Invent. math. 142, 351–395 (2000)
Digital Object Identifier (DOI) 10.1007/s002220000092

Inventiones mathematicae

Universality and scaling of correlations between zeros on complex manifolds*

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Oblatum 17-VI-1999 & 31-III-2000
Published online: 5 June 2000 –

Abstract. We study the limit as $N \to \infty$ of the correlations between simultaneous zeros of random sections of the powers L^N of a positive holomorphic line bundle L over a compact complex manifold M, when distances are rescaled so that the average density of zeros is independent of N. We show that the limit correlation is independent of the line bundle and depends only on the dimension of M and the codimension of the zero sets. We also provide some explicit formulas for pair correlations. In particular, we prove that Hannay's limit pair correlation function for SU(2) polynomials holds for all compact Riemann surfaces.

Introduction

This paper is concerned with the local statistics of the simultaneous zeros of k random holomorphic sections $s_1, \dots, s_k \in H^0(M, L^N)$ of the N^{th} power L^N of a positive Hermitian holomorphic line bundle (L, h) over a compact Kähler manifold M (where $k \le m = \dim M$). The terms 'random' and 'statistics' are with respect to a natural Gaussian probability measure $d\nu_N$ on $H^0(M, L^N)$ which we define below. In the special case where $M = \mathbb{CP}^m$ and L is the hyperplane section bundle $\mathcal{O}(1)$, sections of L^N correspond to holomorphic polynomials of degree N, and $(H^0(\mathbb{CP}^m, \mathcal{O}(N)), d\nu_N)$ is known as the ensemble of SU$(m+1)$ polynomials in the physics literature. To obtain local statistics, we expand a ball U around a given point z^0 by a factor $\sqrt{N}$ so that the average density of sim-

* Research partially supported by NSF grants #DMS-9623214 (first author), #DMS-9800479 (second author), #DMS-9703775 (third author).

ultaneous scaled zeros is independent of N. We then ask whether the simultaneous scaled zeros behave as if thrown independently in $\sqrt{N}U$ or how they are correlated. Correlations between (unscaled) zeros are measured by the *n-point zero correlation function* $K_{nk}^N(z^1,\dots,z^n)$, and those between scaled zeros are measured by the scaled correlation function $K_{nk}^N(\frac{z^1}{\sqrt{N}},\dots,\frac{z^n}{\sqrt{N}})$. Our main result is that the large N limits of the scaled n-point correlation functions $K_{nk}^N(\frac{z^1}{\sqrt{N}},\dots,\frac{z^n}{\sqrt{N}})$ exist and are universal, i.e. are independent of $M,\ L$ and h as well as the point z^0. Moreover, the scaling limit correlation functions can be calculated explicitly. We find that the limit correlations are short range, i.e. that simultaneous scaled zeros behave quite independently for large distances.

To state our problems and results more precisely, we begin with provisional definitions of the correlation functions $K_{nk}^N(z^1,\dots,z^n)$ and of the scaling limit. (See §§1–2 for the complete definitions and notation.) In order to provide a standard yardstick for our universality results, we give M the Kähler metric ω given by the (positive) curvature form of h. The metrics h and ω then induce a Hilbert space inner product on the space $H^0(M,L^N)$ of holomorphic sections of L^N, for each $N\geq 1$. In the spirit of [SZ1] we use this $\mathcal{L}^2$-norm to define a Gaussian probability measure $d\nu_N$ on $H^0(M,L^N)$. When we speak of a random section, we mean a section drawn at random from this ensemble. More generally, we can draw k sections $(s_1,\dots,s_k)$ independently and at random from this ensemble. Let $Z_{(s_1,\dots,s_k)}$ denote their simultaneous zero set and let $|Z_{(s_1,\dots,s_k)}|$ denote the "delta measure" with support on $Z_{(s_1,\dots,s_k)}$ and with density given by the natural Riemannian $(2m-2k)$-volume defined by the metric ω. (See (11) for the precise definition.) To define the n-point zero correlation measure $K_{nk}^N(z^1,\dots,z^n)$ we form the product measure

$$|Z_{(s_1,\dots,s_k)}|^n = \big(\underbrace{|Z_{(s_1,\dots,s_k)}|\times\cdots\times|Z_{(s_1,\dots,s_k)}|}_{n}\big)\ \text{ on }\ M^n := \underbrace{M\times\cdots\times M}_{n}\,. \tag{1}$$

To avoid trivial self-correlations, we puncture out the generalized diagonal in M^n to get the punctured product space

$$M_n = \{(z^1,\dots,z^n)\in M^n : z^p\neq z^q \ \text{ for } p\neq q\}\,. \tag{2}$$

We then restrict $|Z_{(s_1,\dots,s_k)}|^n$ to M_n and define $K_{nk}^N(z^1,\dots,z^n)$ to be the expected value $E(|Z_{(s_1,\dots,s_k)}|^n)$ of this measure with respect to ν_N. When $k=m$, the simultaneous zeros almost surely form a discrete set of points and so this case is perhaps the most vivid. Roughly speaking, $K_{nk}^N(z^1,\dots,z^n)$ gives the probability density of finding simultaneous zeros at $(z^1,\dots,z^n)$.

The first correlation function K_{1kN} just gives the expected distribution of simultaneous zeros of k sections. In a previous paper [SZ1] by two of the authors, it was shown (among other things) that the expected distribution of zeros is asymptotically uniform; i.e.

$$K^N_{1k}(z^0) = c_{mk}N^k + O(N^{k-1})\,,$$

for any positive line bundle (see [SZ1, Prop. 4.4]). The question then arises of determining the higher correlation functions. As was first observed by [BBL] and [Han] for SU(2) polynomials and by [BD] for real polynomials in one variable, the zeros of a random polynomial are non-trivially correlated, i.e. the zeros are not thrown down like independent points. We will do the same for SU$(m+1)$ polynomials and will give formulas for their scaling limit n-point correlation functions (see (91)–(93)). By universality of the scaling limit, the same formulas hold for any M, L, h.

