C_0 \exp(-\gamma'|x|)
\]
where $C_0$ is chosen so large that
\[
\beta(x) \geq 0.
\]
This implies that $(1 - \beta(x))$ satisfies the inequality
\[
\tilde{g}(x) = (2\pi)^{-n} \int_{-\gamma}^{\gamma} \cdots \int_{-\gamma}^{\gamma} \exp(-i\tau x) \tilde{a}^{-1}(\tau) d\tau \leq C_1 \exp(-\gamma' |x|)$. 

Green Functions of Difference Operators

Now suppose that condition (1.1) is satisfied and \( |a(x)| \) admits the estimate
\[
|a(x)| < C_0(1 + |x|)^{-\gamma},
\]
where \( \gamma > v \). We claim that \( \bar{g}(x) \) admits the similar estimate
\[
\bar{g}(x) < C_1(1 + |x|)^{-\gamma}.
\]
To prove this, consider the function
\[
\bar{b}(x) = \begin{cases} |b(x)| & \text{for } |x| < A_0, \\ C_0(1 + |x|)^{-\gamma} & \text{for } |x| \geq A_0,
\end{cases}
\]
where \( A_0 \) is chosen so large that
\[
\sum_{x \in \mathbb{Z}^r} \bar{b}(x) < 1.
\]
This is possible since
\[
\sum_{x \in \mathbb{Z}^r} |b(x)| < 1 \quad \text{and} \quad \gamma > v.
\]
Since \( \bar{b}(x) \geq |b(x)| = \bar{b}(x) \geq 0 \) we have
\[
\bar{g}(x) = 1 + \bar{b}(x) + (\bar{b} \ast \bar{b})(x) + \cdots \geq 1 + \bar{b}(x) + (\bar{b} \ast \bar{b})(x) + \cdots = \bar{g}(x);
\]
therefore it suffices to estimate \( \bar{g}(x) \). The Fourier transform
\[
\beta(\tau) = \sum_{x \in \mathbb{Z}^r} \exp(i\tau x) \bar{b}(x)
\]
of \( \bar{b}(x) \) possesses the following properties:

1. \( \beta(\tau) \) is periodic,
2. \( 1 - \beta(\tau) \neq 0 \) for real \( \tau \),
3. \( \beta(\tau) \) is infinitely differentiable for \( \tau \neq 0 \),
4. for any \( N > 0 \) the function \( \beta(\tau) \) admits the decomposition
\[
\beta(\tau) = \sum_{0 \leq j \leq N} L_j |\tau|^{-v+j} + \beta_N(\tau),
\]
where \( \beta_N(\tau) \in C^N(\mathbb{R}^r) \).

This implies that \((1 - \beta(\tau))^{-1}\) also possesses these properties, and in turn that the function
\[
\bar{g}(x) = (2\pi)^{-v} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \exp(-i\tau x)(1 - \beta(\tau))^{-1} d\tau
\]
satisfies the inequality \( \bar{g}(x) \leq C_1(1 + |x|)^{-\gamma} \). Therefore (2.6) is proved.
Now Theorem 2.1 implies that
\[ |t_\Lambda(x, y)| < C_1 (1 + |x - y|)^{-\gamma}, \quad |t_\Lambda(x, y) - g(x - y)| < C_1 (1 + d)^{-\gamma}, \quad (2.7) \]
where \( C_2 = C_1 L, d = \max\{d(x, \Lambda'), d(y, \Lambda')\} \).

§3. Positive operators

In this section instead of conditions (1.1) and (1.2) we will impose a positivity condition on the operator \( A \), i.e.
\[ (Af, f) = \sum_{x, y \in \mathbb{Z}} (a(x - y)f(y), f(x)) \geq c \|f\|^2, \quad (3.1) \]
where \( c > 0 \). Since \( \|A\| \leq \|a\| < \infty \), we see that \( A \) is a self-adjoint bounded operator. From (3.1) it follows that \( (A_\Lambda f, f) \geq c \|f\|^2 \) for any \( \Lambda \subset \mathbb{Z}^d \); hence \( A_\Lambda \) is invertible and \( \|A_\Lambda^{-1}\| \leq c^{-1} \).

Let
\[ \lambda(t) = \sum_{x \in \Lambda} |a(x)|. \quad (3.2) \]
Suppose that the series
\[ \sum_{n \geq 1} \lambda(n) \quad (3.3) \]
converges. Then for any \( \varepsilon > 0 \) there is a number \( d = d(\varepsilon) \geq 1, d \in \mathbb{Z} \), such that
\[ \sum_{n \geq 1} \lambda(nd) < \varepsilon/2. \quad (3.4) \]
Let
\[ u(n) = \begin{cases} 0 & \text{for } n \leq 0, \\ \varepsilon^{-1} \lambda(nd) & \text{for } n > 0. \end{cases} \quad (3.5) \]
Then \( \|u\|_1 = \sum_{n \geq 1} \varepsilon^{-1} \lambda(nd) < \frac{1}{2} \). Hence the series
\[ q = (e - u)^{-1} = \sum_{j = 0}^{\infty} u \cdots u \quad (\text{j times}) \quad (3.6) \]
converges. Put
\[ v(n) = \begin{cases} 0 & \text{for } n < 0, \\ \sum_{0 \leq j \leq n} q(j)p(n - j) & \text{for } n \geq 0, \end{cases} \quad (3.7) \]
where
\[ p(n) \]
where \( \varepsilon_0 > 0 \). The function \( v \) is

Proposition 3.1. (1) \( v(0) = n, k \geq 0 \).

