DISTRIBUTION OF ENERGY LEVELS OF A QUANTUM FREE PARTICLE ON A SURFACE OF REVOLUTION

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1. Introduction. Let \( M \) be a two-dimensional smooth compact manifold which is homeomorphic to a sphere, and which is a surface of revolution in \( \mathbb{R}^3 \), with an axis \( A \) and poles \( N \) and \( S \) (see Figure 1). The geodesic flow on \( M \) is a classical integrable system due to the Clairaut integral,

\[
r \sin \alpha = \text{const}.
\]

In the present work we are interested in high energy levels of the corresponding quantum system,

\[
-\Delta u_n = E_n u_n.
\]

Let \( s \) be the normal coordinate (the length of geodesic) along the meridian, and let

\[
r = f(s), \quad 0 \leq s \leq L,
\]

be the equation of \( M \), where \( r \) is the radial coordinate. Then

\[
\Delta = f(s)^{-1} \frac{\partial}{\partial s}\left( f(s) \frac{\partial}{\partial s} \right) + f(s)^{-2} \frac{\partial^2}{\partial \varphi^2},
\]

where \( \varphi \) is the angular coordinate.

We assume that \( f(s) \) has a simple structure, so that

\[
f'(s) \neq 0, \quad s \neq s_{\text{max}}; \quad f''(s_{\text{max}}) \neq 0,
\]

where

\[
f(s_{\text{max}}) = \max_{0 \leq s \leq L} f(s) \equiv f_{\text{max}}.
\]

For normalization we put \( f_{\text{max}} = 1 \). Another assumption on \( M \) is the following twist hypothesis.

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Consider the equator on $M$,

$$\gamma_e = \{s = s_{\text{max}}, 0 \leq \phi \leq 2\pi\}$$

(we do not assume that $M$ is symmetric with respect to $\gamma_e$, but we still call $\gamma_e$ the equator keeping a visual interpretation of objects on $M$), and a geodesic $\gamma$ which starts at $x_0 = (s = s_{\text{max}}$, $\phi = 0) \in \gamma_e$ at some angle $-\pi/2 < \phi_0 < \pi/2$ to the north direction. The Clairaut integral on $\gamma$ is $I = \sin \phi_0$, and we can parametrize $\gamma$ by $-1 < I < 1$: $\gamma = (I)$; it follows from the Clairaut integral that $\gamma(I)$ oscillates between two parallels, $s = s_-$ and $s = s_+$, where $f(s_-) = f(s_+) = I$, so $\gamma(I)$ intersects $\gamma_e$ infinitely many times. Let $x_n$ be the $n$th intersection of $\gamma$ with $\gamma_e$, $n \in \mathbb{Z}$. Define

$$\tau(I) = |\gamma(x_0, x_2)|$$

to be the length of $\gamma$ between $x_0$ and $x_2$, and

$$\omega(I) = (2\pi)^{-1}(\phi(x_2) - \phi(x_0))$$

to be the phase of $\gamma$ between $x_0$ and $x_2$ (see Figure 1). Observe that $\omega(I)$ is defined mod 1. To define $\omega(I)$ uniquely, we choose a continuous branch of $\omega(I)$ starting at $\omega(0) = 0$. Then $\omega(0) = -\omega(1)$ and, for $I \geq 0$,

$$\omega(I) = \pi^{-1} \int_{s_0}^{s_+} \frac{d\phi}{ds} ds - 1. \quad (1.6)$$

Define $\tau(I) = \lim_{I \to -0} \tau(I)$ and $\omega(I) = \lim_{I \to -0} \omega(I)$.

It is easy to see that a finite geodesic $\gamma$ with the Clairaut integral $0 < I < 1$ is closed if and only if $\omega(I)$ is rational. More precisely, let $n(\gamma)$ denote the number of revolutions of a closed geodesic $\gamma$ around the axis $A$, and let $m(\gamma)$ denote the number of oscillations of $\gamma$ along meridian. Then

$$\omega(I) = (n(\gamma)/m(\gamma)) - 1.$$ 

To facilitate formulation of subsequent results, we take the convention that a finite geodesic $\gamma$ with $I = 1$, which goes along the equator, is closed if and only if both $n(\gamma)$ and

$$m(\gamma) \equiv \frac{n(\gamma)}{\omega(1) + 1}$$

are integers.

**Twist Hypothesis (TH).** $\omega'(I) \neq 0$, for all $I \in [0, 1]$.  

**Figure 1.** Geodesic on a surface of revolution.
To illustrate TH consider an ellipsoid of revolution,

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1.$$  

If $a < b$ (oblolute ellipsoid), then $\omega'(I) > 0$, if $a > b$ (oblate ellipsoid), then $\omega'(I) < 0$; so TH holds in both cases. Curves (a) and (b) on Figure 2 are the graphs of $\omega(I)$ found with the help of a computer for the ellipsoids of revolution with $a = 1, b = 2$ and $a = 2, b = 1$, respectively. The cross sections of the ellipsoids are shown in the lower part of Figure 2. TH is violated for a sphere ($a = b$), when $\omega'(I) \equiv 0$. TH can be also violated for a bell-like shape of $M$ shown on Figure 2 (see the cross section (c) and the graph (c) of $\omega(I)$ on this figure) and in some other cases.

Let

$$N(R) = \# \{ E_* \subseteq R^2 \}$$

be the counting function of $E_*$. Then the Weyl law says that

$$N(R) = \frac{\text{Vol} M}{4\pi} R^2 + \Lambda(R)$$

(1.7)

Let

$$\omega(I)$$

be the phase function for different surfaces of revolution. Cross sections of the surfaces of revolution are shown in the lower part of the figure.

Figure 2. The phase function $\omega(I)$ for different surfaces of revolution. Cross sections of the surfaces of revolution are shown in the lower part of the figure.

where $\Lambda(R) = o(R^2), R \to \infty$. A general estimate of Hörmander [Hör1] gives

$$\Lambda(R) = O(R).$$

This estimate is sharp for $S^2$ and some other degenerate surfaces for which closed geodesics cover a set of positive Liouville measure in the phase space. If the Liouville measure of the union of all closed geodesics in the phase space is 0, then, as is shown by Duistermaat and Guillemin [DG],

$$\Lambda(R) = o(R).$$

For surfaces of negative curvature, Selberg and Béard [Bé] prove a better estimate:

$$\Lambda(R) = O(R/\log R),$$

and it is a very difficult open problem to show that $\Lambda(R) = O(R^{1-\varepsilon})$ for some $\varepsilon > 0$, even in the case of constant negative curvature. (See the recent works [Sar], [LS], and [HR] where statistics of eigenvalues and eigenfunctions of the Laplace operator on surfaces of constant negative curvature are discussed.)

For a flat torus $\Lambda(R)$ reduces to the error term of the classical circle problem, and the best estimate here is due to Huxley [Hux]:

$$\Lambda(R) = O(R^{1/2} (\log R)^{31/146}).$$

A well-known conjecture of Hardy [Har1],

$$\Lambda(R) = O(R^{1/2} + \varepsilon), \quad \forall \varepsilon > 0,$$

is probably also a very difficult open problem. On the other hand Hardy proves [Har2] that

$$\limsup_{R \to \infty} R^{-1/2} |\Lambda(R)| = \infty,$$

so $(1/2) + \varepsilon$ is the best possible exponent.

Colin de Verdière [CdV1], [CdV2] proves that, for a generic surface of revolution of simple structure,

$$\Lambda(R) = O(R^{3/2}).$$

(1.8)

We prove in the present paper the following result.

**Theorem 1.1.** Assume that $M$ is a surface of revolution of simple structure and that $M$ satisfies TH. Then

$$N(R) = \frac{\text{Vol} M}{4\pi} R^2 + R^{1/2} F(R),$$

(1.9)
where $F(R)$ is an almost periodic function of the Besicovitch class $B^2$, and the Fourier series of $F(R)$ in $B^2$ is

$$
F(R) = \sum_{\text{closed geodesics } \gamma} A(\gamma) \cos(|\gamma| R - \phi),
$$

(1.10)

where summation goes over all closed (in general, multiple) oriented geodesics $\gamma \neq 0$ on $M$, $\phi = (\pi/2) + (\pi/4) \text{sgn } \omega(I)$, and

$$
A(\gamma) = \pi^{-1}(-1)^{m(\gamma)}|\omega(I)|^{-1/2}m(\gamma)^{-3/2}
= \pi^{-1}(-1)^{m(\gamma)}|\omega(I)|^{-1/2}a(\gamma)^{3/2}|\gamma|^{-3/2}, \quad I = I(\gamma).
$$

(1.11)

The equation (1.10) is a trace formula which relates eigenvalues of the Laplace operator to closed geodesics. It is to be observed that (1.10) gives the Fourier series of $F(R)$ in the Besicovitch space $B^2$ which means that

$$
\lim_{T \to \infty} \frac{1}{T} \int_0^T |F(R) - \sum_{\text{closed geodesics } \gamma \text{ with } |\gamma| < N} A(\gamma) \cos(|\gamma| R - \phi)|^2 dR = 0.
$$

So (1.10) is actually an asymptotic trace formula describing the behavior of $F(R)$ for large $R$ (cf. [Gui]). In Theorem 1.2 we extend Theorem 1.1 to the case when TH is violated. In this case we introduce the following.

**Diophantine Hypothesis (DH).** Assume that $\omega(I)$ has at most finitely many critical points $0 < I_1 < \cdots < I_k < 1$ (so that $I = 0, 1$ are not critical) with $\omega(I_k) = 0$ and $\omega'(I_k) \neq 0$, $k = 1, \ldots, K$. Assume, in addition, that for every $k = 1, \ldots, K$, $\omega(I_k)$ is either rational or Diophantine in the sense that there exist $1 > \zeta > 0$ and $C > 0$ such that

$$
|\omega(I_k) - \frac{p}{q}| \geq \frac{C}{q^{2+\zeta}}, \quad \forall \frac{p}{q} \in \mathbb{Q}.
$$

(1.12)

**Theorem 1.2.** Assume that $M$ is a surface of revolution of simple structure and DH holds. Then

$$
N(R) = \frac{\text{Vol } M}{4\pi} R^2 + R^{2.3} \sum_{k=0}^N \Phi_k(R) + R^{1.2}F(R),
$$

(1.13)

where $\Phi_k(R)$ are bounded periodic functions and $F(R)$ is an almost periodic function of the Besicovitch class $B^2$. The Fourier series of $\Phi_k(R)$ is

$$
\Phi_k(R) = (1/2)3^{-2.3} \Gamma(2/3)\pi^{-4.5}a(\gamma_k)^{k+3} \sum_{\gamma \in I_k} (-1)^{m(\gamma)}|\gamma|^{-4.3} \sin(|\gamma| R).
$$

(1.14)

and the Fourier series of $F(R)$ is

$$
F(R) = \sum_{\gamma \in I, \gamma \neq I_k} A(\gamma) \cos(|\gamma| R - \phi(\gamma)),
$$

(1.15)

where $\phi(\gamma) = (\pi/2) + (\pi/4) \text{sgn } \omega(I)$. $I = I(\gamma)$ and $A(\gamma)$ is given in (1.11).

In the works [HB], [BCDL], [Ble1], [Ble2], and [BL] some general results are proved on the existence and properties of a limit distribution of almost any periodic function of the Besicovitch class $B^2$ (see especially Theorems 4.1–4.3 in [Ble1] and Theorems 3.1, 3.3 in [Ble2]). Theorem 1.1 combined with these results lead us to the following.

**Corollary.** Assume that $M$ is a surface of revolution of simple structure and $M$ satisfies $TH$. Then the normalized error function $F(R)(R)$ in the formula (1.9) has a limit distribution $v(t)$, i.e., for every bounded continuous function $g(t)$ on the line,

$$
\lim_{T \to \infty} \frac{1}{T} \int_0^T g(F(R)) dR = \int_{-\infty}^{\infty} g(t)v(t) dt.
$$

If, in addition, the lengths of all primitive closed geodesics on $M$ with $I \geq 0$ are linearly independent over $\mathbb{Z}$, then $v(t)$ is absolutely continuous and the density function $p(t) = v(t)/dt$ is an entire function of $t$ which satisfies on the real axis the estimates

$$
0 \leq p(t) \leq C \exp(-\lambda |t|^4), \quad C, \lambda > 0.
$$

$$
P(-t, 1 - P(t)) \geq C \exp(-\lambda' |t|^4), \quad t \geq 0; \quad C', \lambda' > 0; \quad P(t) = \int_{-\infty}^t p(t') dt'.
$$

Observe that Theorem 1.1 is a particular case of Theorem 1.2, so we need to prove only Theorem 1.2. The plan of the remainder of the paper is as follows. In Section 2 we present a theorem of Colin de Verdière and show how, with the help of this theorem, to reduce Theorem 1.2 to a lattice-point problem for the Bohr-Sommerfeld quasi-classical approximation. Section 3 is auxiliary: here we investigate the asymptotics at infinity of the Fourier transform of the characteristic function of a plane domain with nondegenerate inflection points and angular points. In Section 4 we prove the lattice-point version of Theorem 1.2 for the number of lattice points inside a dilated plane oval with finitely many points of inflection. Section 5 is technical, and there we prove some lemmas used in Section 4. In Section 6 we prove Theorem 1.2. Finally in an appendix we prove a $2/3$-estimate for ovals with semicubic singularity. That estimate is used in the main part of the paper.