To introduce the scaling limit, let us return to the case $k = m$ where the simultaneous zeros form a discrete set of points. Since an m-tuple of sections of L^N will have N^m times as many zeros as m-tuples of sections of L, it is natural to expand U by a factor of $\sqrt{N}$ to get a density of zeros that is independent of N. That is, we choose coordinates $\{z_q\}$ for which $z^0 = 0$ and $\omega(z^0) = \frac{i}{2}\sum_q dz_q \wedge d\bar{z}_q$ and then rescale $z \mapsto \frac{z}{\sqrt{N}}$. Were the zeros thrown independently and at random on U, the conditional probability density of finding a simultaneous zero at a point w given a zero at z would be a constant independent of (z, w). Non-trivial correlations (for any codimension $k \in \{1, \ldots, m\}$) are measured by the difference between 1 and the (normalized) *n-point scaling limit zero correlation function*

$$\widetilde{K}^\infty_{nkm}(z^1, \ldots, z^n) = \lim_{N\to\infty} \left(c_{mk}N^k\right)^{-n} K^N_{nk}\left(\frac{z^1}{\sqrt{N}}, \ldots, \frac{z^n}{\sqrt{N}}\right), \qquad (z^1, \ldots, z^n) \in U_n\,. \tag{3}$$

Our main result (Theorem 3.6) is universality of the scaling limit correlation functions:

> *The n-point scaling limit zero correlation function $\widetilde{K}^\infty_{nkm}(z^1, \ldots, z^n)$ is given by a universal rational function, homogeneous of degree 0, in the values of the function $e^{i\Im(z\cdot\bar{w}) - \frac{1}{2}|z-w|^2}$ and its first and second derivatives at the points $(z, w) = (z^p, z^{p'})$, $1 \le p, p' \le n$. Alternately it is a rational function in $z^p_q, \bar{z}^p_q, e^{z^p\cdot\bar{z}^{p'}}$.*

The function $e^{i\Im(z\cdot\bar{w}) - \frac{1}{2}|z-w|^2}$ which appears in the universal scaling limit is (up to a constant factor) the Szegö kernel $\Pi^{\mathbf{H}}_1(z, w)$ of level one for the reduced Heisenberg group $\mathbf{H}^n_{\rm red}$ (cf. § 1). Its appearance here owes to the fact that the correlation functions can be expressed in terms of the Szegö kernels

$\Pi_N(x, y)$ of L^N. I.e., let X denote the circle bundle over M consisting of unit vectors in L^*; then $\Pi_N(x, y)$ is the kernel of the orthogonal projection $\Pi_N : \mathcal{L}^2(X) \to \mathcal{H}^2_N(X) \approx H^0(M, L^N)$. Indeed we have (Theorem 2.4):

The n-point correlation $\widetilde{K}^N_{nk}(z^1, \dots, z^n)$ is given by the above universal rational function, applied this time to the values of the Szegö kernel Π_N and its first and second derivatives at the points $(z^p, z^{p'})$.

In view of this relation between the correlation functions and the Szegö kernel, it suffices for the proof of the universality theorem 3.6 to determine the scaling limit of the Szegö kernel Π_N and to show its universality. Indeed we shall show the following (Theorem 3.1):

Use the above local coordinates and a well-chosen local frame e_L about a point $z_0 \in M$ to obtain local coordinates $(z_1, \dots, z_m, \theta)$ in a neighborhood $\widetilde{U} \approx U \times S^1$ of $(z^0, e^*_L(z^0)) \in X$. We then have

$$N^{-m}\Pi_N\left(\frac{z}{\sqrt{N}}, \frac{\theta}{N}; \frac{z'}{\sqrt{N}}, \frac{\theta'}{N}\right) = \Pi_1^{\mathbf{H}}(z, \theta; z', \theta') + O(N^{-1/2})\ .$$

A more precise form of this asymptotic formula for Π_N is given in [SZ2, Theorem 2.3], where the formula is also extended to almost complex symplectic manifolds.

The fact that the correlation functions can be expressed in terms of the Szegö kernel may be explained in (at least) two ways. The first is that the correlation functions may be expressed in terms of the joint probability density $D^N_{nk}(x, \xi; z)dxd\xi$ of the (vector-valued) random variables

$$(x, \xi) = (x^1, \dots, x^n, \xi^1, \dots, \xi^n)\ , \quad x^p = (s_1(z^p), \dots, s_k(z^p)),$$
$$\xi^p = (\nabla s_1(z^p), \dots, \nabla s_k(z^p))\ ,$$

given by the values of the k sections and their covariant derivatives at the n points $\{z^p\}$. Our method of computing the correlation functions is based on the following probabilistic formula (Theorem 2.1):

For N sufficiently large so that the density $D^N_{nk}(x, \xi; z)$ is given by a continuous function, we have

$$K^N_{nk}(z) = \int d\xi\ D^N_{nk}(0, \xi; z) \prod_{p=1}^{n} \det\left(\xi^p_j \xi^{p*}_{j'}\right)_{1 \le j, j' \le k}\ ,$$
$$z = (z^1, \dots, z^n) \in M_n\ ,$$

where $\xi = (\xi^1, \dots, \xi^n)$ and $\xi^{p*}_j : L^N_{z^p} \to T_{M,z^p}$ denotes the adjoint to $\xi^p_j : T_{M,z^p} \to L^N_{z^p}$.