Proof. Properties (1) and (2) are obvious. \( q(0) = p(0) = 1 \) and \( v(n) = \sum_{0 \leq j \leq n} q(j)p(n - j) \).

Let us prove (3). We have
\[ v(n + k) = \sum_{0 \leq j \leq n + k} q(j)p(n + k - j) = p(k) \sum_{0 \leq j \leq n} q(j)p(n - j) \]
The proposition is proved.

The central result of this section is

Theorem 3.2. Under the conditions \( L > 0 \), independent of \( \Lambda \), such that
\[ C = \|A\| \leq \|a\|, \quad |t_\Lambda(x, y)| \leq \frac{1}{L} \]
for any \( d \) satisfying (3.4).

Remarks. (1) In some cases the connection of this estimate to an integral operator is

(2) Repeating the proof of the estimate
\[ |t_\Lambda(x, y)t_\Lambda(y, z)| \leq \frac{1}{L} |t_\Lambda(x, z)| \]
which coincides with the estimate
\[ |t_\Lambda(x, y)|t_\Lambda(y, z)| \leq \frac{1}{L} |t_\Lambda(x, z)| \]
which coincides with the estimate
\[ |t_\Lambda(x, y)|t_\Lambda(y, z)| \leq \frac{1}{L} |t_\Lambda(x, z)| \]
\[(x - y) < C_1(1 + d)^{-\gamma}, \quad (2.7)\]

(2) we will impose a positivity condition,
\[f(x) \geq c \|f\|^2, \quad (3.1)\]

where \(A \) is a self-adjoint bounded operator, \(f \in H^1 \) for any \(\Lambda \subset \mathbb{Z}^n\); hence \(A^*_A \)

\[d = d(\epsilon) \geq 1, d \in \mathbb{Z}, \text{ such that} \quad (3.2)\]

\[d = d(\epsilon) \geq 1, d \in \mathbb{Z}, \text{ such that} \quad (3.3)\]

\[d = d(\epsilon) \geq 1, d \in \mathbb{Z}, \text{ such that} \quad (3.4)\]

\[d = d(\epsilon) \geq 1, d \in \mathbb{Z}, \text{ such that} \quad (3.5)\]

\[d = d(\epsilon) \geq 1, d \in \mathbb{Z}, \text{ such that} \quad (3.6)\]

where

\[p(n) = \begin{cases} 0 & \text{for } n < 0, \\ \exp(-\epsilon_0 n) & \text{for } n \geq 0, \end{cases} \]

where \(\epsilon_0 > 0\). The function \(v(n)\) has the following properties.

**Proposition 3.1.** (1) \(v(0) = 1\); (2) \(v(n) \geq p(n)\); (3) \(v(n + k) \geq \exp(-\epsilon_0 k)v(n)\) if \(n, k \geq 0\).

**Proof.** Properties (1) and (2) follow from (3.7) if we take into account that \(q(0) = p(0) = 1\) and \(v(n) = \sum_{0 \leq j \leq n} q(j)p(n - j) \geq q(0)p(n) = p(n)\).

Let us prove (3). We have

\[v(n + k) = \sum_{0 \leq j \leq n + k} q(j)p(n + k - j) \geq \sum_{0 \leq j \leq n} q(j)p(n + k - j)\]

\[= p(k) \sum_{0 \leq j \leq n} q(j)p(n - j) = \exp(-\epsilon_0 k)v(n).\]

The proposition is proved.

The central result of this section is the following theorem.

**Theorem 3.2.** Under the condition (3.3) there are constants \(c > 0, \epsilon_0 > 0, \) and \(L > 0, \) independent of \(\Lambda, \) such that

\[c/C^2 \leq |t_A(y, y)| \leq C/c^2, \]

where \(C = \|A\| \leq \|a\|_1, \)

\[|t_A(x, y)| \leq L|t_A(y, y)|p\left(\left|\frac{x - y}{d} + 1\right| + 1\right).\]

for any \(d\) satisfying (3.4).

**Remarks.** (1) In some cases, e.g. in the passage from a difference Green operator to an integral operator, it is useful to have estimates of \(c, \epsilon_0\) and \(L.\) In this connection note that Theorem 3.2 holds for \(0 < \epsilon < \sqrt{c}/4, \)

\(0 < \epsilon_0 < \frac{3}{2}(L + 1), \)

\(L = 4\|a\|_1/\sqrt{c}.\)

(2) Repeating the proof given above almost word for word we may obtain the estimate

\[|t_A(x, y)t_A^{-1}(y, y)| \leq Lv\left(\left|\frac{x - y}{d} + 1\right| + 1\right),\]

which coincides with the estimate given in the theorem in the scalar case, \(n = 1, \)
but is somewhat stronger than the estimate of the theorem in the vector case, \( n > 1 \). We give this estimate here only because in applications we sometimes need to estimate just \( |t_A(x, y)t_A^{-1}(y, y)| \); see §4.