**2. Quasi-classical approximation.** Colin de Verdière proves in [CdV2] the following result.
THEOREM CdV. If $M$ is a surface of revolution of simple structure, then
\[ \text{Spectrum}(-\Delta) = \{ E_{kl} = Z(k + (1/2), l); k, l \in \mathbb{Z}, |l| \leq k \} \quad (2.1) \]
with $Z(p) = Z(p_1, p_2) \in C^0(\mathbb{R}^2)$ such that
\[ Z(p) = Z_1(p) + Z_0(p) + O(|p|^{-1}), \quad |p| \to \infty. \quad (2.2) \]
where
\[ Z_1(p), Z_0(p) \in C^0(\mathbb{R}^2 \setminus \{0\}); \quad Z_1(p) > 0, p \neq 0 \]
and
\[ Z_j(\lambda p) = \lambda^j Z_j(p), \quad \forall \lambda > 0, p \in \mathbb{R}^2, j = 0, 2. \]
In addition, in the sector $\{|p_1| \geq |p_2|\}$, $Z_1(p)$ satisfies the equation
\[ \pi^{-1} \int_0^\infty \sqrt{Z_2(p) - p_1^2 p_2^2} \, ds = p_1 - |p_2|, \quad (2.4) \]
where $a, b$ are the turning points, i.e.,
\[ Z_2(p) - p_1^2 p_2^2 = 0 \quad \text{for} \quad s = a, b. \quad (2.5) \]
It is to be noted that the Bohr-Sommerfeld quantization rule is
\[ \pi^{-1} \int_0^\infty \sqrt{E_{kl} - l^2} \, ds = k + (1/2) - |l|, \quad k \geq |l|, \quad (2.6) \]
which is equivalent to the approximation
\[ E_{kl} = Z_2(k + (1/2), l). \quad (2.7) \]
(2.1) implies that
\[ N(R) = \# \{ E_{kl} \leq R^2 \} = \# \{ (k, l); Z(k + (1/2), l) \leq R^2, |l| \leq k \}. \quad (2.8) \]
Define
\[ N_{\text{BS}}(R) = \# \{ (k, l); Z_2(k + (1/2), l) \leq R^2, |l| \leq k \}. \quad (2.9) \]
(BS stands for Bohr-Sommerfeld.)
Define \( X(p) = (Z_2(p))^{1/2} \). Then \( X(p) > 0 \) and
\[
X(\lambda p) = \lambda X(p) \quad \forall \lambda > 0, p \in \mathbb{R}^2.
\]
(2.15) implies that
\[
I(n, n') = 0 \quad \text{if} \quad |X(n) - X(n')| > C_0 |n|^{-1}.
\]
(2.18)
In addition, (2.14), (2.15) imply that
\[
I(n, n') = 0 \quad \text{if} \quad X(n) \geq 2T,
\]
(2.19)
and \( \forall n, n' \),
\[
|I(n, n')| \leq CT^{-1} |n|^{-2}.
\]
(2.20)
From (2.18)–(2.20)
\[
\sum_{n, n' \in M} I(n, n') \leq C \sum_{n \in M, X(n) \leq 2T} T^{-1} |n|^{-2} \sum_{n' \in M, |X(n) - X(n')| \leq C_0 |n|^3} 1.
\]
Due to the 2/3-estimate of Sierpinski-Landau-Rancoli-Colin de Verdière [Sie], [Lan], [Ran], [CdV1],
\[
\sum_{n' \in M, |X(n) - X(n')| \leq C_0 |n|^3} 1 \leq C |n|^{2/3}.
\]
and hence
\[
\sum_{n, n' \in M} I(n, n') \leq CT^{-1} \sum_{n \in M, X(n) \leq 2T} |n|^{-4/3} \leq C_0 T^{-4/3}.
\]
(2.21)
Since (2.13) and (2.21) imply (2.10), Theorem 2.1 is proved.

By (2.9), \( N_{\text{ms}}(R) \) is the number of lattice points \( (k + (1/2), l) \) of a shifted square lattice in the sectorial domain
\[
\Omega(R) = \{ p \in \mathbb{R}^2 : Z_2(p) \leq R^2, |p_2| \leq p_1 \}.
\]

Observe that \( \Omega(R) = \Omega(1) \) dilated with the coefficient \( R \), so the problem of finding the asymptotics of \( N_{\text{ms}}(R) \) as \( R \to \infty \) reduces to a lattice-point problem about the asymptotics of the number of lattice points inside \( R\Omega \), where \( \Omega \) is a sectorial domain between diagonals \( p_2 = \pm p_1 \) which is bounded by the curve \( \Gamma = \{ Z_2(p) = 1 \} \). By (2.4) \( \Gamma \) is the graph of the function
\[
p_1 = |p_2| + \pi^{-1} \int_0^R \sqrt{1 - p_2^2 f^{-2}(s)} \, ds.
\]

Figure 3 shows the curve \( \Gamma \) found with the help of a computer for the surfaces of revolution presented on Figure 2 above.

Sections 3–5 below are devoted to the lattice-point problem for an arbitrary sectorial domain between diagonals which is bounded by a “generic” curve \( \Gamma \).

3. Asymptotics of Fourier transform of nonconvex domains. In this section we consider the following auxiliary problem. Let \( \Omega \) be a sectorial domain on a plane which is bounded by two segments \([0, z_0], [0, z_1]\) and by a smooth curve \( \Gamma \) which goes from \( z_0 \) to \( z_1 \) (see Figure 4). We assume that \( \Gamma \) has at most finitely many points of inflection where the curvature \( \sigma(p) \) vanishes and that all the points of inflection are nondegenerate, i.e., \( d\sigma / ds \neq 0 \) at these points. We are interested in the asymptotics of the Fourier transform
\[
\tilde{z}(\xi) = \int_{\Omega} \exp(ip\xi) \, dp
\]
as \( |\xi| \to \infty \). With the help of a partition of unity, this problem can be reduced to a series of local problems of the following type.

Let \( \Gamma \) be a smooth curve near some point \( p^0 \in \Gamma \) and \( \chi(p) = 1 \) from one side of \( \Gamma \) and \( \chi(p) = 0 \) from the other side of \( \Gamma \) (locally). Let \( \varphi(p) \in C_0^\infty \) be a \( C^\infty \)-function

![Figure 3. Curve \( \Gamma \) for the surfaces of revolution shown on Figure 2.](image-url)
with compact support near $p^0$. Assume in addition that $\varphi(p) = 1$ in the vicinity of $p^0$. Then the local problem is: what is the asymptotics of

$$\tilde{\chi}(\xi) = \int_{\mathbb{R}^1} \varphi(p) \chi(p) \exp(i p \xi) \, dp,$$

as $|\xi| \to \infty$.

We are also interested in the case when $\Gamma$ has an angular point at $p^0$, and we consider in a sequence the following case: (a) $\Gamma$ is either convex or concave near $p^0$; (b) $\Gamma$ has an inflection point at $p^0$; (c) $\Gamma$ has an angular point at $p^0$ with one side that is either convex or concave and with the other side that is a straight ray; (d) $\Gamma$ is an angle between two straight rays; and (e) $\Gamma$ is a straight line (see Figure 5).

In the case (a) the answer is the following well-known lemma. Let $\Gamma_0 \subset \Gamma$ be an open arc on $\Gamma$ such that

$$p^0 \in \Gamma_0 \subset \{ p \in \Gamma : \varphi(p) = 1 \},$$

and

$$V = \{ \xi \in \mathbb{R}^2 \setminus \{0\} : 3p = p(\xi) \in \Gamma_0 \text{ such that } n_{\Gamma}(p) = |\xi^{-1} \xi| \},$$

where $n_{\Gamma}(p)$ is the vector of normal to $\Gamma$ at $p \in \Gamma$ which looks in the direction where

**Figure 4.** Sectorial domain with points of inflection on the boundary.

**Figure 5.** Local structures of $\Gamma$. 
\( \chi(p) = 0 \). Define
\[
Y(\zeta) = \zeta \cdot p(\zeta), \quad \zeta \in \mathbb{R}^3 \setminus \{0\}.
\] (3.1)

For the sake of brevity we denote \( \sigma(p(\zeta)) \) by \( \sigma(\zeta) \). We assign a sign to the curvature \( \sigma(p) \), so that \( \sigma(p) > 0 \) if the region \( \{\chi(p) = 1\} \) is convex near \( p \in \Gamma \), and \( \sigma(p) < 0 \) if this region is concave near \( p \). If \( p_2 = f(p_1) \) is the equation of \( \Gamma \) near some \( \theta = \theta(\zeta) \) in the coordinate system with an orthonormal basis \( e_1, e_2 \), such that \( e_2 = n_\theta(p) \), then
\[
\sigma(\theta) = -f''(\theta_1).
\] (3.2)

**Lemma 3.1 (see [Hla]).** If \( \sigma(p) \neq 0 \) on \( \Gamma_0 \), then
\[
\hat{\zeta}(\zeta) = (2\pi)^{-1/2} |\zeta|^{-3/2} |\sigma(\zeta)|^{-1/2} \exp(i(\chi(\zeta) - \phi)) + O(|\zeta|^{-5/2}) \quad \text{if} \quad \zeta \in V,
\] (3.3)

where
\[
\phi = (\pi/2) + (\pi/4) \sigma(\zeta).
\] (3.4)

Assume now that \( p_0 \) is a nondegenerate point of inflection, i.e., \( \sigma(p_0) = 0 \) and \( (d\sigma/ ds)(p_0) \neq 0 \). In this case \( p_0 \) is the turning point for \( n_\theta(p) \), so that, for small deviations of the direction of \( \zeta \) from \( n_\theta(p_0) \), we have either two points \( p = p_\pm(\zeta) \) with \( n_\theta(p) = |\zeta|^{-1} \zeta \) or no such point at all. Define, for \( \zeta \in V \),
\[
\theta(\zeta) = \begin{cases} 1 & \text{if} \exists \zeta \in \Gamma_0 \text{ with } n_\theta(p) = |\zeta|^{-1} \zeta, \\ 0 & \text{otherwise}. \end{cases}
\] (3.5)

To be definite in the choice of \( p_\pm(\zeta) \), we assume that \( \sigma(p_+(\zeta)) > 0 \) and \( \sigma(p_-(\zeta)) < 0 \). Let \( \sigma_+(\zeta) = \sigma(p_+(\zeta)) \) and \( \sigma_-(\zeta) = \sigma(p_-(\zeta)) \). Let \( \theta(\zeta) \) be the angular coordinate of \( \zeta \). Finally, let
\[
\text{Ai}(t) = \int_{-\infty}^{t} \exp(it^2 + it^3/3) dt
\]
be the Airy function. Recall that \( \text{Ai}(t) \in C^\infty(\mathbb{R}) \) and, when \( t \to \infty \),
\[
\text{Ai}(\pm t) = \pm t^{-1/4} \cos(\zeta - (\pi/4)) + O(t^{-7/4}),
\]
\[
\text{Ai}'(\pm t) = \pm t^{-1/2} \sin(\zeta - (\pi/4)) + O(t^{-5/4}), 
\]
\[
|\text{Ai}(t)|, |\text{Ai}'(t)| \leq C \exp(-t),
\]

which implies that, for \( t \neq 0 \),
\[
|\text{Ai}(t) - \theta(\pm t)\pi^{-1/2} |t|^{-1/4} \cos(\zeta - (\pi/4))| \leq C |t|^{-7/4},
\]
\[
|\text{Ai}'(t) - \theta(\pm t)\pi^{-1/2} |t|^{-1/4} \sin(\zeta - (\pi/4))| \leq C |t|^{-5/4},
\] (3.6)

where \( \theta(t) = 1 \) for \( t > 0 \) and \( \theta(t) = 0 \) for \( t < 0 \).