This formula, which is valid in a more general setting, is based on the approach of Kac [Ka] and Rice [Ri] (see also [EK]) for zeros of functions on $\mathbb{R}^1$, and of [Hal] for zeros of (real) Gaussian vector fields. Since our probability measure $d\nu_N$ (on the space of sections) is Gaussian, it follows that D^N_{nk} is also a Gaussian density. It will be proved in §2.3 that the covariance matrix of this Gaussian may be expressed entirely in terms of Π_N and its covariant derivatives. This type of formula for the correlation function of zeros was previously used in [BD], [Han] and the works cited above. We believe that this formula will have interesting applications in geometry.

A second link between correlation functions and Szegö kernels is given by the Poincaré-Lelong formula. In fact, this was our original approach to computing the correlation functions in the codimension 1 case. For the sake of brevity, we will not discuss this approach here; instead we refer the reader to our companion article [BSZ1].

From the universality of our answers, it follows that the scaling limit pair correlation functions depend only on the distance between points:

$$\widetilde{K}^\infty_{2km}(z^1, z^2) = \kappa_{km}(r)\,, \qquad r = |z^1 - z^2|\,, \tag{4}$$

where κ_{km} depends only on the dimension m of M and the codimension k of the zero set. In §4, we give explicit formulas for the limit pair correlation functions κ_{km} in some special cases. In particular, for the hypersurface case we have

$$\kappa_{1m}(r) = \frac{\left[\frac{1}{2}(m^2+m)\sinh^2 t + t^2\right]\cosh t - (m+1)t\sinh t}{m^2\sinh^3 t} + \frac{(m-1)}{2m},$$
$$t = \frac{r^2}{2}\,.$$

Our calculation uses the Heisenberg model, which (although noncompact) is the most natural one since the scaled Szegö kernels are all equal to Π_1, and there is no need in this case to take a limit. We also discuss the hyperplane section bundle $\mathcal{O}(1) \to \mathbb{CP}^m$, which is the most studied, since the sections of its powers are the SU$(m+1)$ polynomials – homogeneous polynomials in $m+1$ variables – and the case $m = 1$ (the SU(2) polynomials) appears frequently in the physics literature (e.g., [BBL,FH,Han,KMW,Pr]). We give expressions for the zero correlations K^N_{nk} for the SU$(m+1)$ polynomials and by letting $N \to \infty$, we obtain an alternate derivation of our universal formula for the scaling limit correlation.

We show (Theorem 4.1) that $\kappa_{km}(r) = 1 + O(r^4 e^{-r^2})$ as $r \to +\infty$, and hence these correlations are short range in that they differ from the case of independent random points by an exponentially decaying term. We observe that when $\dim M = 1$, there is a repulsion between nearby zeros in the sense that $\kappa_{11}(r) \to 0$ as $r \to 0$, as was noted by Hannay [Han] and

Bogomolny-Bohigas-Leboeuf [BBL] for the case of SU(2) polynomials. These asymptotics are illustrated in Fig. 1 below (see also [Han]):

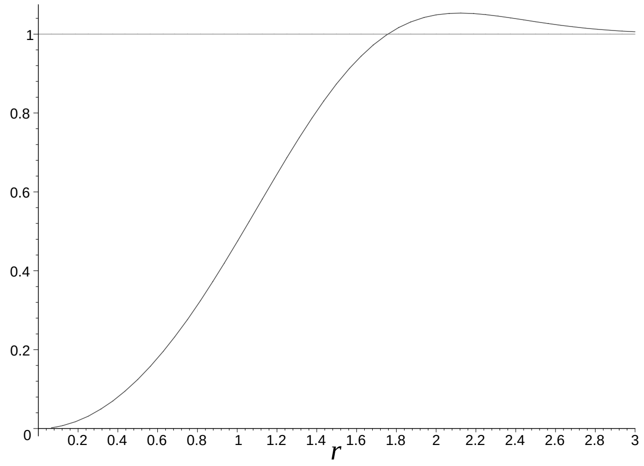

Fig. 1 The 1-dimensional limit pair correlation function κ_{11}

For $\dim M = 2$, we have instead $\kappa_{22}(r) \to \frac{3}{4}$ as the scaled distance $r \to 0$. The graph of the limit pair correlation function $\kappa_{22}(r)$ for the simultaneous scaled zeros of a random pair (s_1, s_2) of sections on any complex projective surface is illustrated in Fig. 2 below.

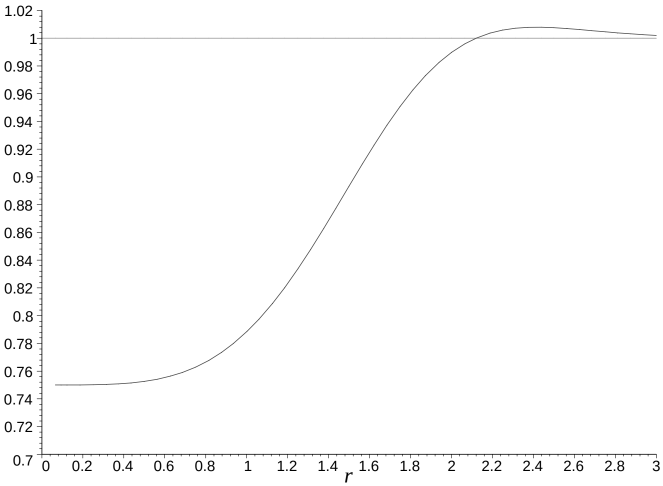

Fig. 2 The limit pair correlation function κ_{22}

The function $\kappa_{mm}(r)$ can be interpreted as the normalized conditional probability of finding a zero near a point z^1 given that there is a zero at a second point a scaled distance r from z^1 (in the case of discrete zeros in m dimensions). These graphs show that for dimensions 1 and 2, there is a unique scaled distance where this probability is maximized. It would be interesting to explore the behavior of the correlations in higher dimensions.