**Proof.** Put

\[
s_0 = \{y\}, \quad s_i = \{x \in \Lambda \mid (i - 1)d < |x - y| \leq id\},
\]

\[
w_i = \sqrt{\sum_{x \in s_i} |t_A(x, y)|^2}, \quad S_i = \bigcup_{0 \leq j \leq i} s_j, \quad S_i^c = \Lambda \setminus S_i.
\]

Let us prove that

\[
c/C^2 \leq w_0 \leq C/c^2, \tag{3.7'}
\]

\[
\sum_{i \geq n + 1} w_i^2 \leq c^{-1} \left( \sum_{0 \leq i \leq n} \lambda(id)w_{i-1}^2 \right)^2. \tag{3.8}
\]

Let \( f(x) = t_A(x, y) \) and let \( y \) be fixed. Then \((A_{\Lambda}f)(x) = \delta(x - y)\) and

\[
C \|f\| \geq 1 = \|A_{\Lambda}f\| \geq c \|f\|,
\]

\[
c/C^2 \leq c \|f\|^2 \leq w_0 = |t_A(x, y)| = |(A_{\Lambda}f, f)| \leq C \|f\|^2 \leq C/c^2.
\]

The relation (3.7') is proved. Let

\[
t_A(x, y) = t_A'(x, y) + t_A''(x, y),
\]

where \( t_A'(x, y) = 0 \) for \( x \in S_n^c \) and \( t_A''(x, y) = 0 \) for \( x \in S_n \). Let \( f'(x) = t_A'(x, y), f''(x) = t_A''(x, y) \), for a fixed \( y \), and \( f = f' + f'' \). Then \((A_{\Lambda}f)(x) = \delta(x - y)\); hence \( A_{\Lambda}f'(x) = \delta(x - y) - A_{\Lambda}f''(x) \).

The "projection" of this equation onto \( S_n^c \) gives

\[
(A_{S_n}f')(x) = -(P_{S_n}A_{\Lambda}f')(x) = -\sum_{z \in S_n} a(x - z)t_A(z, y),
\]

where \( x \in S_n^c \) and

\[
P_{\Omega}g(x) = \begin{cases} g(x) & \text{for } x \in \Omega, \\ 0 & \text{for } x \notin \Omega. \end{cases}
\]

Thus

\[
c \sum_{i \geq n + 1} w_i^2 = c \sum_{x \in S_n} |t_A(x, y)|^2 \leq c \|f''\|^2 \leq \|A_{S_n}f''\|^2
\]

\[
= \sum_{x \in S_n} \left| \sum_{z \in S_n} a(x - z)t_A(z, y) \right|^2. \tag{3.9}
\]

Now let us estimate the last sum:

\[
\sum_{x \in S_n} \left| \sum_{z \in S_n} a(x - z)t_A(z, y) \right| \leq \sum_{x \in S_n} \left( \sum_{z \in S_n} |a(x - z)| \right)^{1/2} \left( \sum_{z \in S_n} |t_A(z, y)| \right)^{1/2}
\]

\[
= \sum_{x \in S_n} \left( \sum_{0 \leq i \leq n} \lambda(id)w_{i-1}^2 \right)^{1/2} \left( \sum_{0 \leq i \leq n} \lambda(id)w_{i-1}^2 \right)^{1/2}
\]

But by the Schwarz inequality

\[
\sum_{z \in S_i} |a(x - z)| \cdot |t_A(z, y)| \leq \left( \sum_{z \in S_i} |a(x - z)| \right)^{1/2} \left( \sum_{z \in S_i} |t_A(z, y)| \right)^{1/2}
\]

If \( x \in S_n^c, z \in s_i \), then \( |x - z| \leq \sum_{i \geq n + 1} w_i \), hence

\[
\sum_{x \in S_n} \left| \sum_{z \in S_n} a(x - z)t_A(z, y) \right| \leq \sum_{x \in S_n} \left( \sum_{0 \leq i \leq n} \lambda(id)w_{i-1}^2 \right)^{1/2} \left( \sum_{0 \leq i \leq n} \lambda(id)w_{i-1}^2 \right)^{1/2}
\]

(We have made use of the Schwartz inequality.)
the theorem in the vector case. In applications we sometimes

\[ |x - y| \leq d \],

\[ S_i = \Lambda \setminus S_i. \]

\[
(3.7')
\]

\[
(3.8')
\]

Now let us estimate the last expression.

\[
\sum_{x \in S_n} \left| \sum_{z \in S_n} a(x - z) t_A(z, y) \right|^2
\]

\[
\leq \sum_{x \in S_n} \left( \sum_{z \in S_n} |a(x - z) t_A(z, y)| \right)^2
\]

\[
= \sum_{x, z \in S_n} \sum_{0 \leq i \leq n} \sum_{x \in S_i, z \in S_j} |a(x - z) t_A(z, y)| |a(x - z') t_A(z', y)|.
\]

But by the Schwarz inequality

\[
\sum_{z \in S_n} |a(x - z)| \cdot |t_A(z, y)| \leq \sqrt{\sum_{z \in S_n} |a(x - z)|^2} \sqrt{\sum_{z \in S_n} |t_A(z, y)|^2}.
\]