**Lemma 3.2.** Assume that \( \sigma(p_0) = 0 \) and \( (d\sigma/ ds)(p_0) \neq 0 \). Then there exist real-valued functions \( a(x), b(x) \) near \( a_0(x) = f(n_\theta(p^0)) \) such that \( a(a_0) = 0 \), \( a'(a_0) \neq 0 \), \( b(a_0) = Y(n_\theta(p^0)) \), and
\[
\hat{\zeta}(\zeta) = -i \exp(i|\zeta| b(a(\zeta))) \left[ \xi^{-4/3} \text{Ai}_p(|\xi|^{2/3} a(\zeta)) u(a(\zeta)) \right] + O(|\zeta|^{-7/3}) \quad \text{if} \quad \zeta \in V,
\] (3.7)

where \( u(x), v(x) \) are \( C^6 \) functions and
\[
u(\zeta_0) = \left| \frac{1}{2} \frac{d\sigma}{ds}(p_0) \right|^{1/3}.
\] (3.8)

**Corollary.** If \( a(\zeta) \neq \zeta_0 \), then
\[
\chi(\zeta) - i \hat{\zeta}(\zeta) (\zeta^2 + \chi(\zeta)) = C|\zeta|^{-5/2} [a(\zeta) - a_0(\zeta)]^{-1/2}
\] (3.9)

where
\[
\chi^\pm(\zeta) = \left| \zeta \right|^{-3/2} 2\sigma\pm(\zeta))^{-1/2} \exp(i(\chi_\pm(\zeta) - \phi_\pm(\zeta))
\]
\[
\phi_\pm = (\pi/4) \pm (\pi/4).
\] (3.10)

**Proof of the corollary.** From (3.6) and (3.7), we obtain that, for \( \zeta(\infty) = \zeta \neq \zeta_0 \),
\[
\chi_\infty(\zeta) = -i \exp(i|\zeta| b(a(\zeta))) \left[ \xi^{-4/3} \text{Ai}_p(|\xi| a_0(\zeta)) - (\pi/4) \right] + O(|\zeta|^{-5/2} x - a_0(\zeta))^{-7/4}
\]
\[
\left. \right. \text{with some } a_0(\zeta), u_0(\zeta), \text{and } v_0(\zeta). \text{ On the other hand, Lemma 3.1 gives us that, for a fixed } a(\zeta) = \zeta \neq \zeta_0, \right.
\]
\[
\chi_\infty(\zeta) = \theta(\zeta) \left[ \chi_\infty^+(\zeta) + \chi_\infty^-(\zeta) \right] + O(|\zeta|^{-5/2}), \quad |\zeta| \to \infty.
\] (3.11)

Comparing (3.11) with (3.12) we obtain that the main terms in these two asymptotics coincide, and hence
\[
\chi_\infty(\zeta) = \theta(\zeta) \left[ \chi_\infty^+(\zeta) + \chi_\infty^-(\zeta) \right] + O(|\zeta|^{-5/2} x - a_0(\zeta))^{-7/4},
\]
which proves the corollary.
Proof of Lemma 3.2. Consider an orthonormal basis $e_1, e_2$ on the plane with $e_2 = n_1(p^0)$. Let $p = p^0 + p_1 e_1 + p_2 e_2$, $\xi = \xi_1 e_1 + \xi_2 e_2$, and $p_2 = f(p_1)$ be the equation of $\Gamma$ near $p^0$. Observe that $\Phi(0) = f'(0) = f''(0) = 0$ and we can choose the direction of $e_1$ in such a way that $f''(0) = (d\sigma/ds)(p^0)$. Integrating by parts in $p_2$, we obtain that

$$\tilde{\gamma}(\xi) = \int_{\mathbb{R}} \varphi(p)\chi(p) \exp(ip\xi) \, dp$$

$$= -i\xi_2^{-1} \exp(ip\xi) \int_{-\infty}^{\infty} \varphi(p_1, f(p_1)) \exp(ip_1 \xi_1 + if(p_1) \xi_2) \, dp_1 + O(|\xi|^{-N})$$

$$= -i\xi_2^{-1} \exp(ip\xi) \int_{-\infty}^{\infty} \varphi(p_1, f(p_1)) \exp[i\xi_1 \Phi(p_1, x)] \, dp_1 + O(|\xi|^{-N}),$$

where $\Phi(p_1, x) = p_1 \sin(x - x_0) + f(p_1) \cos(x - x_0)$ is a $C^\infty$ real-valued function with

$$\Phi = \frac{\partial \Phi}{\partial p_1} = \frac{\partial^2 \Phi}{\partial p_1^2} = 0, \quad \frac{\partial^3 \Phi}{\partial p_1^3} = \frac{\partial}{\partial p_1} = ds(p^0), \quad \text{and} \quad \frac{\partial^3 \Phi}{\partial p_1 \partial x_2} \neq 0,$$

at $p_1 = 0, x = x_0$. (3.7) follows now from Theorem 7.7.18 in [Hör2]. Lemma 3.2 is proved.

Let us turn now to angular points. Assume that $\Gamma$ is a smooth curve near $p^0 \in \Gamma$ and $L$ is a straight line which intersects $\Gamma$ at $p^0$ transversely. Then, near $p^0$, $\Gamma$ and $L$ divide the plane into four parts. Let $\chi(p) = 1$ in one of these parts and $\chi(p) = 0$ in the remainder. Again we are interested in asymptotics of $\tilde{\gamma}(\xi)$ as $|\xi| \to \infty$.

Define the auxiliary functions

$$P_\pm(y) = (2\pi)^{-1/2} \exp(\pm i \pi/4) \int_{-\infty}^{\infty} \exp(\mp i t^2/2) \, dt,$$

which are $C^\infty$ bounded functions on $\mathbb{R}^1$ such that

$$P_\pm(-\infty) = 0, \quad P_\pm(0) = 1/2, \quad P_\pm(\infty) = 1;$$

$$P_+ (y) = \overline{P_-(y)}, \quad P_+ (y) + P_-(y) = 1;$$

$$|P_\pm(y) - \theta(y)| \leq C(1 + |y|)^{-1} \tag{3.14}$$

Lemma 3.3. Assume $\sigma(p) \neq 0$ in the vicinity of $p^0 \in \Gamma$ and $L$ intersects $\Gamma$ at $p^0$ transversally. Then there exists $C^\infty$ real-valued functions $a(x)$ and $b(x)$ near $x_0 = z(n_1(p^0))$ such that $a(x_0) = 0, \ a'(x_0) \neq 0, \ b(x_0) = Y(n_1(p^0))$, and

$$\tilde{\gamma}(\xi) = |\xi|^{-3/2} \exp[i \xi_1 a(x_0)] - i\phi) P_+(i \xi_1^2 a(x_0)) + O(|\xi|^{-N}) \tag{3.15}$$

where $\phi = (\pi/2 + \pi/4) \sign(\sigma(p^0)), \ \chi = \text{sign}(\sigma(p^0))$ and $u(a)$ is a $C^\infty$ function near $x_0$ with $u(x_0) = 2\pi \sigma(p^0)^{1/2}$.

**Corollary.** If $\xi \in V$ and $a(x) \neq x_0 = z(n_1(p^0))$, then

$$|\tilde{\gamma}(\xi) - |\xi|^{-3/2} 2\pi \sigma(p^0) \text{sign}(\sigma(p^0)) \exp[i(y(x) - \phi)]| \leq C|\xi|^{1/2} (a(x) - x_0)^{-1}. \tag{3.16}$$

where $\theta(x) = 1$ if $\exists \beta = p(\xi) \in \Gamma$ with $n_1(p) = |\xi|^{-1} |\xi|$, and $\theta(x) = 0$ otherwise.

**Proof of the corollary.** From (3.14), (3.15), we obtain that for $a(x) = x \neq x_0$

$$\tilde{\gamma}(\xi) = |\xi|^{-3/2} \exp[i \xi_1 b(x) - i\phi] \theta(x) + O(|\xi|^{-N}), \quad x \in V,$$

On the other hand, when $a(x) = x \neq x_0$ is fixed, Lemma 3.1 proves that

$$\tilde{\gamma}(\xi) = |\xi|^{-3/2} 2\pi \sigma(p^0)^{-1} \theta(x) \exp(i(y(x) - \phi)) + O(|\xi|^{-N}), \quad |\xi| \to \infty, \ xi \in V.$$ 

Comparing these two asymptotics we conclude that the main terms in them coincide, and therefore (3.16) holds.

**Proof of Lemma 3.2.** Consider a basis $e_1, e_2$ on the plane, where $e_1$ is a unit tangent vector to $\Gamma$ at $p^0$ and $e_2$ is a unit tangent vector to $L$ at $p^0$. Let $e_1^* e_2^*$ be a dual basis, $e_1^* e_2^* = \delta_{12}$, and $p = p^0 + p_1 e_1 + p_2 e_2, \ \xi = \xi_1 e_1^* + \xi_1 e_2^*$. Then

$$\tilde{\gamma}(\xi) = \int_{\mathbb{R}} \varphi(p) \chi(p) \exp(ip\xi) \, dp$$

$$= J \exp(p^0 \xi) \int_{-\infty}^{\infty} \varphi(p_1, f(p_1)) \exp(ip_1 \xi_1 + if(p_1) \xi_2) \, dp_1$$

$$+ O(|\xi|^{-N}) \tag{3.17}$$

where $p_2 = f(p_1)$ is the equation of $\Gamma$ and $J$ is the Jacobian. Observe that $p_1 \xi_1 + f(p_1) \xi_2 = |\xi| \Phi(p_1, x)$ where $\Phi(p_1, x)$ is a $C^\infty$ real-valued function with

$$\Phi = \frac{\partial \Phi}{\partial p_1} = 0, \quad \frac{\partial^2 \Phi}{\partial p_1^2} \neq 0, \quad \frac{\partial^2 \Phi}{\partial p_1 \partial x_2} \neq 0,$$

at $p_1 = 0, x = x_0$.
4. Lattice-point problem. Let $Z(p) \in C^0(\mathbb{R}^2 \setminus \{0\})$ be a $C^0$ positive function homogeneous of degree 2. Define

\[
N(R) = \# \{ n = (n_1, n_2) \in \mathbb{Z}^2 : Z(n_1 + (1/2), n_2) \leq R^2, |n_2| \leq n_1 \},
\]

\[
\Omega(R) = \{ p \in \mathbb{R}^2 : Z(p) \leq R^2, |p_2| \leq p_1 \},
\]

\[
A(R) = AR^2 = \text{Area}(\Omega(R)).
\]

(4.1)

The lattice-point problem we are interested in is to evaluate $N(R) - A(R)$ as $R \to \infty$. Let

\[
\Gamma = \{ p \in \mathbb{R}^2 : Z(p) = 1, |p_2| \leq p_1 \},
\]

and let $z_0, z_1 \in \Gamma$ be the endpoints of $\Gamma$ with $z_{01} = -z_{02} > 0$ and $z_{11} = z_{12} > 0$. For $p \in \Gamma$ denote by $n_1(p)$ the vector of the outer normal to $\Gamma$ at $p$. Observe that

\[
n_1(p) = |\text{grad } Z(p)|^{-1} \text{ grad } Z(p),
\]

and

\[
p \cdot \text{grad } Z(p) = 2Z(p) > 0.
\]

(4.2)

hence

\[
p \cdot n_1(p) = 2|\text{grad } Z(p)|^{-1} \text{ grad } Z(p) > 0.
\]

(4.3)

Denote by $\sigma(p)$ the curvature of $\Gamma$ at $p \in \Gamma$ with a sign, so that $\sigma(p) > 0$ if $\Omega$ is convex near $p$ and $\sigma(p) < 0$ if $\Omega$ is concave near $p$. In what follows we assume the following.

**HYPOTHESIS D.** (i) $\sigma(p) \neq 0$ everywhere on $\Gamma$ except, maybe, a finite set $W = \{ w_1, \ldots, w_K \}, z_0, z_1 \notin W$, and

\[
\frac{d\sigma}{ds}(w_k) \neq 0, \quad k = 1, \ldots, K,
\]

(4.5)

where $s$ is the natural coordinate on $\Gamma$. (ii) For all $w_k \in W$ the vector $v_k = n_1(w_k)$ is either rational, i.e., $n \cdot v_k = 0$ for some $n \in \mathbb{Z}^2$, $n \neq 0$, or Diophantine in the sense that $\exists l > 0$ and $C > 0$ such that

\[
|n \cdot v_k| > \frac{C}{|n|^{1/l}}, \quad \forall n \in \mathbb{Z}^2, n \neq 0.
\]

(4.6)
Without loss of generality we may assume that
\[ 0 = s(z_0) < s(w_1) < \cdots < s(w_k) < s(z_1), \]  
(4.7)
where \( s(p) \) is the natural coordinate of \( p \in \Gamma \).

We call \( \xi \in \mathbb{R}^2 \setminus \{0\} \) rational if \( n \cdot \xi = 0 \) for some \( n \in \mathbb{Z}^2 \setminus \{0\} \). It is to be noted that \( \xi \) is rational if and only if the set
\[ L(\xi) = (\mathbb{Z}^2 \setminus \{0\}) \cap \{n \xi, \lambda \in \mathbb{R}\} \]
is nonempty. Let
\[ \Gamma_{\text{rat}} = \{p \in \Gamma; n_\xi(p) \text{ is rational}\} \]

For \( p \in \Gamma \) define
\[ Y(p) = p \cdot n_\xi(p). \]  
(4.8)
By (4.3), \( Y(p) > 0 \).

**Theorem 4.1.** Assume Hypothesis D holds. Then
\[ N(R) = AR^2 + R^{2/3} \sum_{k: \frac{1}{\nu_\xi} \text{is rational}} \Phi_k(R) + R^{1/2} F(R), \]  
(4.9)
where \( \Phi_k(R) \) are continuous periodic functions,
\[ \Phi_k(R) = (1/2)3^{-2/3} \Gamma(2/3)\pi^{-4/3} \left| \frac{d\sigma}{ds}(w_k) \right|^{-1/3} \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} (-1)^n |n|^{-4/3} \sin(2\pi Y(w_k) |n| R - n \cdot z)), \]  
(4.10)
and \( F(R) \in B^2 \). The Fourier series of \( F(R) \) is
\[ F(R) = \pi^{-1} \sum_{p \in \Gamma_{\text{rat}}} \theta(p)|\sigma(p)|^{-1/2} \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} (-1)^n |n|^{-32} \cos(2\pi Y(p) |n| R - \phi(p)), \]  
(4.11)
where
\[ \theta(p) = \begin{cases} 1 & \text{if } p \neq z_0, z_1, \\ (1/2) & \text{if } p = z_0, z_1. \end{cases} \]
and \( \phi(p) = (\pi/2) + (\pi/4) \text{ sgn } \sigma(p) \).