When $k < m$, the zero sets are subvarieties of positive dimension $m-k$; in this case the expected volume of the zero set in a small spherical shell of radius r and thickness ε about a point in the zero set must be $\gtrsim \varepsilon r^{2m-2k-1}$. Hence we have $\kappa_{km}(r) \gtrsim r^{-2k}$, for small r. The graph of the limit correlation function for the case $m = 2, k = 1$ is given in Fig. 3 below.

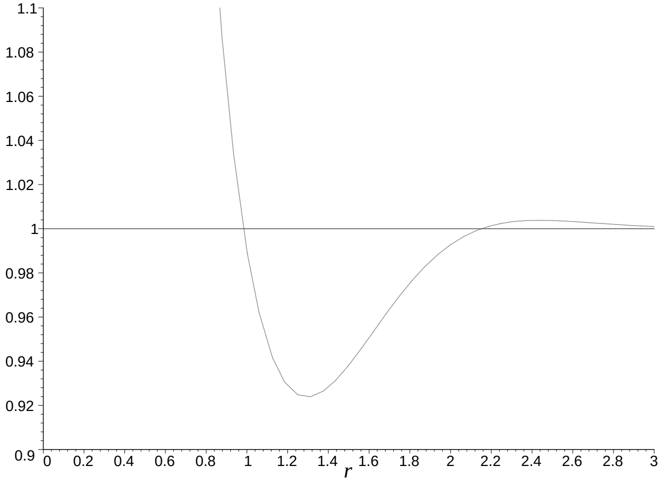

Fig. 3 The limit pair correlation function κ_{12}

To end this introduction, we would like to link our methods and results to a long tradition of (largely heuristic) results on universality and scaling in statistical mechanics (cf. [FFS]). One may view the rescaling transformation on U as generating a renormalization group. The intuitive picture in statistical mechanics is that the renormalization group should carry a given system (read "$L \to M$") to the fixed point of the renormalization group, i.e. to the scale invariant situation. We observe that the local rescaling of U is nothing other than the Heisenberg dilations $\delta_{\sqrt{N}}$ on $\mathbf{H}^m_{\mathrm{red}}$. Since these dilations are automorphisms of the (unreduced) Heisenberg group, the Szegö kernel of $\mathbf{H}^m$ is invariant under these dilations; i.e., it is the fixed point of the renormalization group. As predicted by this intuitive picture, we find that in the scaling limit all the invariants of the line bundle, in particular its zero-point correlation functions, are drawn to their values for the fixed point system (read "Heisenberg model").

Contents

1. Notation

We begin with some notation and basic properties of sections of holomorphic line bundles, their zero sets, Szegö kernels, and Gaussian measures. We also provide two examples that will serve as model cases for studying correlations of zeros of sections of line bundles in the high power limit.

1.1. Sections of holomorphic line bundles. In this section, we introduce the basic complex analytic objects: holomorphic sections and the currents of integration over their zero sets. We also introduce Gaussian probability measures on spaces of holomorphic sections. For background in complex geometry, we refer to [GH].

Let M be a compact complex manifold and let $L \to M$ be a holomorphic line bundle with a smooth Hermitian metric h; its curvature 2-form Θ_h is given locally by

$$\Theta_h = -\partial\bar\partial \log \|e_L\|_h^2 \ , \tag{5}$$

where e_L denotes a local holomorphic frame (= nonvanishing section) of L over an open set $U \subset M$, and $\|e_L\|_h = h(e_L, e_L)^{1/2}$ denotes the h-norm of e_L. We say that (L, h) is positive if the (real) 2-form $\omega = \frac{\sqrt{-1}}{2}\Theta_h$ is positive, i.e., if ω is a Kähler form. We henceforth assume that (L, h) is positive, and we give M the Hermitian metric corresponding to the Kähler form ω and the induced Riemannian volume form

$$dV_M = \frac{1}{m!}\omega^m \ . \tag{6}$$

Since $\frac{1}{\pi}\omega$ is a de Rham representative of the Chern class $c_1(L) \in H^2(M, \mathbb{R})$, the volume of M equals $\frac{\pi^m}{m!}c_1(L)^m$.

The space $H^0(M, L^N)$ of global holomorphic sections of $L^N = L \otimes \cdots \otimes L$ is a finite dimensional complex vector space. (Its dimension, given by the Riemann-Roch formula for large N, grows like N^m. By the Kodaira embedding theorem, the global sections of L^N give an embedding into a projective space for $N \gg 1$, and hence M is a *projective algebraic* *manifold*.) The metric h induces Hermitian metrics h^N on L^N given by $\|s^{\otimes N}\|_{h^N} = \|s\|_h^N$. We give $H^0(M, L^N)$ the Hermitian inner product

$$\langle s_1, s_2\rangle = \int_M h^N(s_1, s_2)dV_M \qquad \big(s_1, s_2 \in H^0(M, L^N)\big)\,, \tag{7}$$

and we write $|s| = \langle s, s\rangle^{1/2}$.