If \( x \in S_n, z \in S_i \), then \( |x - z| \geq (n - i)d \). Therefore

\[
\sum_{z \in S_n} |a(x - z)| \leq \sum_{|z| \geq (n - i)d} |a(z)| = \lambda((n - i)d);
\]

hence

\[
\sum_{x \in S_n} \left| \sum_{z \in S_n} a(x - z) t_A(z, y) \right|^2
\]

\[
\leq \sum_{x \in S_n} \sum_{0 \leq i \leq n} \sqrt{\lambda((n - i)d) \cdot \lambda((n - j)d)}
\]

\[
\times \sqrt{\sum_{z \in S_n} |a(x - z)| \cdot |t_A(z, y)|^2} \sqrt{\sum_{z \in S_n} |a(x - z')| \cdot |t_A(z', y)|^2}
\]

\[
\leq \sum_{0 \leq i \leq n} \sqrt{\lambda((n - i)d) \cdot \lambda((n - j)d)} \sum_{x \in S_n} \sum_{z \in S_n} |a(x - z)| \cdot |t_A(z, y)|^2
\]

\[
\times \sqrt{\sum_{x \in S_n} \sum_{z \in S_n} |a(x - z')| \cdot |t_A(z', y)|^2}.
\]

(We have made use of the Schwarz inequality once more) but

\[
\sum_{x \in S_n} |a(x - z)| \leq \sum_{|z| \geq (n - i)d} |a(z)| = \lambda((n - i)d) \quad \text{for } z \in S_i
\]

and

\[
\sum_{z \in S_i} |t_A(z, y)|^2 = w_i^2.
\]
therefore
\[ \sum_{x \in S_n} \left| \sum_{z \in S_n} a(x-z) t_\lambda(z, y) \right|^2 \leq \sum_{0 \leq i, j \leq n} \lambda((n-i)d) \cdot \lambda((n-j)d) w_i w_j \]
\[ = \left( \sum_{0 \leq i \leq n} \lambda((n-i)d) w_i \right)^2. \]

Thus, (3.8) is proved.

Now let us prove that there is a sequence $0 = i_0 < i_1 < \cdots$ such that $\max_m (i_{m+1} - i_m) < \infty$ and
\[ w_i \leq w_0 v(i) \quad \text{for } i = i_0, i_1, \ldots, \tag{3.11} \]
\[ w_i \leq L w_0 v(i), \tag{3.12} \]
where $L = 4 \|a\|_1 / \sqrt{c}$ for all $i$. The proof is by induction on $i_m$. For $m = 0$ the statement is evident since $v(0) = 1$. Suppose that numbers $0 = i_0 < i_1 < \cdots < i_m$ have been constructed so that (3.11) holds for $i_0, i_1, \ldots, i_m$ and (3.12) holds for all $i, 0 \leq i \leq i_m$. Let us construct $i_{m+1}$. Let $n = i_m$. Then (3.8), (3.11), (3.12) imply that
\[ \sum_{i \geq n+1} w_i^2 \leq c^{-1}(\lambda(0) w_0 + \sum_{1 \leq i \leq n} \lambda(id) w_{n-i})^2 \]
\[ \leq c^{-1}(\|a\|_1 w_0 v(n) + 6L w_0 \sum_{1 \leq i \leq n} u(i)v(n-i))^2 \]
\[ = c^{-1}w_0^2(\|a\|_1 v(n) + 6L(u \ast v)(n))^2. \]

By (3.6) and (3.7) we have
\[ (e - u) \ast v = (e - u) \ast q \ast p = p, \]
implying that $u \ast v = v - p$; therefore the last inequality can be rewritten in the form
\[ \sum_{i \geq n+1} w_i^2 \leq c^{-1}w_0^2(\|a\|_1 v(n) + 6L(v(n) - p(n)))^2 \leq c^{-1}w_0^2(\|a\|_1 + 6L)^2v^2(n). \tag{3.13} \]

Let
\[ 0 < \varepsilon < \|a\|_1 / L = \sqrt{c}/4. \tag{3.14} \]

Then
\[ \sqrt{\sum_{i \geq n+1} w_i^2} \leq 2w_0 \|a\|_1 v(n)/\sqrt{c} = L w_0 v(n)/2. \tag{3.15} \]

In particular, for $i \geq n+1$ we have
\[ v(i) \geq v(n) \exp(-\varepsilon) \geq v(n) \exp(-\varepsilon) \]
and
\[ w_0 v(i_{m+1}). \]

By Proposition 3.1 we have
\[ v(i) \geq v(n) \exp(-\varepsilon) \geq v(n) \exp(-\varepsilon) \]
for $n \leq i \leq i_{m+1}$; hence (3.12) holds for $i_{m+1}$. 

It remains to note that
\[ \|t_\lambda(x, y)\| \leq \|a\|_1 / L = \sqrt{c}/4. \]

Therefore
\[ \|t_\lambda(x, y)\| \leq \sqrt{c}/4. \]

Theorem 3.2 is proved.

Now let us consider some compactly supported and the function $|a(x)|$ (it has compact support) then, additionally and
\[ |t_\lambda(x, y)| \leq \sqrt{c}/4. \]

for some $C_1, \gamma \geq 0.$
Green Functions of Difference Operators

In particular, for \( i \geq n + 1 \) we have

\[
- \lambda((n - j)d) \cdot (n - i)d \cdot w_i w_j \leq L w_0 v(n)/2.
\]

In addition, there is an \( i = i_{m+1} \) such that

\[
0 < i_{m+1} - n < L + 1
\]

and

\[
w_{m+1} \leq w_0 v(n)/2.
\]

By Proposition 3.1 we have \( v(n + k) \geq \exp(-\varepsilon_0 \varsigma(n)) \) for \( k \geq 0 \). Let

\[
0 < \varepsilon_0 < 1/(L + 1).
\]

Then

\[
v(i) \geq v(n) \exp(-\varepsilon_0 (i_{m+1} - n)) \geq v(n) \exp(-\varepsilon_0 (L + 1))
\]

\[
\geq v(n) \exp(-\frac{1}{2}) \geq v(n)/2
\]

for \( n \leq i \leq i_{m+1} \); hence (3.16), (3.18) imply that \( w_i \leq L w_0 v(i), \ w_{m+1} \leq w_0 v(i_{m+1}) \).