We need Theorem 4.1 to prove our main Theorem 1.2. For the sake of completeness we want to formulate another theorem which is not needed for the proof of Theorem 1.2, but which is of interest by itself.

Let \( x \in \mathbb{R}^2 \) be a fixed point on the plane. Define
\[ N(R; x) = \# \{n \in \mathbb{Z}^2; z(n + x) \subseteq \mathbb{R}^2\}. \]

**Theorem 4.2.** Assume Hypothesis D holds. Then
\[ N(R; x) = AR^2 + R^{2/3} \sum_{k: \frac{1}{\nu_\xi} \text{is rational}} \Phi_k(R; x) + R^{1/2} F(R; x), \]
where \( \Phi_k(R; x) \) are continuous periodic functions of \( R, \)
\[ \Phi_k(R; x) = (1/2)3^{-2/3} \Gamma(2/3)\pi^{-4/3} \left| \frac{d\sigma}{ds}(w_k) \right|^{-1/3} \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} |n|^{-4/3} \sin(2\pi Y(w_k) |n| R - n \cdot z)), \]
and \( F(R; x) \in B^2 \) in \( R \). The Fourier series of \( F(R; x) \) is
\[ F(R; x) = \pi^{-1} \sum_{p \in \Gamma_{\text{rat}}} |\sigma(p)|^{-1/2} \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} |n|^{-32} \cos(2\pi Y(p) |n| R - \phi(p, n, x)), \]  
(4.12)
where \( \phi(p, n, x) = (\pi/2) + (\pi/4) \text{ sgn } \sigma(p) + 2\pi n \cdot z). \)

Theorem 4.2 is a generalization of Theorem 1.1 in [Ble1] to nonconvex domains. **Proof of Theorem 4.1.** \( N(R) \) can be written as
\[ N(R) = \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} \chi(n + x; R), \quad x = (1/2, 0), \]  
(4.13)
where \( \chi(p; R) \) is the characteristic function of \( \Omega(R) \). Define, for \( \delta > 0, \)
\[ N_\delta(R) = \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} \chi_\delta(n + x; R), \]  
(4.14)
where
\[ \chi_\delta(p; R) = \int_{\partial \Omega} \delta^{-2} \phi(\delta^{-1}(p - p')) dp' = \chi(\cdot; R) \ast (\delta^{-2} \phi(\delta^{-1} \cdot))(p), \]  
(4.15)
and
\[ \phi(p) \in C^\infty_0(\mathbb{R}^2); \phi(p) = \phi_0(|p|); \phi(p) \geq 0; \int_{\mathbb{R}^2} \phi(p) dp = 1; \phi(p) = 0 \text{ when } |p| \geq 1. \]
LEMMA 4.3. For $T > 1$,

$$
\frac{1}{T} \int_{-1}^{1} |N(R) - N(R)| R^{-1} dR \leq C \delta T^{-1/3} + \delta. \quad (4.16)
$$

Proof of this and all subsequent lemmas is given in the next section. For what follows we put

$$
\delta = T^{-1}. \quad (4.17)
$$

In this case (4.16) reduces to

$$
\frac{1}{T} \int_{-1}^{1} |N(R) - N(R)| R^{-1} dR \leq C T^{-1/3}. \quad (4.18)
$$

By the Poisson summation formula,

$$
N(R) - AR^2 = R^2 \sum_{n \in \mathbb{Z} \setminus \{0\}} \tilde{\phi}(2\pi n R) \tilde{\Omega}(2\pi n R)(-1)^{n}, \quad (4.19)
$$

where

$$
\tilde{\Omega}(\zeta) = \int_{\Omega} \exp[ip\zeta \xi] d\xi, \quad \Omega = \{p \in \mathbb{R}^2; Z(p) \leq 1, |p_2| \leq p_1\}.
$$

Let us consider a partition of unity on the projective line $\mathbb{RP}^1$,

$$
\sum_{i=1}^{N} \psi_i(\xi) = 1, \quad 0 \leq \psi_i(\xi) \leq 1, \quad \psi_i(\xi) \in C^\infty(\mathbb{RP}^1),
$$

and lift it to $\mathbb{R}^2 \setminus \{0\}$ putting

$$
\psi(\lambda, \xi) = \psi(\xi) \quad \forall \lambda \in \mathbb{R} \setminus \{0\}.
$$

Let

$$
\Gamma^0 = \{p \in \Gamma; n_r(p) \in \supp \psi_i(\xi)\},
$$

and $\Gamma_j^0$, $j = 1, ..., J$, be connected components of $\Gamma^0$. Without loss of generality we may assume that each $\Gamma_j^0$ contains at most one singular point $z_0, z_1, w_1, ..., w_k$.

Then, for a given $l$, consider a partition of unity in the $p$-plane,

$$
\sum_{m=1}^{M} \chi_m(p) = 1, \quad 0 \leq \chi_m(p) \leq 1, \quad \chi_m(p) \in C^\infty(\Omega),
$$

such that, for each $\Gamma^0_j$, $1 \leq j \leq J$, a unique $\chi_m(p)$ which is $\neq 0$ on $\Gamma^0_j$. Without loss of generality we may also assume that, for each $m$, $\chi_m(p)$ contains at most one singular point.

Define

$$
E_{lm}(R) = R^2 \sum_{n \in \mathbb{Z} \setminus \{0\}} \psi_n(n) \tilde{\phi}(2\pi n \xi) \tilde{\Omega}(2\pi n R)(-1)^{n}, \quad (4.20)
$$

where

$$
\tilde{\Omega}(\xi) = \int_{\Omega} \psi_n(p) \exp(ip\xi \xi) d\xi.
$$

Then by (4.19)

$$
N(R) - AR^2 = \sum_{l \in \Gamma} E_{lm}(R).
$$

Now we evaluate $E_{lm}(R)$. Let $\Gamma_m = \Gamma \cap \supp \chi_m$.

LEMMA 4.4 ($\chi_m$ out of $\Gamma^0$). If $\Gamma_m \cap \Gamma^0 = \emptyset$, then

$$
\sup_{1 \leq i \leq T} |E_{lm}(R)| \leq C \log^2 T.
$$

Next we consider the case when $\Gamma^0 \subset \Gamma_m$ and $\Gamma_m$ contains no singular point. In this case, for all $\xi \in \supp \psi$, there exists a unique $p(\xi) \in \Gamma^0$ such that $n_r(p(\xi)) = |\xi|^{-1} \xi$. Denote by

$$
\gamma(\xi) = \xi \cdot n_r(p(\xi)), \quad \sigma(\xi) = \sigma(p(\xi)).
$$

LEMMA 4.5 (regular $\chi_m$ on $\Gamma$). Assume that $z_0, z_1, w_1, ..., w_k \notin \Gamma_m$ and $\Gamma^0 \subset \Gamma_m$. Then $E_{lm}(R) = \pi^{-1} F_{lm}(R)$, where $F_{lm}(R) \in B^2$ and

$$
F_{lm}(R) = \pi^{-1} \sum_{n \in \mathbb{Z} \setminus \{0\}} \psi_n(n) |n|^{-1/2} |\sigma_n(n)|^{-1/2} \cos(2\pi Y_n(n) R - \phi). \quad (4.21)
$$

where $\phi = (\pi/2) + (\pi/4) \sgn \sigma(p)$, $p \in \Gamma_m$.

Assume now that some inflection point $w_k$ lies inside $\Gamma^0$ and $\Gamma^0 \subset \Gamma_m$. Then, for every $\xi \in \supp \psi(\xi)$, there are two possibilities exist: either there exist two points $p_1(\xi) \in \Gamma^0$ such that $n_r(p_1(\xi)) = |\xi|^{-1} \xi$ or there is no such point at all. Define the function $\theta(\xi)$ which is equal to 1 in the first case and which is equal to 0 in the second case. For sake of definiteness we will assume that $\pm \sigma(p_1(\xi)) > 0$. Denote by

$$
Y_1(\xi) = \xi \cdot n_r(p_1(\xi)), \quad \sigma_1(\xi) = \sigma(p_1(\xi)).
$$
LEMMA 4.6 (ψm at inflection point). If \( v_k = n_\ell(w_k) \in \Gamma^0 \subset \Gamma_m \) and \( v_k \) is rational, then
\[
E_{\ell \mu}(R) = R^{\frac{1}{2}} \Phi_{\ell \mu}(R) + R^{\frac{1}{2}} F_{\ell \mu}(R),
\]
where \( \Phi_{\ell \mu}(R) \) is a periodic continuous function which is given in (4.10) and \( F_{\ell \mu}(R) \in B^2 \).

The Fourier series of \( F_{\ell \mu}(R) \) is
\[
F_{\ell \mu}(R) = \pi^{-\frac{1}{2}} \sum_{n \in \mathbb{Z}} \frac{\psi(n) \theta(n)}{n^{\frac{1}{2}} |\sigma(n)|^{\frac{1}{2}}} \cos(2\pi Y_\ell(n)R - \phi) \tag{4.25}
\]
where \( \phi = (\pi/2) \pm (\pi/4) \).

If \( v_k \) is Diophantine, then \( E_{\ell \mu}(R) = R^{\frac{1}{2}} F_{\ell \mu}(R) \) where \( F_{\ell \mu}(R) \in B^2 \), and the Fourier series of \( F_{\ell \mu}(R) \) coincides with (4.25).

LEMMA 4.7 (angular \( \chi_m \)). If \( n_\ell(z_0) \in \Gamma^0 \subset \Gamma_m \), then \( E_{\ell \mu}(R) = R^{\frac{1}{2}} F_{\ell \mu}(R) \), where \( F_{\ell \mu}(R) \in B^2 \), and the Fourier series of \( F_{\ell \mu}(R) \) is
\[
F_{\ell \mu}(R) = \pi^{-\frac{1}{2}} \sum_{n \in \mathbb{Z}^+} \frac{\psi(n) \theta(n)}{n^{\frac{1}{2}} |\sigma(n)|^{\frac{1}{2}}} \cos(2\pi Y_\ell(n)R - \phi) \tag{4.26}
\]
where
\[
\theta(n) = \begin{cases} 
1 & \text{if } n = \lambda n_\ell(p) \text{ for some } p \in \Gamma^0 \setminus \{z_0\}, \\
(1/2) & \text{if } n = \lambda n_\ell(z_0), \\
0 & \text{otherwise.}
\end{cases}
\]
and \( \phi = (\pi/2) + (\pi/4) \text{sgn}(z_0) \).

Proof of Lemmas 4.3–4.7 is given in the next section.

End of the proof of Theorem 4.1. Let us fix \( l \). By Lemma 4.4,
\[
\sum_{m \in \Gamma_m 
mid n_\ell \in \Gamma^0 \setminus \{z_0\}} \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau}^{\tau} |R^{\frac{1}{2}} E_{\ell \mu}(R)|^2 dR = 0,
\]
and, by Lemmas 4.5–4.7,
\[
\sum_{m \in \Gamma_m 
mid n_\ell \in \Gamma^0 \setminus \{z_0\}} E_{\ell \mu}(R) = R^{\frac{1}{2}} \sum_{m \in \Gamma_l(0)} \Phi_{\ell \mu}(R) + R^{\frac{1}{2}} \sum_{m \in \Gamma_l(0)} F_{\ell \mu}(R),
\]
where
\[
I_1(l) = \{ m : \exists k, j \text{ such that } w_k \in \Gamma^0 \subset \Gamma_m \text{ and } v_k \text{ is rational}, \nmid \}

I_2(l) = \{ m : \Gamma_m \cap \Gamma^0 \neq \emptyset \}.
\]

Making a summation over \( l \) we obtain that, if we define
\[
\Phi(R) = \sum_{\ell \in \Gamma_l(0)} \Phi_{\ell \mu}(R), \quad F(R) = \sum_{\ell \in \Gamma_l(0)} F_{\ell \mu}(R),
\]
then
\[
\lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau}^{\tau} |N_\ell(R) - AR^2 - R^{\frac{1}{2}} \Phi(R) - R^{\frac{1}{2}} F(R)|^2 R^{-1} dR = 0.
\]

Lemma 4.3 implies that the same is true for \( N(R) \):
\[
\lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau}^{\tau} |N(R) - AR^2 - R^{\frac{1}{2}} \Phi(R) - R^{\frac{1}{2}} F(R)|^2 R^{-1} dR = 0.
\]
Since \( F_{\ell \mu}(R) \in B^2 \), \( F(R) \in B^2 \) as well (see [Bes]). The Fourier series for \( \Phi(R) \) and \( F(R) \) are the sum of the Fourier series for \( \Phi_{\ell \mu}(R) \) and \( F_{\ell \mu}(R) \), respectively. This proves the formulas (4.11), (4.12). Theorem 4.1 is proved.

Theorem 4.2 is proved in the same way.

5. Proof of lemmas

Proof of Lemma 4.3. We have
\[
N_\ell(R) - N(R) = \sum_{n \in \mathbb{Z}^+} (\chi_\ell(n + \alpha; R) - \chi(n + \alpha; R)).
\]

The support of \( \chi_\ell(p; R) - \chi(p; R) \) is concentrated in the \( \delta \)-neighborhood of \( \partial \Omega(R) \). Observe that \( \partial \Omega(R) \) consists of the curve \( RT \) and two segments \([0, Rz_\ell] \) and \([0, Rz_\ell] \) oriented along the diagonals \( p_1 \pm p_2 = 0 \). If \( \delta \) is small, then the \( \delta \)-neighborhood of \([0, Rz_\ell] \cup [0, Rz_\ell] \) does not contain points \( n + \alpha, n \in \mathbb{Z}^+ \), \( \alpha = (1/2, 0) \), and it does not contribute to \( N_\ell(R) - N(R) \).