We now explain our concept of a "random section". We are interested in expected values and correlations of zero sets of k-tuples of holomorphic sections of powers L^N. Since the zeros do not depend on constant factors, we could suppose our sections lie in the unit sphere in $H^0(M, L^N)$ with respect to the Hermitian inner product (7), and we pick random sections with respect to the spherical measure. Equivalently, we could suppose that s is a random element of the projectivization $\mathbb{P}H^0(M, L^N)$. Another equivalent approach is to use Gaussian measures on the entire space $H^0(M, L^N)$. We shall use the third approach, since Gaussian measures seem the best for calculations. Precisely, we give $H^0(M, L^N)$ the complex Gaussian probability measure

$$d\nu_N(s) = \frac{1}{\pi^m}e^{-|c|^2}dc\,, \qquad s = \sum_{j=1}^{d_N} c_j S_j^N\,, \tag{8}$$

where $\{S_j^N\}$ is an orthonormal basis for $H^0(M, L^N)$ and dc is $2d_N$-dimensional Lebesgue measure. This Gaussian is characterized by the property that the $2d_N$ real variables $\Re c_j, \Im c_j$ ($j = 1, \dots, d_N$) are independent random variables with mean 0 and variance $\frac{1}{2}$; i.e.,

$$\mathbf{E}\, c_j = 0, \quad \mathbf{E}\, c_j c_k = 0, \quad \mathbf{E}\, c_j \bar{c}_k = \delta_{jk}\,.$$

Here and throughout this paper, $\mathbf{E}$ denotes expectation.

In general, a *complex Gaussian measure* (with mean 0) on a finite dimensional complex vector space V is a measure ν of the form (8), where the c_j are the coordinates with respect to some basis. Explicitly, the complex Gaussian measures on $\mathbb{C}^m$ are the probability measures of the form

$$\frac{e^{-\langle \Delta^{-1}z, z\rangle}}{\pi^m \det \Delta}dz \tag{9}$$

where $\Delta = (\Delta_k^j)$ is a positive definite Hermitian matrix and

$$\langle \zeta, z\rangle = \zeta \cdot \bar{z} = \sum_{q=1}^{m} \zeta_q \bar{z}_q$$

denotes the standard Hermitian inner product in $\mathbb{C}^m$. For the Gaussian measure (9), we have

$$\mathbf{E}\,(z_j z_k) = 0, \qquad \mathbf{E}\,(z_j \bar{z}_k) = \Delta_k^j\,. \tag{10}$$

If ν is a complex Gaussian on V and $\tau : V \to \widetilde{V}$ is a surjective linear transformation, then $\tau_*\nu$ is a complex Gaussian on $\widetilde{V}$. In particular, if $\widetilde{V} = \mathbb{C}^m$, then, $\tau_*\nu$ is of the form (9), where the covariance matrix Δ is given by (10) with $z_j = z_j \circ \tau : V \to \mathbb{C}$.

We shall consider the space $\mathcal{S} = H^0(M, L^N)^k$ $(1 \le k \le m)$ with the probability measure $d\mu = d\nu \times \cdots \times d\nu$, which is also Gaussian. Picking a random element of $\mathcal{S}$ means picking k sections of $H^0(M, L^N)$ independently and at random. For $s = (s_1, \dots, s_k) \in \mathcal{S}$, we let

$$Z_s = \{z \in M : s_1(z) = \cdots = s_k(z) = 0\}$$

denote the zero set of s. Note that if N is sufficiently large so that L^N is base point free, then for μ-a.a. $s \in \mathcal{S}$, we have codim $Z_s = k$. (Indeed, the set of s where codim $Z_s < k$ is a proper algebraic subvariety of $H^0(M, L^N)^k$. In fact, by Bertini's theorem, the Z_s are smooth submanifolds of complex dimension $m - k$ for almost all s, provided N is large enough so that the global sections of L^N give a projective embedding of M, but we do not need this fact here.) For these s, we let $|Z_s|$ denote Riemannian $(2m-2k)$-volume along the regular points of Z_s, regarded as a measure on M:

$$(|Z_s|, \varphi) = \int_{Z_s^{\mathrm{reg}}} \varphi d\,\mathrm{Vol}_{2m-2k} = \frac{1}{(m-k)!} \int_{Z_s^{\mathrm{reg}}} \varphi \omega^{m-k} . \tag{11}$$

It was shown by Lelong [Le] (see also [GH]) that the integral in (11) converges. (In fact, $|Z_s|$ can be regarded as the total variation measure of the closed current of integration over Z_s.) We regard $|Z_s|$ as a measure-valued random variable on the probability space $(\mathcal{S}, d\mu)$; i.e., for each test function $\varphi \in \mathcal{C}^0(M)$, $(|Z_s|, \varphi)$ is a complex-valued random variable.

1.2. Szegö kernels. As in [Ze,SZ1] we now lift the analysis of holomorphic sections over M to a certain S^1 bundle $X \to M$. This is a useful approach to the asymptotics of powers of line bundles and goes back at least to [BG].

We let L^* denote the dual line bundle to L, and we consider the circle bundle $X = \{\lambda \in L^* : \|\lambda\|_{h^*} = 1\}$, where h^* is the norm on L^* dual to h. Let $\pi : X \to M$ denote the bundle map; if $v \in L_z$, then $\|v\|_h = |(\lambda, v)|$, $\lambda \in X_z = \pi^{-1}(z)$. Note that X is the boundary of the disc bundle $D = \{\lambda \in L^* : \rho(\lambda) > 0\}$, where $\rho(\lambda) = 1 - \|\lambda\|^2_{h^*}$. The disc bundle D is strictly pseudoconvex in L^*, since Θ_h is positive, and hence X inherits the structure of a strictly pseudoconvex CR manifold. Associated to X is the contact form $\alpha = -i\partial\rho|_X = i\bar\partial\rho|_X$. We also give X the volume form

$$dV_X = \frac{1}{m!}\alpha \wedge (d\alpha)^m = \alpha \wedge \pi^* dV_M . \tag{12}$$

The setting for our analysis of the Szegö kernel is the Hardy space $\mathcal{H}^2(X) \subset \mathcal{L}^2(X)$ of square-integrable CR functions on X, i.e., functions that are annihilated by the Cauchy-Riemann operator $\bar\partial_b$ (see [St, pp. 592–594]) and are $\mathcal{L}^2$ with respect to the inner product

$$\langle F_1, F_2\rangle = \frac{1}{2\pi}\int_X F_1\overline{F_2}dV_X\,, \quad F_1, F_2 \in \mathcal{L}^2(X)\,. \tag{13}$$