It remains to note that

\[
|t_{A} \sum_{x \in \varepsilon_i} |t_{A}(x, y)|^2 = w_i\]

\[
|t_{A}(x, y)| \leq \sqrt{\sum_{x \in \varepsilon_i} |t_{A}(x, y)|^2} = w_i
\]

and

\[
\left\lfloor \frac{|x - y| + 1}{d} \right\rfloor + 1 = i \quad \text{for } x \in \varepsilon_i;
\]

therefore

\[
|t_{A}(x, y)| \leq L w_0 v\left(\left\lfloor \frac{|x - y| + 1}{d} \right\rfloor + 1\right).
\]

Theorem 3.2 is proved.

Now let us consider some corollaries of Theorem 3.2. If the condition (3.1) is satisfied and the function \( |a(x)| \) admits an exponential estimate (in particular, has compact support) then, as in §2, we can show that \( v(n) \) decreases exponentially and

\[
|t_{A}(x, y)| \leq C_1 \exp(-\gamma' |x - y|)
\]

for some \( C_1, \gamma' \geq 0 \).
If we have only a power estimate for $|a(x)|$, i.e., $|a(x)| < C_0(1 + |x|)^{-\gamma}$ for $\gamma > v + 1$, then Theorem 3.2 produces an estimate

$$|t_\Lambda(x, y)| < C_1(1 + |x - y|)^{-\gamma - v},$$

which is somewhat worse than that of \S2 (see (2.7)).

Theorem 3.2 also makes it possible to prove the following statement:

**Proposition 3.3.** If

$$\sum_{x \in \mathbb{Z}^v} |x|^v|a(x)| < \infty$$

then

$$\sum_{y \in \Lambda} |t_\Lambda(x, y)| \leq C_0,$$

where $C_0$ does not depend on $x$ and $\Lambda$.

**Proof.** We will prove the implications

$$\sum_{x \in \mathbb{Z}^v} |x|^v|a(x)| < \infty \Rightarrow \sum_{n \geq 1} n^{v-1} \lambda(n) < \infty

\Rightarrow \sum_{n \geq 1} n^{v-1} u(n) < \infty

\Rightarrow \sum_{n \geq 1} n^{v-1} q(n) < \infty

\Rightarrow \sum_{n \geq 1} n^{v-1} q(n) < \infty

\Rightarrow \sup_{x, \Lambda} \sum_{y \in \Lambda} |t_\Lambda(x, y)| < \infty,$$

which evidently imply the statement we were to prove. We have

$$\sum_{n \geq 1} n^{v-1} \lambda(n) = \sum_{n \geq 1} n^{v-1} \sum_{|x| = n} |a(x)| = \sum_{x \in \mathbb{Z}^v} \gamma(|x|)|a(x)|,$$

where $\gamma(|x|) = \sum_{n \leq |x|} n^{v-1} \sim \text{const} \cdot |x|^v$, which proves the first implication. The second is evident. To prove the third and the fourth implications note that if, in the space of nonnegative functions $w(n)$ (i.e., $w(n) \geq 0$) such that $w(n) = 0$ for $n < 0$, functionals $S_j(w) = \sum_{n \geq 0} n^j w(n)$ are introduced, then

$$S_j(w_1 \ast w_2) = \sum_{n \geq 0} \sum_{0 \leq m \leq n} n^j w_1(m) w_2(u - m)$$

$$= \sum_{n \geq 0} \sum_{0 \leq m \leq n} \sum_{0 \leq i \leq j} C_i^j n^j (m - n)^{j-i} w_1(m) w_2(n - m)$$

$$= \sum_{0 \leq i \leq j} C_i^j \sum_{m \geq 0} m^i w_1(m) \sum_{n \geq 0} (n - m)^{j-i} w_2(n - m)$$

$$= \sum_{0 \leq i \leq j} C_i^j S_i(w_1) S_{j-i}(w_2),$$

(3.21)

implying that

$$S_{\gamma - 1}(q) = S_{\gamma - 1}(q_{\gamma - 1}) + \sum_{0 \leq i \leq j} C_i^j S_i(w_1) S_{j-i}(w_2),$$

or

$$S_{\gamma - 1}(q) = (1 - S_{\gamma - 1}(q_{\gamma - 1})).$$

Since $S_0(u) = \sum_{n \geq 1} u(n) \leq 1$, estimates of $S_{\gamma - 2}(q), \ldots, S_{\gamma - \gamma}(q)$ reduce to those of $S_{\gamma - \gamma}(q)$ estimating $S_0(q) = \sum_{n \geq 0} q(n)$.

The relation (3.21) makes $S_{\gamma - 1}(v) = \infty$ finite, then so is $S_{\gamma - 1}(v) = \infty$.

Finally, Theorem 3.2 implies

$$\sum_{y \in \Lambda} |t_\Lambda(x, y)|$$

This completes the proof of the proposition.