Now,
\[
I \equiv \frac{1}{\tau} \int_{-\tau}^{\tau} |N_\ell(R) - N(R)|^2 R^{-1} dR = \sum_{n, n'} I_{\ell}(n, n'),
\]
where
\[
I_{\ell}(n, n') = \frac{1}{\tau} \int_{-\tau}^{\tau} (\chi_\ell(n + \alpha; R) - \chi(n + \alpha; R))(\chi_\ell(n' + \alpha; R) - \chi(n' + \alpha; R)) R^{-1} dR.
\]

Observe that \( I_{\ell}(n, n') = 0 \) unless both \( n + \alpha \) and \( n' + \alpha \) lie in the \( \delta \)-neighborhood
of $R \Gamma$ for some $R$, and hence there exist $C > 0$ such that

$$I_0(n, n') = 0, \text{ if either } |X(n + x) - X(n' + x)| \geq C \delta \text{ or } X(n + x) \geq CT,$$

where $X(p) = (Z(p))^{1/2}$. In addition, for all $n, n'$,

$$I_0(n, n') \leq CT^{-1} \delta X(n) X(n')^{-1} \leq C_0 T^{-1} \delta |n|^{-1}.$$

Therefore,

$$I \leq \sum_{n, |X(n + x)| < CT} C_0 T^{-1} \delta |n|^{-1} \sum_{n', |X(n' + x)| < CT} 1.$$

By $2/3$-estimate (see, e.g., [CdV1])

$$\sum_{n', |X(n + x)| < CT} 1 \leq C(n^{2/3} + \delta |n|),$$

and hence

$$I \leq C_1 T^{-1} \delta \sum_{n, |X(n + x)| < CT} |n|^{-1} (|n|^{2/3} + \delta |n|) \leq C_2 \delta T(T^{-1/3} + \delta).$$

Lemma 4.3 is proved.

**Proof of Lemma 4.4.** Let us consider two cases: when $0 \in \text{supp } \chi_m(p)$ and when $z_0 \in \Gamma_m$ (or $z_1 \in \Gamma_m$) and $\Gamma_m \cap \Gamma^0 = \emptyset$; all the other cases are simpler. Due to (4.20),

$$E_{lm} = \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} K(n)$$

with

$$K(n) = \phi(2\pi \delta) \phi(n) R^2 \hat{\chi}_m(2\pi Rn)(-1)^a.$$  \hspace{1cm} (5.1)

By Lemma 3.4, if $0 \in \text{supp } \chi_m(p)$, then

$$R^2 |\hat{\chi}_m(R\xi)| \leq C |\xi|^{-1} \max \{R^{-1} + |\xi_1 + \xi_2|^{-1}, R^{-1} + |\xi_1 - \xi_2|^{-1}\},$$

and so, if $n \in \Lambda$, where

$$\Lambda = \{n \in \mathbb{Z}^2: |n_1 + n_2| > |n_1 - n_2| > 0\},$$

then

$$R^2 |\hat{\chi}_m(2\pi Rn)| \leq C |n|^{-1} |n_1 - n_2|^{-1}.$$

Therefore $I_0 \equiv \sum_{n \in \Lambda} K(n)$ is estimated as

$$I_0 \leq C \sum_{n \in \Lambda} |\phi(2\pi n \delta)| |n|^{-1} |n_1 - n_2|^{-1}.$$

Since $\phi(p) \in C_0^0(\mathbb{R}^2)$,

$$|\phi(\xi)| \leq C(1 + |\xi|)^{-3},$$

and

$$\sum_{|n| > T^2} |\phi(2\pi \delta)| \leq C_0, \quad \delta = T^{-1}. \hspace{1cm} (5.2)$$

On the other hand

$$\sum_{|n| > T^2} |n|^{-1} |n_1 - n_2|^{-1} \leq C \log^2 T. \hspace{1cm} (5.3)$$

Hence $I_0 \leq C \log^2 T$. The sum over $\{|n_1 - n_2| \geq |n_1 + n_2| > 0\}$ is estimated similarly, and hence we obtain

$$\left| \sum_{n_1, n_2 \neq 0, n_1 + n_2 \neq 0} K(n) \right| \leq C \log^2 T.$$

By Lemma 3.4

$$\sum_{n \in \mathbb{Z}^2 \setminus \{0\}} K(n) = C_1 R \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} \phi(2\pi \delta) \phi(n) n_1^{-a} (1)^{n_1} + O(1).$$

Since $\phi(\xi)$ and $\psi_i(\xi)$ are even functions,

$$\sum_{n \in \mathbb{Z}^2 \setminus \{0\}} \phi(2\pi \delta) \phi(n) n_1^{-a} (1)^{n_1} = 0,$$

and thus

$$\sum_{n \in \mathbb{Z}^2 \setminus \{0\}} K(n) = O(1).$$

Similarly,

$$\sum_{n \in \mathbb{Z}^2 \setminus \{0\}} K(n) = O(1).$$

As a result,

$$|E_{lm}(R)| = \left| \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} K(n) \right| \leq C \log^2 T,$$

which was stated.
Assume now that \( z_0 \in \Gamma_\alpha \) and \( \Gamma_\alpha \cap \Gamma^0 = \emptyset \). Let \( \Omega_\alpha \) be the angle with the vertex at \( z_0 \) and the sides which go along \([z_0, 0]\) and the tangent vector to \( \Gamma \) at \( z_0 \), so that \( \Omega_\alpha \) is a linear approximation to \( \Omega \) near \( z_0 \). Assume for the sake of definiteness that \( \sigma(z_0) > 0 \). Then \( \Omega_\alpha \supset \Omega \) near \( z_0 \). Let
\[
\tilde{Z}_m^{(0)}(\zeta) = \int_{\Omega_\alpha} Z_m(p) \exp(ip \zeta) \, dp.
\]
Then
\[
\tilde{Z}_m^{(0)}(\zeta) - \tilde{Z}_m(\zeta) = \int_{\Omega_\alpha} Z_m(p) \exp(\overline{p} \zeta) \, dp
\]
is estimated as follows. Integrating in the direction orthogonal to \( \zeta \), we obtain that
\[
\tilde{Z}_m^{(0)}(\zeta) - \tilde{Z}_m(\zeta) = \int_{t_0}^{t_1} \tilde{Z}_m(t) \exp(it \zeta) \, dt,
\]
where \( \tilde{Z}_m(t) \) is equal to zero in vicinity of \( t_0 \) and
\[
\tilde{Z}_m(t) = a_1(t-t_0)^2 + a_2(t-t_0)^3 + \cdots
\]
in vicinity of \( t_0 \). This implies that \( |\tilde{Z}_m^{(0)}(\zeta) - \tilde{Z}_m(\zeta)| \leq C(1 + |\zeta|)^{-3} \), and hence
\[
\sum_{n \in \mathbb{Z}^2 \setminus \{0\}} (K(n) - K^{(0)}(n)) = O(R^{-1}),
\]
where
\[
K^{(0)}(n) = \phi(2\pi n \delta) \psi(n \eta R^2 \tilde{Z}_m^{(0)}(2\pi n R)) (-1)^n.
\]
Now \( \Omega_\alpha \) is an angular domain. Using the same argument as in the case \( 0 \in \text{supp} \, Z_m \), we obtain that
\[
\sum_{n \in \mathbb{Z}^2 \setminus \{0\}} K^{(0)}(n) = O(\log^2 T).
\]
This proves that
\[
\sup_{1 < R < T} |E_m(R)| \leq C \log^2 T.
\]
Lemma 4.4 is proved.

We omit proof of Lemma 4.5 and pass now to more complicated Lemmas 4.6, 4.7. Lemma 4.5 is proved similarly, with some simplifications (see also the proof of Theorem 3.1 in [Ble]).

**Proof of Lemma 4.6.** Assume \( \lambda_0 \in \Gamma^{0} \subset \Gamma_\alpha \). Let us split \( E_m(R) \) into two parts:
\[
E_m^{(1)}(R) = \sum_{n \in \mathbb{Z}^2 \setminus \mathbb{Z}^2 \setminus \{0\}} K(n), \quad E_m^{(2)}(R) = \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} K(n),
\]
where \( K(n) \) is defined in (5.1) and
\[
L_{\gamma} = \{ \zeta \in \mathbb{R}^2 : \text{dist}(\zeta, L) \leq \gamma \}, \quad L = \{ \zeta = \lambda \nu, \lambda \in \mathbb{R}^1 \},
\]
with some \( \gamma > 0 \) which will be chosen later. Let us evaluate \( E_m^{(1)}(R) \).
Assume first that \( \lambda_0 \) is rational. Then we can choose \( \gamma > 0 \) such that \( L_{\gamma} \setminus L \) contains no integer points. In this case
\[
E_m^{(1)}(R) = \sum_{n \in \mathbb{Z}^2 \setminus \mathbb{Z}^2 \setminus \{0\}} \phi(2\pi n \delta) R^2 \tilde{Z}_m^{(0)}(2\pi n R) (-1)^n.
\]
By Lemma 3.5, if \( \zeta \in L \), then
\[
\tilde{Z}_m^{(0)}(\zeta) = -i \exp(i Y(\zeta)) |\zeta|^{-4/3} \text{Ai}(0)(1/2) \sigma'(\lambda_0)|^{-1/3} + O(|\zeta|^{-5/3}),
\]
\[
\text{Ai}(0) = 3^{-2/3} \Gamma^{-1}(2/3),
\]
and hence
\[
E_m^{(1)}(R) = R^{2-3} \Phi_m(R; \delta) + O(R^{1/2}),
\]
with
\[
\Phi_m(R; \delta) = \sum_{n \in \mathbb{Z}^2 \setminus \mathbb{Z}^2 \setminus \{0\}} \phi(2\pi n \delta) (-i) \sin(2\pi Y(n) R)
\]
\[
\times |2\pi n|^{-4/3} \text{Ai}(0)(1/2) \sigma'(\lambda_0)^{-1/3} (-1)^n.
\]
Since
\[
\sum_{n \in \mathbb{Z}^2 \setminus \{0\}} |1 - \phi(2\pi q \delta)| q^{-4/3} \leq C \delta^{1/3} = CT^{-1/3},
\]
we obtain that
\[
E_m^{(1)}(R) = R^{2-3} \Phi_m(R) + O(R^{1/3} + R^{2/3} T^{-1/3}),
\]
where
\[
\Phi_m(R) = (1/2) 3^{-2/3} \pi^{-4/3} \Gamma^{-1}(2/3) \sigma'(\lambda_0)^{-1/3} \sum_{n \in \mathbb{Z}^2 \setminus \mathbb{Z}^2 \setminus \{0\}} (-1)^n |n|^{-4/3} \sin(2\pi Y(n) R)
\]
is a periodic function of \( R \).
Assume now that $v_k$ is Diophantine,
\[ |n \cdot v_k| \geq C|n|^{-\zeta}, \quad 0 < \zeta < 1. \quad \text{(5.7)} \]
In this case we put $\gamma = 1$. Define
\[ E_{\text{in}}^{(R; N)} = \sum_{n \in \mathbb{Z}^2 \cap L \cdot n \leq N} K(n). \]
Let us prove that
\[ \sup_{1 < R < \infty} R^{-1/2} \left| E_{\text{in}}^{(R; N)} - E_{\text{in}}^{(R)} \right| \leq C N|n|^{-1/4}. \quad \text{(5.8)} \]
Indeed, $|\text{Ai}(y)| \leq C|y|^{-1/4}$ and $|\text{Ai}(z)| \leq C_0|z - \xi(n)|$, and hence (3.7) implies that
\[ |\zeta_{n}(\zeta)| \leq C|\zeta|^{-3/2} |\xi(\zeta)| - |\xi(\zeta)|^{-1/4} \]
and
\[ |K(n)| \leq C R^{1/2} |n|^{-3/2} |\xi(n)| - |\xi(\zeta)|^{-1/4}. \]
Therefore
\[ R^{-1/2} \left| E_{\text{in}}^{(R; N)} - E_{\text{in}}^{(R)} \right| = R^{-1/2} \left| \sum_{n \in \mathbb{Z}^2 \cap L \cdot n > N} K(n) \right| \leq C R^{-1/2} \sum_{n \in \mathbb{Z}^2 \cap L \cdot n > N} |n|^{-3/2} |\xi(n)| - |\xi(\zeta)|^{-1/4}. \quad \text{(5.9)} \]
Due to the Diophantine condition (5.7),
\[ |\xi(n)| - |\xi(v_k)| \geq C|n|^{-\zeta + 1/4}. \]
Let $J \geq N$. Order all $n \in \mathbb{Z}^2 \cap L$ with $1 \leq |n| \leq 2J$ and $n \cdot v_k > 0$ in the increasing order of $|n \cdot v_k| = |n_k v_{k1} - n_1 v_{k2}|$:
\[ |n^{(1)} \cdot v_k| \leq |n^{(2)} \cdot v_k| \leq \ldots. \]
Then (5.7) implies that
\[ |n^{(1)} \cdot v_k| \geq C_0 J^{-1/2}. \]
and
\[ |n^{(j+1)} \cdot v_k| \geq |n^{(j)} \cdot v_k| + C_0 J^{-1/2}, \quad j \geq 1. \]
so that
\[ |n^{(j)} \cdot v_k| \geq C_0 J^{-1/2}, \]
and hence
\[ |\xi(n^{(0)}) - \xi(v_k)| \geq C_1 J^{-1/2}. \]
and
\[ \sum_{n \in \mathbb{Z}^2 \cap L \cdot n > N} |n|^{-3/2} |\xi(n)| - |\xi(v_k)|^{-1/4} \leq C J^{-3/2} J^{12 + 2c} \sum_{J} j^{-1/4} \leq C_1 J^{-1/4}. \]
This implies
\[ \sum_{n \in \mathbb{Z}^2 \cap L \cdot n > N} |n|^{-3/2} |\xi(n)| - |\xi(v_k)|^{-1/4} \leq C N|n|^{-1/4}, \]
and hence (5.8) follows from (5.9). Let us evaluate now $E_{\text{in}}^{(J)}(R)$.
From (3.9),
\[ |E_{\text{in}}^{(J)}(R) - E_{\text{in}}^{(J)}(R) - E_{\text{in}}^{(R)}| \leq C R^2 \sum_{n \in \mathbb{Z}^2 \cap L} |\psi(n)| |\phi(2\pi n \delta)| \leq C R^{-5/2} |n| - |\xi(v_k)|^{-1/4}. \quad \text{(5.11)} \]
where
\[ E_{\text{in}}^{(J)}(R) = \sum_{n \in \mathbb{Z}^2 \cap L} K^\pm(n) \]
and
\[ K^\pm(n) = R^2 \pi^{-1} |\psi(n)| |\phi(2\pi n \delta)| |n|^{-3/2} |\sigma_\pm(n)|^{-1/2} (1)^\chi \cos(2\pi R Y_\delta(n) - \phi_\delta). \quad \text{(5.12)} \]
The right-hand side in (5.11) is estimated as follows. Let $d(\zeta) = \text{dist}(\zeta, L)$. Then
\[ \text{RHS} \leq C R^{-1/2} \sum_{n \in \mathbb{Z}^2 \cap L} |\psi(n)| |\phi(2\pi n \delta)| |n|^{-3/2} d(n)^{-1/4} |n|^{-7/4} \]
\[ = C R^{-1/2} \sum_{n \in \mathbb{Z}^2 \cap L} |\psi(n)| |\phi(2\pi n \delta)| |n|^{-3/2} d(n)^{-7/4}. \]
Since, for $p = 1, 2, \ldots$,
\[ \sum_{n \in \mathbb{Z}^2 \cap L, p \leq |n| < p+1} |\psi(n)| d(n)^{-7/4} \leq C, \]
and