Equivalently, $\mathcal{H}^2(X)$ is the space of boundary values of holomorphic functions on D that are in $\mathcal{L}^2(X)$. We let $r_\theta x = e^{i\theta}x$ ($x \in X$) denote the S^1 action on X and denote its infinitesimal generator by $\frac{\partial}{\partial\theta}$. The S^1 action on X commutes with $\bar\partial_b$; hence $\mathcal{H}^2(X) = \bigoplus_{N=0}^\infty \mathcal{H}^2_N(X)$ where $\mathcal{H}^2_N(X) = \{F \in \mathcal{H}^2(X) : F(r_\theta x) = e^{iN\theta}F(x)\}$. A section s of L determines an equivariant function $\hat s$ on L^* by the rule $\hat s(\lambda) = (\lambda, s(z))$ ($\lambda \in L^*_z, z \in M$). It is clear that if $\tau \in \mathbb{C}$ then $\hat s(z, \tau\lambda) = \tau\hat s$. We henceforth restrict $\hat s$ to X and then the equivariance property takes the form $\hat s(r_\theta x) = e^{i\theta}\hat s(x)$. Similarly, a section s_N of L^N determines an equivariant function $\hat s_N$ on X: put

$$\hat s_N(\lambda) = \left(\lambda^{\otimes N}, s_N(z)\right)\,, \quad \lambda \in X_z\,,$$

where $\lambda^{\otimes N} = \lambda \otimes \cdots \otimes \lambda$; then $\hat s_N(r_\theta x) = e^{iN\theta}\hat s_N(x)$. The map $s \mapsto \hat s$ is a unitary equivalence between $H^0(M, L^N)$ and $\mathcal{H}^2_N(X)$. (This follows from (12)–(13) and the fact that $\alpha = d\theta$ along the fibers of $\pi : X \to M$.)

We let $\Pi_N : \mathcal{L}^2(X) \to \mathcal{H}^2_N(X)$ denote the orthogonal projection. The level-N Szegö kernel $\Pi_N(x, y)$ is defined by

$$\Pi_N F(x) = \int_X \Pi_N(x, y)F(y)dV_X(y)\,, \quad F \in \mathcal{L}^2(X)\,. \tag{14}$$

It can be given as

$$\Pi_N(x, y) = \sum_{j=1}^{d_N} \widehat{S}_j^N(x)\overline{\widehat{S}_j^N(y)}\,, \tag{15}$$

where $S_1^N, \ldots, S_{d_N}^N$ form an orthonormal basis of $H^0(M, L^N)$. Pick a local holomorphic frame e_L for L over an open subset $U \subset M$, let e_L^* denote the dual frame, and write $h(z) = h(e_L(z), e_L(z)) = \|e_L\|_h^2$. The map $(z, e^{i\theta}) \mapsto e^{i\theta}h(z)^{1/2}e_L^*(z)$ gives an isomorphism $U \times S^1 \approx \pi^{-1}(U) \subset X$, and we use the coordinates (z, θ) to identify points of $\pi^{-1}(U)$. For $s \in H^0(M, L^N)$, we have

$$\hat s(z, \theta) = \left\langle s(z), e^{iN\theta}h(z)^{N/2}e_L^*(z)\right\rangle = e^{iN\theta}h(z)^{N/2}f(z), \quad s = fe_L^{\otimes N}\,. \tag{16}$$

Although the level-N Szegö kernel Π_N is defined on X, its absolute value is well-defined on M as follows: writing $S_j^N = f_j^N e_L^{\otimes N}$, we have

$$\begin{aligned}\Pi_N(z, \theta; w, \varphi) &= e^{iN(\theta-\varphi)}\Pi_N(z, 0; w, 0) \qquad (17)\\ &= e^{iN(\theta-\varphi)}h(z)^{N/2}h(w)^{N/2}\sum_{j=1}^{d_N} f_j^N(z)\overline{f_j^N(w)}\,,\end{aligned}$$

for $z, w \in U$. (Here we may take U to be the disjoint union of connected neighborhoods of z and w, if z is not close to w.) Thus we can write

$$|\Pi_N(z, w)| = |\Pi_N(z, 0; w, 0)| \, ,$$

which is independent of the choice of local frame e_L. On the diagonal we have

$$\Pi_N(z, z) = \Pi_N(z, \theta; z, \theta) = \sum_{j=1}^{d_N} \|S_j^N(z)\|_{h^N}^2 \, .$$

The Hermitian connection ∇ on L induces the decomposition $T_X = T_X^H \oplus T_X^V$ into horizontal and vertical components, and we let t^H denote the horizontal lift (to X) of a vector field t in M. We consider the horizontal operators on X:

$$d_{z_q}^H \stackrel{\text{def}}{=} d_{(\partial/\partial z_q)^H} \, , \quad d_{\bar{z}_q}^H \stackrel{\text{def}}{=} d_{(\partial/\partial \bar{z}_q)^H} \, ,$$

where $z_1, \ldots, z_m$ denote local holomorphic coordinates on M. We note that

$$d_{z_q}^H \hat{s} = \left(\nabla_{z_q}^N s\right)\widehat{\ }, \quad s \in H^0(M, L^N) \, , \tag{18}$$