In concluding this section we observe that

$$|a(x)| \leq C_0,$$

where $L_1 = C_0 \|a\|$, and $\eta(t)$ is a word for worst case of Theorem 2.1, and therefore.

**§4. Applications**

Here we discuss several problems of support.

**4.1. Problem**

Suppose that, outside a finite support, the distribution is defined, and that the distri-
implying that
\[ S_{\nu-1}(q) = S_{\nu-1}(\delta_{x,0} + u \ast q) = S_{\nu-1}(\delta_{x,0}) + S_0(u)S_{\nu-1}(q) + \sum_{0 \leq i \leq \nu-2} C_i^\nu S_i(q)S_{\nu-i-1}(u), \]
or
\[ S_{\nu-1}(q) = (1 - S_0(u))^{-1}(S_{\nu-1}(\delta_{x,0}) + \sum_{0 \leq i \leq \nu-2} C_i^\nu S_i(q)). \]

Since \( S_0(u) = \sum_{n=1} u(n) < \frac{1}{2} \) this inequality reduces the estimate of \( S_{\nu-1}(q) \) to estimates of \( S_{\nu-2}(q), \ldots, S_0(q) \). Similarly the estimate of \( S_{\nu-2}(q) \) can be reduced to those of \( S_{\nu-3}(q), \ldots, S_0(q) \), etc. Eventually everything reduces to estimating \( S_0(q) = \sum_{n=0} q(n) \), which is finite. This proves the third implication.

The relation (3.21) makes it possible to prove that if \( S_{\nu-1}(q) \) and \( S_{\nu-1}(p) \) are finite, then so is \( S_{\nu-1}(v) = S_{\nu-1}(q \ast p) \). Hence the fourth implication is proved.

Finally Theorem 3.2 implies that
\[ \sum_{x \in \mathbb{Z}^r} v(x) |\mathcal{A}(x, y)| \leq \text{const} \cdot \sum_{x \in \mathbb{Z}^r} v \left( \left\lfloor \frac{|x - y| + 1}{d} \right\rfloor + 1 \right) \leq \text{const} \cdot \sum_{n \geq 0} (1 + n)^{-1} v(n). \]

This completes the proof of Proposition 3.3.

In concluding this section, we note that if \( \sup_{x \in \Lambda} \sum_{y \in \Lambda} |\mathcal{A}(x, y)| = C_0 \) is finite then
\[ |\mathcal{A}(x, y) - g(x - y)| \leq L_1 \eta(d), \]
where \( L_1 = C_0 \|a\|_1 \) and \( \eta(t) \) and \( d \) are defined in Theorem 2.1. The proof of this statement is a word for word repetition of the corresponding part of the proof of Theorem 2.1, and therefore we omit it.

§4. Applications

Here we discuss several problems considered in [2] for functions with compact support.

4.1. Problem of linear interpolation

Suppose that, outside a finite set \( \Lambda \subset \mathbb{Z}^r \), a configuration \( \{f^k(x) \in \mathbb{R}^r, x \in \Lambda^c \} \) is defined, and that the distribution of the random quantities \( \sigma(x) \in \mathbb{R}^r \), where
\( x \in \Lambda \), is defined by the Gaussian density
\[
p_{\Lambda}(f | f^0) = \Xi_{\Lambda,F}^{-1} \cdot \exp(-H_{\Lambda}(f | f^0)),
\]
where
\[
\Xi_{\Lambda,F} = \int \exp(-H_{\Lambda}(f | f^0)) \, d\mu(f),
\]
\[
H_{\Lambda}(f | f^0) = \frac{1}{2} \sum_{x \in \Lambda} \sum_{y \in \Lambda} (a(x - y)f(y), f(x)) + \sum_{x \in \Lambda} \sum_{y \in \Lambda'} (a(x - y)f^0(y), f(x)),
\]
i.e. is the conditional Gibbs distribution with potential \( a(x) \). Then
\[
\langle \sigma(x) \rangle = \int f(x)p_{\Lambda}(f | f^0) \, d\mu(f)
\]
for \( x \in \Lambda \) (interpolation of \( \langle \sigma(x) \rangle \) is linear in \( f \)). The problem consists of the estimation of the coefficients \( s_{\Lambda}(x, y) \). Here we surely suppose that the operator \( A : f(x) \to \sum_{y \in \Lambda} a(x - y)f(y) \) is nonnegative. We will assume that \( (Af, f) \geq c \|f\|^2 \), where \( c > 0 \).

We have the following simple formula which reduces this problem to one already considered:
\[
s_{\Lambda}(x, y) = t_{\Lambda \cup \{y\}}(x, y) t_{\Lambda \cup \{y\}}^{-1}(y, y). \tag{4.1}
\]
Let us prove this. Let
\[
f^0(y) = \delta_{y,0} E \quad \text{for } y \in \Lambda'.
\]
Let us write \( H_{\Lambda}(f | f^0) \) in the form
\[
H_{\Lambda}(f | f^0) = (A_{\Lambda}f, f)/2 - (f, g) = (A_{\Lambda}(f + T_{\Lambda}g), f + T_{\Lambda}g)/2 - (T_{\Lambda}g, g)/2,
\]
where \( T_{\Lambda} = A_{\Lambda}^{-1} \),
\[
g(x) = \sum_{z \in \Lambda'} a(x - z)f^0(z) = a(x - y).
\]
From this formula we see that \( \langle \sigma(x) \rangle + T_{\Lambda}g(x) \geq 0 \), i.e. that \( s_{\Lambda}(x, y) = \langle \sigma(x) \rangle = -T_{\Lambda}g(x) \). Applying the operator \( A_{\Lambda} \) to both parts of the last equality, we deduce that
\[
\sum_{z \in \Lambda} a(x - z)s_{\Lambda}(z, y) = -g(x) = -a(x - y)
\]
or
\[
\sum_{z \in \Lambda} a(x - z)s_{\Lambda}(z, y) \neq a(x - y) = 0. \tag{4.2}
\]