\[ \sup_{p \in [0, 1]} |\psi(2\pi p\delta)| \leq C(1 + p\delta)^{-\delta}, \]

we obtain

\[ \text{RHS} \leq CR^{-1/2} \sum_{p \geq 1} (1 + p\delta)^{-\delta} p^{-3/4} = C_0 R^{-1/2} \delta^{-1/4} = C_0 R^{-1/2} T^{1/4}, \]

so that

\[ |E_{im}(R) - E_{im}^*(R) - E_{im}(R)| \leq CR^{-1/2} T^{1/4}. \quad (5.13) \]

Define

\[ F_{im}^\delta(R) = R^{-1/2} E_{im}^\delta(R) = R^{-1/2} \sum_{n \in \mathbb{Z}^2 \setminus \mathbb{L}_e} K^\delta(n), \]

\[ F_{im}(R; N, \delta) = R^{-1/2} \sum_{n \in \mathbb{Z}^2 \setminus \mathbb{L}_e, |n| \leq N} K^\delta(n). \quad (5.14) \]

The central point in our proof is the following.

**Lemma 5.1.** For all \( N, T \geq 1, \)

\[ \frac{1}{T} \int_0^T |F_{im}(R) - F_{im}(R; N, \delta)|^2 \, dR \leq C(N^{-1/3} + T^{-1/4}), \quad \delta = T^{-1}. \]

We give the proof of Lemma 5.1 below, in the end of this section. Now let us derive Lemma 4.6 from Lemma 5.1.

Assume that \( \alpha \) is rational. Define

\[ F_{im}(R) = R^{-1/2}(E_{im}(R) - R^{2\delta} \Phi_{im}(R)) \]

with \( \Phi_{im}(R) \) given in (5.6). Then by (5.5)

\[ F_{im}(R) = R^{-1/2} E_{im}^\delta(R) + O(R^{-1/6}), \quad 1 \leq R \leq T, \]

and, by (5.13),

\[ R^{-1/2} E_{im}^\delta(R) = F_{im}(R) + F_{im}(R) + O(R^{-1/3} T^{1/4}), \quad 1 \leq R \leq T, \]

so that

\[ F_{im}(R) = F_{im}(R) + F_{im}(R) + O(R^{-1/6} + R^{-1/4} T^{1/4}), \quad 1 \leq R \leq T, \]

which implies that

\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T |F_{im}(R) - F_{im}^\delta(R) - F_{im}(R)|^2 \, dR = 0. \quad (5.15) \]

Define

\[ F_{im}^\delta(R; N) = \sum_{n \in \mathbb{Z}^2, |n| \leq N} K(n, \delta) \]

where

\[ K(n) = \pi^{-1} \psi(n) |\theta(n)| n^{-3/2} |\sigma_{\pm}(n)|^{-1/2} (-1)^n \cos(2\pi R Y_{\pm}(n) - \Phi_{\pm}). \quad (5.16) \]

Observe that \( F_{im}^\delta(R; N) \) is a finite trigonometric sum, and hence

\[ |F_{im}^\delta(R; N, \delta) - F_{im}^\delta(R; N)| \leq C(N) \delta = C(N) T^{-1}. \]

Therefore Lemma 5.1 implies that

\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T |F_{im}^\delta(R) - F_{im}(R; N)|^2 \, dR \leq CN^{-1/3}, \]

and hence from (5.15) we deduce that

\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T |F_{im}(R) - F_{im}(R; N) - F_{im}(R; N)|^2 \, dR \leq CN^{-1/3}. \quad (5.17) \]

This implies that \( F_{im}(R) \in B^2 \) and (4.19) is the Fourier expansion of \( F_{im}(R) \). For rational \( \alpha \) Lemma 4.6 is proved.

In the case of Diophantine \( \alpha \) we define \( F_{im}(R) = R^{-1/2} E_{im}(R) \). Then

\[ F_{im}(R) = F_{im}^\delta(R) + F_{im}^{1/2}(R), \quad \delta = R^{-1/2} E_{im}(R), \quad j = 1, 2, \]

and similarly to (5.17) we have that

\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T |F_j(R) - F_j(R; N) - F_j(R; N)|^2 \, dR \leq CN^{-1/3} \quad (5.18) \]

with \( F_j(R; N) \) defined in (5.16).

Then (5.8) implies that

\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T |F_{im}^\delta(R) - F_{im}^{1/2}(R; N)| \, dR \leq CN^{-1/3}. \quad (5.19) \]
where

\[ F^{(1)}_{im}(R; N) = R^{32} \sum_{n \in \mathbb{Z} \cap [0, N]} \psi_i(n) \bar{\phi}(2\pi n\delta) \bar{\phi}(2\pi n R)(-1)^{n}. \]

This is a finite sum and from (3.9) we obtain that

\[ |F^{(1)}_{im}(R; N) - \tilde{F}^{(1)}_{im}(R; N) - \tilde{F}^{(1)}_{im}(R; N)| \leq C(N)(R^{-1} + T^{-1}) \]  \hspace{1cm} (5.20)

with

\[ \tilde{F}^{(1)}_{im}(R) = \sum_{n \in \mathbb{Z} \cap [0, N]} K_1(n). \]

It follows from (5.18)–(5.20) that

\[ \limsup_{T \to \infty} \frac{1}{T} \int_0^T |F_{im}(R) - F_{im}(R; N) - \tilde{F}^{(1)}_{im}(R; N) - \tilde{F}^{(1)}_{im}(R; N)|^2 dR \leq C N^{(c-1)/4}, \]

which proves Lemma 4.6 for Diophantine \( v_2 \).

**Proof of Lemma 5.1.** To simplify notations we will omit ± in sub- and superscripts. From (5.12) and (5.14)

\[ \frac{1}{T} \int_0^T |F_{im}(R) - F_{im}(R; N, \delta)|^2 dR \]

\[ = \pi^{-2} \sum_{n, n' \in \mathbb{Z} \cap [0, N], n' > N} \bar{\psi}_i(n)\psi_i(n') \times \bar{\phi}(2\pi n\delta) \bar{\phi}(2\pi n'\delta) \delta(n)\delta(n') |n|^{-32} |n'|^{-32} |\sigma(n)|^{1/2} |\sigma(n')|^{1/2} I(n, n'), \]  \hspace{1cm} (5.21)

where

\[ I(n, n') = \frac{1}{T} \int_0^T \cos(2\pi Y(n)R - \phi(n)) \cos(2\pi Y(n')R - \phi(n')) dR. \]

Observe that

\[ |I(n, n')| \leq \min\{1, T^{-1} |Y(n) - Y(n')|^{-1}\}. \]  \hspace{1cm} (5.22)

In addition,

\[ |\bar{\phi}(2\pi n\delta)| \leq C(1 + Y(n)\delta)^{-5}, \quad C|n| \leq Y(n) \leq C|n|, \]

and the RHS of (5.21) is symmetric in \( n, n' \). This implies that

\[ \frac{1}{T} \int_0^T |F_{im}(R) - F_{im}(R; N, \delta)|^2 dR \leq C_0 \sum_{n, n' \in \mathbb{Z} \cap [0, N], n' > N} H(n, n') \]  \hspace{1cm} (5.23)

with

\[ H(n, n') = (1 + Y(n)\delta)^{-5} |\sigma(n)|^{1/2} |\sigma(n')|^{1/2} \min\{1, T^{-1} |Y(n) - Y(n')|^{-1}\} \]

and \( \beta > 0 \). First we estimate \( \sum_{n, n' \in \mathbb{Z} \cap [0, N]} H(n, n') \) with

\[ S(N) = \{n, n' \in \mathbb{Z} \cap [0, N], Y(n) \geq Y(n') > \beta N; Y(n) - Y(n') \geq 1\}. \]  \hspace{1cm} (5.24)

Let us fix some \( n \) with \( Y(n) \geq \beta N \) and define the layers

\[ S(N, n, j) = \{n' \in \mathbb{Z} \cap [0, N], Y(n') \geq \beta N; j \geq 1 \geq Y(n) - Y(n') \geq j\}, \]

\[ 1 \leq j \leq Y(n) - \beta N. \]

Observe that \( |\sigma(n)| \leq C |x(n) - x(v_k)|^{-1/2} \), and hence

\[ \sum_{n, n' \in \mathbb{Z} \cap [0, N]} \psi_i(n) \delta(n') \sigma(n')^{1/2} \leq C \text{Area } S(N, n, j) \leq C_0 (Y(n) - j) \]

and therefore

\[ \sum_{n, n' \in \mathbb{Z} \cap [0, N]} \psi_i(n) \delta(n') \sigma(n')^{1/2} Y(n')^{-32} T^{-3/2} (Y(n) - Y(n'))^{-1} \]

\[ \leq C \sum_{j=1}^{Y(n) - j} (Y(n) - j)^{-1/2} T^{-j} \]

\[ \leq C_0 T^{-1} \log Y(n), \]

where \( S(N, n) = \bigcup_{j=0}^\infty S(N, n, j). \) Making a summation in \( n \) we obtain now that

\[ \sum_{(n, n') \in S(N)} H(n, n') \leq CT^{-1} \sum_{n, n' \in \mathbb{Z} \cap [0, N], Y(n') > \beta N} \psi_i(n) (1 + Y(n)\delta)^{-3} \times Y(n')^{-3} \log Y(n)|\sigma(n)|^{-1/2}. \]

Define the layers

\[ S_j = \{n \in \mathbb{Z} \cap [0, N], j \leq Y(n) \leq j + 1\}, \quad j > \beta N. \]
Then
\[ \sum_{n \in S_j} \psi(n)\theta(n)|\sigma(n)|^{-1/2} \leq C \text{ Area } S_j \leq C_0 j, \]
and hence
\[ \sum_{(n,n') \in S(N)} H(n,n') \leq C T^{-1} \sum_{j \geq N} (1 + j\delta)^{-5} j^{-1} \log j \]
\[ \leq C_0 T^{-1} |\log \delta|^2 \]
\[ = C_0 T^{-1} \log^2 T. \] (5.25)

It remains to estimate \( \sum_{n \in S(N)} H(n,n') \) where
\[ S_0(N) = \{ n, n' \in \mathbb{Z}^2 \mid L_n; Y(n) \geq Y(n') > \beta N, 1 \geq Y(n) - Y(n') \geq 0 \}. \]

Let us fix some \( n \) with \( T \geq Y(n) > \beta N \). Define the layers
\[ S_0(N, n, j) = \{ n' \in \mathbb{Z}^2 \mid L_j; Y(n') > \beta N; (j + 1) T^{-1} \geq Y(n) - Y(n') \geq j T^{-1}) \}, \]
\[ j = 0, 1, \ldots, T. \]

To estimate the sum over \( S_0(N, n, j) \) we use the following.