where ∇^N is the induced connection on L^N. We then have

$$\begin{aligned} d_{z_q}^H \Pi_N(z, \theta; w, \varphi) &= \sum_{j=1}^{d_N} \left(\nabla_{z_q}^N S_j^N\right)\widehat{\ }(x)\overline{\widehat{S}_j^N(y)} \\ &= e^{iN(\theta-\varphi)} h(z)^{N/2} h(w)^{N/2} \sum_{j=1}^{d_N} f_{j;q}^N(z)\overline{f_j^N(w)} \, , \\ d_{z_p}^H d_{\bar{w}_q}^H \Pi_N(z, \theta; w, \varphi) &= \sum_{j=1}^{d_N} \left(\nabla_{z_p}^N S_j^N\right)\widehat{\ }(x)\overline{\left(\nabla_{w_q}^N S_j^N\right)\widehat{\ }(y)} \\ &= e^{iN(\theta-\varphi)} h(z)^{N/2} h(w)^{N/2} \sum_{j=1}^{d_N} f_{j;p}^N(z)\overline{f_{j;q}^N(w)} \, , \end{aligned} \tag{19}$$

$$\nabla_{z_q}^N = \nabla_{\partial/\partial z_q}^N \, , \qquad f_{j;q}^N = \frac{\partial f}{\partial z_q} + N f_j^N h^{-1} \frac{\partial h}{\partial z_q} \, .$$

We can also use (16) and (18) to describe the horizontal lift in local coordinates:

$$d_{z_q}^H = \frac{\partial}{\partial z_q} - \frac{i}{2} \frac{\partial \log h}{\partial z_q} \frac{\partial}{\partial \theta} \, . \tag{20}$$

1.3. Model examples. In two special cases we can work out the Szegö kernels and their derivatives explicitly, namely for the hyperplane section bundle over $\mathbb{CP}^m$ and for the Heisenberg bundle over $\mathbb{C}^m$, i.e. the trivial line bundle with curvature equal to the standard symplectic form on $\mathbb{C}^m$. These cases will be important after we have proven universality, since scaling limits of correlation functions for all line bundles coincide with those of the model cases.

In fact, the two models are locally equivalent in the CR sense. In the case of $\mathbb{CP}^m$, the circle bundle X is the $2m+1$ sphere S^{2m+1}, which is the boundary of the unit ball $B^{2m+2} \subset \mathbb{C}^{m+1}$. In the case of $\mathbb{C}^m$, the circle bundle is the reduced Heisenberg group $\mathbf{H}^m_{\rm red}$, which is a discrete quotient of the simply connected Heisenberg group $\mathbb{C}^m \times \mathbb{R}$. As is well-known, the latter is equivalent (in the CR and contact sense) to the boundary of B^{2m+2} (see [St]).

1.3.1. SU$(m+1)$*-polynomials.* For our first example, we let $M = \mathbb{CP}^m$ and take L to be the hyperplane section bundle $\mathcal{O}(1)$. Sections $s \in H^0(\mathbb{CP}^m, \mathcal{O}(1))$ are linear functions on $\mathbb{C}^{m+1}$; the zero divisors Z_s are projective hyperplanes. The line bundle $\mathcal{O}(1)$ carries a natural metric $h_{\rm FS}$ given by

$$\|s\|_{h_{\rm FS}}([w]) = \frac{|(s,w)|}{|w|}\,, \qquad w = (w_0,\dots,w_m) \in \mathbb{C}^{m+1}\,, \tag{21}$$

for $s \in \mathbb{C}^{m+1*} \equiv H^0(\mathbb{CP}^m, \mathcal{O}(1))$, where $|w|^2 = \sum_{j=0}^m |w_j|^2$ and $[w] \in \mathbb{CP}^m$ is the complex line through w. The Kähler form on $\mathbb{CP}^m$ is the Fubini-Study form

$$\omega_{\rm FS} = \frac{\sqrt{-1}}{2}\Theta_{h_{\rm FS}} = \frac{\sqrt{-1}}{2}\partial\bar\partial \log |w|^2\,. \tag{22}$$

The dual bundle $L^* = \mathcal{O}(-1)$ is the affine space $\mathbb{C}^{m+1}$ with the origin blown up, and $X = S^{2m+1} \subset \mathbb{C}^{m+1}$. The N-th tensor power of $\mathcal{O}(1)$ is denoted $\mathcal{O}(N)$. Elements $s_N \in H^0(\mathbb{CP}^m, \mathcal{O}(N))$ are homogeneous polynomials on $\mathbb{C}^{m+1}$ of degree N, and $\hat s_N = s_N|_{S^{2m-1}}$. The monomials

$$s^N_J = \left[\frac{(N+m)!}{\pi^m j_0!\cdots j_m!}\right]^{\frac12} z^J\,,\ z^J = z_0^{j_0}\cdots z_m^{j_m},\ J=(j_0,\dots,j_m),\ |J| = N \tag{23}$$

form an orthonormal basis for $H^0(\mathbb{CP}^m, \mathcal{O}(N))$. (See [SZ1, §4.2]; the extra factor $\left(\frac{m!}{\pi^m}\right)^{1/2}$ in (23) comes from the fact that here $\mathbb{CP}^m$ has the usual volume $\frac{\pi^m}{m!}$, whereas in [SZ1], the volume of $\mathbb{CP}^m$ is normalized to be 1.) Hence the Szegö kernel for $\mathcal{O}(N)$ is given by

$$\Pi_N(x,y) = \sum_{|J|=N} \frac{(N+m)!}{\pi^m j_0!\cdots j_m!} x^J \bar y^J = \frac{(N+m)!}{\pi^m N!}\langle x, y\rangle^N\,. \tag{24}$$

Note that the total Szegö kernel is given by

$$\Pi(x,y) = \sum_{N=1}^{\infty} \Pi_N(x,y) = \frac{m!}{\pi^m}(1 - \langle x, y\rangle)^{-(m+1)}$$
$$= 2\pi \times [\text{classical Szegö kernel on } S^{2m+1}]\,.$$

(The factor 2π is due to our normalization (13).)