In \( \Lambda \cup \{y\} \) define the function
\[
\tilde{s}(x) = s_{\Lambda \cup \{y\}}(x, y).
\]
Then the equality (4.2) can be written
\[
A_{\Lambda \cup \{y\}} \tilde{s}(x) = 0.
\]
Since \( A_{\Lambda \cup \{y\}} \) is invertible, \( \tilde{s}(x) = 0 \) because, by definition, \( A_{\Lambda \cup \{y\}} \tilde{s}(x) = 0 \).

Considering this equality for \( \tilde{s}(x) \) and identity (4.1) and Theorem 41
\[
|s_{\Lambda}(x, y)| \leq C'
\]
if (1.1) holds;
\[
|s_{\Lambda}(x, y)| \leq C
\]
if (1.2) holds; and

in the general case (see Remarks 41).

4.2. Estimation of two initial points

Let \( \Lambda, \Omega \subset \mathbb{Z}' \). It is necessary to present \( t_{\Lambda}(x, y) - t_{\Omega}(x, y) \).

Let us present \( t_{\Lambda}(x, y) - t_{\Omega}(x, y) = -t_{\Omega}(x, y) \).

It is not difficult to see that
\[
t_{\Lambda}(x, y) - t_{\Omega}(x, y) = -t_{\Omega}(x, y) \neq 0.
\]
In \( \Lambda \cup \{y\} \) define the function

\[
\tilde{s}(x) = \begin{cases} 
s_{\Lambda}(x, y) & \text{for } x \in \Lambda, \\
E & \text{for } x = y.
\end{cases}
\]

Then the equality (4.2) can be interpreted as follows:

\[
A_{\Lambda \cup \{y\}} \tilde{s}(x) = \begin{cases} 
0 & \text{for } x \in \Lambda, \\
\sum_{z \in \Lambda \cup \{y\}} a(y - z) \tilde{s}(z) & \text{for } x = y.
\end{cases}
\]

Since \( A_{\Lambda \cup \{y\}} \) is invertible, \( \tilde{s}(x) \) coincides with \( t_{\Lambda \cup \{y\}}(x, y) \) up to a multiple, because, by definition, \( A_{\Lambda \cup \{y\}} t_{\Lambda \cup \{y\}}(x, y) = \delta_{x,y} E \). Since also \( \tilde{s}(y) = E \), we have

\[
\tilde{s}(x) = t_{\Lambda \cup \{y\}}(x, y) \tilde{s}(y).
\]

Considering this equality for \( x \in \Lambda \), we arrive at the desired identity (4.1).

Identity (4.1) and Theorems 2.1, 2.2 and 3.2 imply that

\[
|s_{\Lambda}(x, y)| \leq C \frac{C}{d} \tilde{g}(x - y) / c
\]

if (1.1) holds;

\[
|s_{\Lambda}(x, y)| \leq C \frac{C}{d} |g(x - y)| + (|g| \ast \hat{\mu} \ast |g|)(x - y) / c
\]

if (1.2) holds; and

\[
|s_{\Lambda}(x, y)| \leq Lp \left( \left[ \frac{|x - y| + 1}{a} \right] + 1 \right)
\]

in the general case (see Remark 2 on Theorem 3.2).

### 4.2. Estimate of the difference of the Green functions of two intersecting domains

Let \( \Lambda, \Omega \subset \mathbb{Z}^d \). It is necessary to estimate \( |t_{\Lambda}(x, y) - t_{\Omega}(x, y)| \) for \( x, y \in \Lambda \cap \Omega \).

Let us present \( t_{\Lambda}(x, y) - t_{\Omega}(x, y) \) in a convenient form. If we use the identity

\[
t_{\Lambda}^{-1}(x, y) - t_{\Omega}^{-1}(x, y) = - \sum_{z \in \Lambda} t_{\Lambda}(x, z) \left( \sum_{z' \in \Lambda \cap \Omega} a(z - z') t_{\Omega}(z', y) - \delta_{x,y} \right),
\]

it is not difficult to see that

\[
t_{\Lambda}(x, y) - t_{\Omega}(x, y) = - \sum_{z \in \Lambda \cap \Omega} t_{\Lambda}(x, z) \left( \sum_{z' \in \Lambda \cap \Omega} a(z - z') t_{\Omega}(z', y) \right.
\]