**LEMMA 5.2 (2/3-estimate).** Let
\[ Y(\xi) = |\xi| f(|\xi| \cdot \alpha - x_0)|^{1/2}, \] (5.26)
where \( f(t) \in C^\infty([0, 1]), \alpha > 0, f(t) > 0, f'(0) = 0, f''(0) \neq 0 \). If \( \phi(x) \in C^\infty([x - 0, x_0 + \varepsilon]) \) with supp \( \phi(x) \) near \( x_0 \), then
\[ \sum_{n \in \Pi(R, R^{1/3}(x_0, L_j))} \psi(n)|\sigma(n)|^{-1/2} \leq C^{1/3}, \] (5.27)
where
\[ \Pi(R) = \{ n \in \mathbb{Z}^2 \mid x_0 \leq x(n) \leq x_0 + \varepsilon; R \leq Y(n) \leq R + R^{1/3} \} \]
and \( L = \{ \xi \mid \phi(\xi) = \phi_0 \}. \)

**Remark.** If we step away from \( x_0 \) putting \( x_0 + \varepsilon \leq x(n) \leq x_0 + \varepsilon, \varepsilon > 0 \), in \( \Pi(R) \) (instead of \( x_0 \leq x(n) \leq x_0 + \varepsilon \)), then (5.27) reduces to a well-known 2/3-estimate (see, e.g., [CdV1]).

**Proof of Lemma 5.2.** is given in the appendix to the paper.

With the help of Lemma 5.2, we obtain that, if \( T^3 \geq Y(n) \), then
\[ \sum_{n \in S_0(N, n, j)} \psi(n')\theta(n')|\sigma(n')|^{-1/2} \leq CY(n)^{2/3} \]
(observe that \( |\sigma(n')|^{-1/2} \leq C|z(n') - x(n')|^{-1/2} \leq C|z(n') - x(n)|^{-1/2} \), and
\[ \sum_{n \in S_0(N, n, j)} \psi(n')\theta(n')|\sigma(n')|^{-1/2} Y(n')^{3/2} \min \{ 1, T^{-1}(Y(n) - Y(n'))^{-1} \} \]
\[ \leq CY(n)^{-3/2} Y(n)^{3/2} T^{-1} j^{-1} Y(n) \]
\[ = CY(n)^{1/2} T^{-1} j^{-1}, \quad \text{if } T \geq j \geq 2, \]
and \( \leq CY(n)^{1/5} j \) if \( j = 0, 1 \). Hence
\[ \sum_{n \in S_0(N, n, j)} \psi(n')\theta(n')|\sigma(n')|^{-1/2} Y(n')^{3/2} \min \{ 1, T^{-1}(Y(n) - Y(n'))^{-1} \} \]
\[ \leq C(Y(n)^{1/3} T^{-1} \log T + Y(n)^{1/5}), \]
where
\[ S_0(N, n) = \{ n' \in \mathbb{Z}^2 \mid L_j; Y(n') > \beta N, 1 \geq Y(n) - Y(n') \geq 0 \}. \]

Making a summation over \( n \), we obtain
\[ \sum_{(n,n') \in S_0(N, n, j) \in S^3} H(n,n') \leq C \sum_{n \in \mathbb{Z}^2 \mid L_j; Y(n) > \beta N} \psi(n)(1 + Y(n)\delta)^{-5} \theta(n)|\sigma(n)|^{-1/2} \times |\sigma(n)|^{-1/2} (Y(n)^{1/3} T^{-1} \log Y(n) + Y(n)^{1/5}) = C_0. \]

Consider the layers
\[ S_j = \{ n \in \mathbb{Z}^2 \mid L_j; j + 1 \geq Y(n) \geq j \}, \quad T^3 \geq j \geq \beta N. \]

The width of \( S_j \) is of order \( 1 \), and a simple argument shows that
\[ \sum_{n \in S_j} \psi(n)\theta(n)|\sigma(n)|^{-1/2} \leq C j. \]

This implies
\[ I_0 \leq C \sum_{j \geq N} (1 + j\delta)^{-5} j^{3/2} j^{1/6} T^{-1} \log j + j^{1/5} \]
\[ \leq C_0(\delta^{-2/3} |\log \delta| T^{-1} + N^{-1/3}) \]
\[ = C_0(T^{-1} \log T + N^{-1/3}). \]
Thus
\[ \sum_{(n, n') \in S_0(N), Y(n) \in T^3} H(n, n') \leq C(T^{-1/3} \log T + N^{-1/3}). \] (5.28)

The final step is to estimate
\[ \sum_{(n, n') \in S_0(N), Y(n) \in T^3} H(n, n') \]
and this is quite simple. Since, for \((n, n') \in S_0(N), H(n, n') \leq C(1 + Y(n)\delta)^{-3}Y(n)^{-2} |\sigma(n)|^{-1/2} |\sigma(n')|^{-1/2}\]
and
\[ \sum_{n: Y(n) > Y_0} |\sigma(n)|^{-1/2} \leq CY(n), \]
we obtain
\[ \sum_{(n, n') \in S_0(N), Y(n) \in T^3} H(n, n') \]
\[ \leq C \sum_{(n, n') \in S_0(N), Y(n) \in T^3} (1 + Y(n)\delta)^{-3}Y(n)^{-2} |\sigma(n)|^{-1/2} \]
\[ \leq C_0 \sum_{j = 1}^{T^2} (1 + j\delta)^{-3}j \]
\[ \leq C_1 T^{-2}. \] (5.29)

From (5.23), (5.25), (5.28), and (5.29), Lemma 5.1 follows.

Proof of Lemma 4.7. The proof of Lemma 4.7 is similar in its main steps to the proof of Lemma 4.6. Assume \(a_0 = \eta(\epsilon_0) \in \Gamma_0 \subset \Gamma_\infty\). By (4.15),
\[ \sum_{n \in \mathbb{Z}^3 \setminus \{0\}} K(n) \]
with
\[ K(n) = R^{3\epsilon_2} \psi(2\pi n\delta)\phi_\infty(n) \sum_{m = 1}^{T^2} (2\pi Rm)^{-\epsilon_1}. \]
Define
\[ L = \{ \lambda \in \mathbb{R}, \lambda \} \subset \mathbb{R}, \quad L_1 = \{ n \in \mathbb{Z}: \text{dist}(n, L) \leq 1 \}, \]
\[ F^{(1)}_{1, m}(R) = \sum_{n \in L_1, n \neq 0} K(n), \quad F^{(2)}_{1, m}(R) = \sum_{n \in L_1} K(n), \]
so that \( F_{1, m}(R) = F^{(1)}_{1, m}(R) + F^{(2)}_{1, m}(R)\). By (3.15),
\[ \sum_{n \in \mathbb{Z}^3 \setminus \{0\}} K(n) \leq C(n)^{-3\epsilon_2}, \]
and hence
\[ \sum_{n \in \mathbb{Z}^3 \setminus \{0\}} \sum_{n \neq L_1} |K(n)| \leq C N^{-1/2}. \] (5.31)

Define
\[ K_\infty(n) = n^{-1 |n|^{-3\epsilon_2}} \psi(2\pi \epsilon n\delta) |\sigma(n)|^{-1/2} \]
and
\[ \psi(2\pi \epsilon n\delta) \sum_{n \in \mathbb{Z}^3 \setminus \{0\}} \sum_{n \in \mathbb{Z}^3 \setminus \{0\}} |K(n)| \leq C R^{-1/2} \sum_{n \in \mathbb{Z}^3 \setminus \{0\}} |\psi(2\pi \epsilon n\delta)| |n|^{-\epsilon_2} |\sigma(n)|^{-1/2} \]
\[ \leq R^{-1/2} \log^2 T. \] (5.33)

The following lemma holds:

**Lemma 5.3.** For all \(N, T \geq 1\),
\[ \sum_{n \in \mathbb{Z}^3 \setminus \{0\}} \left| \sum_{n \in \mathbb{Z}^3 \setminus \{0\}} K_\infty(n) \right|^2 \leq C(N^{-1/3} + T^{-1/4}). \]

We omit the proof of this lemma since it is basically the same as the proof of Lemma 3.3 in [Ble1]. (See also the proof of Lemma 5.1 above, where a similar statement is proved in a more complicated situation.)

Lemma 3.3 implies that, for a fixed \(n \in \mathbb{Z}^3 \setminus \{0\}, K(n)\) converges to \(K_\infty(n)\) as \(R \to \infty\), and hence
\[ \lim_{T \to \infty} \sum_{n \in \mathbb{Z}^3 \setminus \{0\}} |K(n) - K_\infty(n)|^2 dR = 0, \]
and by Lemma 5.3,
\[ \lim_{T \to \infty} \sum_{n \in \mathbb{Z}^3 \setminus \{0\}} |F_{1, m}(R) - \sum_{n \in \mathbb{Z}^3 \setminus \{0\}} K_\infty(n)|^2 dR = 0. \]

This proves Lemma 4.7.
6. Energy levels and closed geodesics. In this section we prove Theorem 1.2. Let
\[ N(R) = \# \{ E_n \leq R \}, \quad N_{\text{ad}}(R) = \# \{ n \in \mathbb{Z}^2 : Z_2(n_1 + 1/2, n_2) \leq R^2, |n_2| \leq n_1 \} \]  
(6.1)

and
\[ \Gamma = \{ p \in \mathbb{R}^2 : Z_2(p) = 1, |p_2| \leq p_1 \}. \]  
(6.2)

Recall that DH is defined by (1.12) and Hypothesis D by (4.5), (4.6).

**Lemma 6.1.** DH implies Hypothesis D for \( \Gamma \).

**Proof.** By (2.4), \( \Gamma \) is the graph of the function
\[ p_1 = g(p_2) = |p_2| + \pi^{-1} \int_a^b (1 - p_2^2 f^{-2}(s))^{1/2} ds, \quad f(s) = f(b) = |p_2|, \quad |p_2| \leq f_{\text{max}}. \]  
(6.3)

\( g(p_2) \) is an even \( C^\infty \) function, so we assume \( p_2 \geq 0 \). The function
\[ \frac{dp_1}{dp_2} = 1 - \pi^{-1} \int_a^b p_2 f^{-2}(s)(1 - p_2^2 f^{-2}(s))^{-1/2} ds \]  
(6.4)

has a nice geometric interpretation.

**Proposition 6.2.**
\[ \frac{dp_1}{dp_2} |_{p_1 = 1} = -\omega(I), \quad I \geq 0. \]  
(6.5)

**Proof.** An equation of \( \gamma(I) \) is
\[ \frac{d\phi}{dl} = If^{-2}(s), \quad \frac{ds}{dl} = (1 - I^2 f^{-2}(s))^{1/2}, \]  
(6.6)

where \( l \) is the normal coordinate on \( \gamma \). Hence
\[ \frac{d\phi}{ds} = If^{-2}(s)(1 - I^2 f^{-2}(s))^{-1/2} \]
and
\[ \omega(I) = \pi^{-1} \int_a^b \frac{d\phi}{ds} ds = 1 = \pi^{-1} \int_a^b If^{-2}(s)(1 - I^2 f^{-2}(s))^{-1/2} ds - 1. \]

Comparing this with (6.4) we obtain (6.5). Proposition 6.2 is proved.

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Observe that an inflection point on \( \Gamma \) is characterized by \( d^2 p_1/dp_2^2 = 0 \). By (6.5) this is equivalent to \( \omega'(I) = 0 \). Similarly, the nondegeneracy of the inflection point is characterized by \( \omega''(I) \neq 0 \). In addition, the Diophantine condition (4.6) is equivalent to (1.12), and hence DH implies Hypothesis D. Lemma 6.1 is proved.

Lemma 6.1 implies that Theorem 4.1 holds for \( N_{\text{ad}}(R) \). Our goal now is to find geometric interpretation of frequencies and amplitudes in formulas (4.11), (4.12).

Consider a finite geodesic \( \gamma \) which starts at \( x_0 = (s_{\text{max}}, 0) \) at some angle \(-\pi/2 \leq x \leq \pi/2\) to the north direction. \( \gamma \) is uniquely determined by \( I = \sin x \) and \( I = |\gamma| \).

Let \( G \) be the set of all \( \gamma(I, l) \), \( -1 \leq I \leq 1, l > 0 \).

Assume that \( \Gamma \) is defined as in (6.3). Define two maps:
\[ p : G \to \Gamma, \quad \xi : G \to \mathbb{R}^3 \setminus \{0\}, \]  
(6.6)

where
\[ p : \gamma = \gamma(I, l) \to p(\gamma) = (p(I, l)), \quad \xi : \gamma = \gamma(I, l) \to \xi(\gamma) = (l \tau^{-1}(I), \omega(I) \tau^{-1}(I)). \]  
(6.7)

**Proposition 6.3.** \( p(\gamma) \) and \( \xi(\gamma) \) satisfy
\[ n_4(p(\gamma)) = |\xi(\gamma)|^{-1} \xi(\gamma) \]  
(6.8)

and
\[ p(\gamma) \cdot \xi(\gamma) = (2\pi)^{-1} |\gamma|. \]  
(6.9)

**Proof.** (6.5) implies that \( n_4(p(\gamma)) \) is colinear with the vector \( (I, \omega(I)) \) as well as \( \xi(\gamma) \), and hence (6.8) follows. To prove (6.9) observe that the both sides of (6.9) depend linearly on \( |\gamma| \), so it is sufficient to prove (6.9) in the particular case when \( |\gamma| = \tau(I) \).