\[
\left. + \sum_{z \in \Omega \setminus \Lambda} t_{\Lambda}(x, z) \sum_{z' \in \Omega \setminus \Lambda} a(z - z') t_{\Omega}(z', y). \right)
\]
Let
\[ d_0 = \min \{ d(x, \Lambda \setminus \Omega), d(y, \Omega \setminus \Lambda) \}. \]
Then the last identity and Theorem 2.1 imply that
\[ |t_\Lambda(x, y) - t_\Omega(x, y)| \leq L_n(d_0), \]
where \( L = \|a\|_1 \cdot \|g\|_1, \eta(t) = \sup_{|t| \leq \epsilon} \hat{g}(x) \) if (1.1) is satisfied.
When the positivity condition (3.1) is satisfied and in addition it is known that
\[ \sup_{x, y \in \Lambda} \sum_{y' \in \Lambda} |t_\Lambda(x, y)| < \infty, \tag{4.3} \]
the estimate
\[ |t_\Lambda(x, y) - t_\Omega(x, y)| < L_0 \zeta(d_0) \]
holds, where \( L_0 \) is a constant independent of \( \Lambda \) and \( \Omega \) and
\[ \zeta(t) = \sup_{t' \leq t} \left( \left[ \frac{t' + 1}{d} \right] + 1 \right). \]
As was already mentioned, to satisfy (4.3) it suffices that
\[ \sum_{x \in \mathbb{Z}^s} |x|^{k} \cdot |a(x)| < \infty. \]

4.3. Computation of unconditional distributions for Gaussian fields

Suppose that we use finite sets \( \Omega \subset \Lambda \subset \mathbb{Z}^s \) and a field of random quantities \( \sigma(x) \) in \( \Lambda \) defined by the Gaussian density
\[ p_\Lambda(f) = \Xi_\Lambda^{-1} \exp \left( -\frac{1}{2} \sum_{x, y \in \Lambda} (c_{\Lambda, \Omega}(x, y)f(y), f(x)) \right). \]
Then the unconditional distribution of the random quantities \( \sigma(x) \) in \( \Omega \) is also Gaussian. Suppose that its density is
\[ p_{\Lambda, \Omega}(f) = \Xi_{\Lambda, \Omega}^{-1} \exp \left( -\frac{1}{2} \sum_{x, y \in \Omega} (c_{\Lambda, \Omega}(x, y)f(y), f(x)) \right). \]
The problem is to estimate \( |a(x - y) - c_{\Lambda, \Omega}(x, y)| \). The trick used in the proof of Theorem 2.2 (see also [2], [7]) makes it simple to obtain the formula
\[ a(x - y) - c_{\Lambda, \Omega}(x, y) = \sum_{z \in \Lambda \setminus \Omega} a(x - z)t_{\Lambda \setminus \Omega}(z, z')a(z' - y). \tag{4.4} \]
From this formula and Theorems 2.1 and 3.2 we derive
\[ |a(x - y) - c_{\Lambda, \Omega}(x, y)| \leq (|a| * \hat{g} * |a|)(x - y) \]
if (1.1) holds, and
\[ |a(x - y) - c_{\Lambda, \Omega}(x, y)| \leq L_1 \eta(t) \]
in the general case.
Finally, if \( \Omega \subset \Lambda \cap \Lambda' \), then
\[ |c_{\Lambda, \Omega}(x, y) - c_{\Lambda', \Omega}(x, y)| \leq L_2 \eta(t) \]
where \( L_0 \) does not depend on
\[ \chi(t) = \sup_{|t| \leq \epsilon} \hat{g}(x) \]
if (1.1) holds. In the general case it is known that
\[ \sup_{x \in \Omega} \zeta(t) \]
the similar estimate
\[ |c_{\Lambda, \Omega}(x, y) - c_{\Lambda', \Omega}(x, y)| \leq L_1 \eta(t) \]
holds, where \( L_1 \) does not depend on
\[ \chi(t) \]

References

[7] B. Simon, The \( P(q) \), Euclidean (Q)
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if (1.1) holds, and

\[ |a(x - y) - c_{\Lambda, \Omega}(x, y)| \leq L(|a| \ast \tilde{v} \ast |a|)(x - y), \]

where

\[ \tilde{v}(x) = v\left(\left\lceil \frac{|x - y| + L}{d} \right\rceil + 1 \right) \]

in the general case.

Finally, if \( \Omega \subset \Lambda \cap \Lambda' \), then by making use of (4.4) it is easy to see that

\[ |c_{\Lambda, \Omega}(x, y) - c_{\Lambda', \Omega}(x, y)| \leq L_0(\alpha(d_0) + \eta(d_0)), \]

where \( L_0 \) does not depend on \( x, y, \Lambda, \Lambda', \Omega \) and

\[ \alpha(t) = \sup_{|x| \leq t} |a(x)|, \quad \eta(t) = \sup_{|x| \leq t} |\tilde{g}(x)|, \]

\[ d_0 = \min_{x \in \Omega} d(x, \Lambda \setminus \Lambda') = d(\Omega, \Lambda \setminus \Lambda') \]

if (1.1) holds. In the general case, under the extra assumption that

\[ \sup_{x, y \in \Lambda} \sum_{y' \in \Lambda} |l_\Lambda(x, y')| < \infty, \]

the similar estimate

\[ |c_{\Lambda, \Omega}(x, y) - c_{\Lambda', \Omega}(x, y)| \leq L_1(\alpha(d_0) + \xi(d_0)) \]

holds, where \( L_1 \) does not depend on \( x, y, \Lambda, \Lambda', \Omega \) and

\[ \xi(t) = \sup_{r \leq t} v\left(\left\lceil \frac{r' + 1}{d} \right\rceil + 1 \right) \]

References