In this case (6.9) reduces to
\[ g(I) + 1 \omega(I) = (2\pi)^{-1} \tau(I). \]  
(6.10)

Since
\[ g(I) = 1 + \pi^{-1} \int_a^b (1 - I^2 f^{-2}(s))^{1/2} ds, \]
\[ \omega(I) = \pi^{-1} \int_a^b If^{-2}(s)(1 - I^2 f^{-2}(s))^{-1/2} ds - 1, \]
and by (6.6)
\[ \tau(I) = 2 \int_a^b ds = 2 \int_a^b (1 - I^2 f^{-2}(s))^{-1/2} ds, \]
(6.10) follows. Proposition 6.3 is proved.
(6.8) implies that
\[ \xi(\gamma) \in L_{n_1}(\pi(\gamma)) = \{ \lambda n_1(\pi(\gamma)) \}, \lambda > 0 \}.
\]
Hence we can define the map \( \pi : G \to N_\gamma \Gamma \), where
\[ N_\gamma \Gamma = \bigcup_{\gamma \in \Gamma} L_{n_1}(\pi(\gamma)), \]
as \( \gamma \mapsto (\pi(\gamma), \xi(\gamma)) \). Observe that \( \pi \) is one-to-one.

**Proposition 6.4.** \( \gamma \in G \) is a closed geodesic if and only if \( \xi(\gamma) \in \mathbb{Z}^2 \). In this case
\[
Y(\pi(\gamma))|\xi(\gamma)| = (2\pi)^{-1} |\gamma|,
\]
\[
|\sigma(\pi(\gamma))|^{-1/2} |\xi(\gamma)|^{-3/2} = |\omega(I)|^{-1/2} |\tau(I)| |\gamma|^{-3/2},
\]
\[
\text{sgn} \sigma(\pi(\gamma)) = \text{sgn} \omega(I), \quad \gamma = \gamma(I, I).
\]

**Proof.** Observe that \( \gamma(I, I, I) \neq 0 \) is a closed geodesic with \( n_1 \) revolutions around the axis and \( n_2 \) oscillations along the meridian if and only if \( I = |\gamma| = n_1 \tau(I) \) and \( \omega(I) = (n_2, n_2) - 1 \), so that \( \xi(\gamma) = (n_2, n_1 - n_2) \in \mathbb{Z}^2 \). Similarly, \( \gamma = \gamma(0, 0) \) is a closed geodesic if and only if \( I = |\gamma| = n_2 \tau(I) \), so that \( \xi(\gamma) = (n_1, 0) \). This proves the first part of Proposition 6.4.

To prove (6.11) let us notice that \( Y(\pi(\gamma))|\xi(\gamma)| = \pi(\gamma)^* \xi(\gamma) \), and hence (6.11) follows from (6.9). Let us prove (6.12). We have
\[
\sigma(I) = \frac{g'(I)}{(1 + g'(I))^2} |\xi(I)| |I|^{-1/2}(1 + \omega(I))^2 |\tau(I)|,
\]
and, since \( g'(I) = -\omega(I) \),
\[
\sigma(\pi(\gamma)) = \frac{\omega(I)}{(1 + \omega(I))^2} |\tau(I)| |\gamma|^{-3/2}.
\]

On the other hand, by (6.7)
\[
|\xi(\gamma)| = |\gamma| |\tau(I)|^{-1}(1 + \omega(I))^2 |\tau(I)|,
\]
and hence
\[
|\sigma(\pi(\gamma))|^{-1/2} |\xi(\gamma)|^{-3/2} = |\omega(I)|^{-1/2} |\tau(I)|^{-3/2}(1 + \omega(I))^2 |\tau(I)|^{-3/2}.
\]

(6.12) is proved. (6.13) follows from (6.14). Proposition 6.4 is proved.
where \( \phi(\xi) \in C_0^\infty(\mathbb{R}^2) \), \( \phi(\xi) \ni 0, \phi(\xi) = 0 \) if \( |\xi| > 1 \) and \( \int_{\mathbb{R}^2} \phi(\xi) \, d\xi = 1 \), and
\[
I_d(R) = \sum_{n \in \mathbb{Z}^2} Q_d(n; R).
\]

Put
\[
\delta = R^{-1/3}. \tag{A.2}
\]

Observe that
\[
Q_d(\xi; R) \leq C Q_d(\xi; R),
\]
so that \( I(R) \leq CI_d(R) \) and to prove (A.1) it is sufficient to show that
\[
I_d(R) \leq CR^{1/6}. \tag{A.3}
\]

By the Poisson summation formula,
\[
I_d(R) = \sum_{n \in \mathbb{Z}^2} \phi(2\pi n) \tilde{Q}(2\pi n; R). \tag{A.4}
\]

The term \( n = 0 \) is
\[
\tilde{Q}(0; R) = \int_{\Pi(R)} Q(\xi; R) \, d\xi \leq CR^{1/3} \int_0^R \left| \xi_1 \right|^{-1/2} d\xi_2 \leq C_d R^{1/6},
\]
and hence we may consider only \( n \neq 0 \). Let \( p_1 = p \cdot e_1, p_2 = p \cdot e_2 \). If \( p_2 \gg \gamma |p_1|, \gamma > 0 \), then
\[
\tilde{Q}(p; R) = \int_{\Pi(R)} Q(\xi; R) \exp(ip\xi) \, d\xi \tag{A.5}
\]
can be estimated in the following way.

First we integrate in (A.5) by the lines \( |p|^{-1} p \cdot \xi = c \) and then in \( c \), so that
\[
\tilde{Q}(p; R) = \int_{-\infty}^\infty \exp(i|p|c) S(c; R) \, dc,
\]
where
\[
S(c; R) = \int_{\Pi(R) \cap \{ |p|^{-1} p \cdot \xi = c \}} \tilde{\lambda}(\xi_1) \psi(R^{-1} \xi_1) |\xi_2|^{-1/2} \, d\xi.
\]

If \( \epsilon \ll \gamma \), then the lines \( |p|^{-1} p \cdot \xi = c \) cross \( \Pi(R) \) transversally, which implies that
\[
S(c; R) = R^{-1/3} \lambda_0(c - c_0) \psi_0(R^{-1} (c - c_0); R) |c - c_0|^{-1/2},
\]
where \( c_0 = R |p|^{-1} p \cdot \xi_0 \), \( \lambda_0(t) \in C^\infty \), \( \lambda_0(t) = 0 \) in the vicinity of 0, \( \lambda_0(t) = 1 \) in the vicinity of \( \infty \), and \( \psi_0(\cdot; R) \in C_0^\infty([0, \infty)) \) has a limit in \( C_0^\infty \)-topology as \( R \to \infty \). Therefore
\[
|\tilde{Q}(p; R)| \leq CR^{-1/3} |p|^{-5}
\]
and
\[
\left| \sum_{n \in \mathbb{Z}^2 \cap \{ |p|^{-1} p \cdot \xi = c, |p| \neq 0 \}} \tilde{Q}(2\pi n; R) \right| \leq CR^{-1/3}. \tag{A.6}
\]

The main difficulty is to estimate \( \tilde{Q}(p; R) \) when \( |p_2| < \gamma |p_1|, \gamma > 0 \). We have
\[
\tilde{Q}(p; R) = J \int_0^\infty dt \exp(ip_2 t) \lambda(t) \psi(R^{-1} t) \int_0^b ds \exp(ip_1 s), \tag{A.7}
\]
\[
t = \xi_1, s = \xi_1, a = h(t; R), b = h(t; R + R^{-1/3}),
\]
where \( J \) is Jacobian and \( h(t; R) = Rh(R^{-1} t) \). Let
\[
\tilde{Q}_0(p; R) = J \int_0^\infty dt \exp(ip_2 t) \psi(R^{-1} t) \int_0^b ds \exp(ip_1 s). \tag{A.8}
\]
The difference \( \tilde{Q}_1(p; R) = \tilde{Q}(p; R) - \tilde{Q}_0(p; R) \) can be estimated as follows.
Observe that
\[
\int_0^\infty dt \exp(i\phi(t)(1 - \lambda^2)\psi(R^{-1}t^2)) \leq C(1 + p_2)^{-1/2},
\]
and hence
\[
|\tilde{Q}_o(p; R)| \leq C(1 + p_2)^{-1/2} R^{-1/3}
\]
and
\[
\sum_{n \in \mathbb{Z}, \alpha \in [\mathbb{N}]} |\tilde{Q}_o(2\pi n; R)| \leq CR^{-1/3} \sum_{n \in \mathbb{Z}} |\hat{\phi}(2\pi n\delta)| (1 + |n \cdot e_z|)^{-1/2} \leq C_0 R^{-1/3}\delta^{-3/2} = C_0 R^{1/6}.
\]
Thus it remains to estimate a similar sum with \(\tilde{Q}_o(2\pi n; R)\).

Let us integrate in \(s\) in (A.8) and make the change of variable \(u = (R^{-1}t)^{1/2}\). This gives
\[
\tilde{Q}_o(p; R) = 2J(i\phi(1)^{-1}R^{1/2}(W(p; R) - U(p; R))\]
where
\[
U(p; R) = \int_0^\infty du \exp(iR p_2 u^2) \psi(u^2) \exp(iR p_1 h(u)),
\]
\[
W(p; R) = \int_0^\infty du \exp(iR p_2 u^2) \psi(u^2) \exp[iR(1 + R^{-4/3})p_1 h((1 + R^{-4/3})u)].
\]

Let us evaluate first
\[
U(p; R) = \int_0^\infty du \exp[iR p_1(y u^2 + h(u))] \psi(u^2), \quad y = p_2/p_1.
\]
There exists a \(C^\infty\) change of variable \(T = T(u, y)\) such that \(T(0) = 0, (\partial T/\partial u)(0) = 1\) and
\[
yu^2 + h(u) = b(y) + a(y) T + T^{3/3},
\]
where \(a(y), b(y) \in C^\infty, a(0) = b(0) = 0\) (see [Hör2]). In addition,
\[
a(y) + T^{2}(0, y) = 0,
\]
which follows if we differentiate both sides of (A.11) at \(u = 0\). After this change of variable, \(U(p; R)\) reduces to
\[
U(p; R) = \exp(iR p_1 b(y)) \int_0^\infty \exp(iR p_1 (-c(y) T + T^{3/3})) \psi_0(T, y) \frac{\partial u}{\partial T}(T, y) dT,
\]
with \(c(y) = T(0, y)\) and \(\psi_0(T, y) = \psi(T^2(T, y))\). Following [Hör2] let us divide \((\partial u/\partial T)(T, y)\) by \(-c(y) + T^2\) with a remainder:
\[
\frac{\partial u}{\partial T}(T, y) = r(T, y)(-c(y) + T^2) + r_o(y) + r_1(y) T.
\]
Then the first term after substitution into (A.12) allows integration by parts, which gives an extra \((R p_1)^{-1}\), and the other two terms give the main contribution to \(U(p; R)\),
\[
U(p; R) = \exp(iR p_1 b(y)) \{((R p_1)^{-1/3} V(c(y)(R p_1)^{1/3} r_o(y)) + (R p_1)^{-2/3} V_6(c(y)(R p_1)^{1/3} r_1(y)) + O((R p_1)^{-1})\}
\]
where
\[
V(x) = \int_x^\infty \exp[i(-x^2 T + T^{3/3})] dT, \quad V_0(x) = \int_x^\infty T \exp[i(-x^2 T + T^{3/3})] dT.
\]
The method of stationary phase gives the asymptotics of \(V(x), V_0(x)\) when \(|x| \to \infty\), and from this asymptotics we obtain \(|V(x)| \leq C(1 + |x|)^{-1/2}\) and \(|V_0(x)| \leq C(1 + |x|)^{1/2}\). Therefore
\[
|U(p; R)| \leq C |R p_1|^{-1/3} \min\{1, |(p_2/p_1)| (R p_1)^{1/3}\}
\]
\[
= C \min\{R^{-1/3} |p_2|^{-1/3}, R^{-1/3} |p_1|^{-1/3}\}.
\]
A similar estimate holds for \(W(p; R)\), and finally we obtain
\[
|\tilde{Q}_o(p; R)| \leq C |p_1|^{-1} \min\{|p_2|^{-1}, R^{1/3} |p_1|^{-1/3}\}.
\]
Hence
\[
\sum_{n \in \mathbb{Z}, \alpha \in [\mathbb{N}]} |\hat{\phi}(2\pi n\delta)| |\tilde{Q}_o(2\pi n; R)| \leq CR^{1/6} \sum_{n \in \mathbb{Z}, \alpha \in [\mathbb{N}]} |n \cdot e_z|^{-4/3} \leq C_\alpha R^{1/6},
\]
and
\[
\sum_{1 < n < e^{1/2}} \sum_{n \cdot e_1 \leq n} \phi(2\pi n) \frac{\phi(2\pi n : R)}{|n \cdot e_1|^{1/2}} \leq C \sum_{1 < n < e^{1/2}} \sum_{n \cdot e_1 \leq n} \frac{\phi(2\pi n)}{|n \cdot e_1|^{1/2}} \\
\leq C_d e^{-1/2} = C_d R^{1/6}.
\]

Lemma 5.2 is proved.

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