1.3.2. The Heisenberg model. Our second example is the linear model $\mathbb{C}^m \times \mathbb{C} \to \mathbb{C}^m$ for positive line bundles $L \to M$ over Kähler manifolds and their associated Szegö kernels. It is most illuminating to consider the associated principal S^1 bundle $\mathbb{C}^m \times S^1 \to \mathbb{C}^m$, which may be identified with the boundary of the disc bundle $D \subset L^*$ in the dual line bundle. This S^1 bundle is the *reduced Heisenberg group* $\mathbf{H}^m_{\text{red}}$ (cf. [Fo], p. 23).

Let us recall its definition and properties. We start with the usual (simply connected) Heisenberg group $\mathbf{H}^m$ (cf. [Fo] [St]; note that different authors differ by factors of 2 and π in various definitions). It is the group $\mathbb{C}^m \times \mathbb{R}$ with group law

$$(\zeta, t)\cdot(\eta, s) = (\zeta + \eta, t + s + \Im(\zeta\cdot\bar\eta)).$$

The identity element is $(0,0)$ and $(\zeta,t)^{-1} = (-\zeta,-t)$. Abstractly, the Lie algebra of $\mathbf{H}_m$ is spanned by elements $Z_1,\dots,Z_m,\bar Z_1,\dots,\bar Z_m, T$ satisfying the canonical commutation relations $[Z_j,\bar Z_k] = -i\delta_{jk}T$ (all other brackets zero). Below we will select such a basis of left invariant vector fields.

$\mathbf{H}^m$ is a strictly convex CR manifold which may be embedded in $\mathbb{C}^{m+1}$ as the boundary of a strictly pseudoconvex domain, namely the upper half space $\mathcal{U}^m := \{z \in \mathbb{C}^{m+1} : \Im z_{m+1} > \frac12\sum_{j=1}^m |z_j|^2\}$. The boundary of $\mathcal{U}^m$ equals $\partial\mathcal{U}^m = \{z \in \mathbb{C}^{m+1} : \Im z_{m+1} = \frac12\sum_{j=1}^m |z_j|^2\}$. $\mathbf{H}^m$ acts simply transitively on $\partial\mathcal{U}^m$ (cf. [St], XII), and we get an identification of $\mathbf{H}^m$ with $\partial\mathcal{U}^m$ by:

$$[\zeta,t] \to (\zeta, t + i|\zeta|^2) \in \partial\mathcal{U}^m.$$

The Szegö projector of $\mathbf{H}^m$ is the operator $\Pi : \mathcal{L}^2(\mathbf{H}^m) \to \mathcal{H}^2(\mathbf{H}^m)$ of orthogonal projection onto boundary values of holomorphic functions on $\mathcal{U}^m$ which lie in $\mathcal{L}^2$. The kernel of Π is given by (cf. [St], XII §2 (29))

$$\Pi(x,y) = K(y^{-1}x), \qquad K(x) = -C_m\frac{\partial}{\partial t}[t + i|\zeta|^2]^{-m} \in \mathcal{D}'(\mathbf{H}^m)\,.$$

The linear model for the principal S^1 bundle described in §1.2 is the so-called reduced Heisenberg group $\mathbf{H}^m_{\text{red}} = \mathbf{H}^m/\{(0,2\pi k) : k \in \mathbb{Z}\} = \mathbb{C}^m \times S^1$ with group law

$$(\zeta, e^{it})\cdot(\eta, e^{is}) = (\zeta + \eta, e^{i(t+s+\Im(\zeta\cdot\bar\eta))}).$$

It is the principal S^1 bundle over $\mathbb{C}^m$ associated to the line bundle $L_{\mathbf{H}} = \mathbb{C}^m \times \mathbb{C}$. The metric on $L_{\mathbf{H}}$ with curvature $\Theta = \partial\bar\partial|z|^2$ is given by setting $h_{\mathbf{H}}(z) = e^{-|z|^2}$; i.e., $|f|_{h_{\mathbf{H}}} = |f|e^{-|z|^2/2}$. The reduced Heisenberg group $\mathbf{H}^m_{\mathrm{red}}$ may be viewed as the boundary of the dual disc bundle $D \subset L^*_{\mathbf{H}}$ and hence is a strictly pseudoconvex CR manifold.

It seems most natural to approach the analysis of the Szegö kernels on $\mathbf{H}^m_{\mathrm{red}}$ from the representation-theoretic point of view. Let us begin with the case $N = 1$. We thus consider the space $\mathcal{V}_1 \subset \mathcal{L}^2(\mathbf{H}^m_{\mathrm{red}})$ of functions f satisfying $\frac{1}{i}\frac{\partial}{\partial\theta} f = f$, which forms a (reducible) representation of $\mathbf{H}^m_{\mathrm{red}}$ with central character $e^{i\theta}$. By the Stone-von Neumann theorem there exists a unique (up to equivalence) representation (V_1, ρ_1) with this character and by the Plancherel theorem, $\mathcal{V}_1 \cong V_1 \otimes V_1^*$.

The space of CR functions in $\mathcal{V}_1$ is an irreducible invariant subspace. Here, by CR functions we mean the functions satisfying the left-invariant Cauchy-Riemann equations $\bar{Z}^L_q f = 0$ on $\mathbf{H}^m_{\mathrm{red}}$. Here, $\{\bar{Z}^L_q\}$ denotes a basis of the left-invariant anti-holomorphic vector fields on $\mathbf{H}^m_{\mathrm{red}}$. Let us recall their definition: we first equip $\mathbf{H}^m_{\mathrm{red}}$ with its left-invariant contact form $\alpha^L = \sum_q (u_q dv_q - v_q du_q) + d\theta$ $(\zeta = u + iv)$. The left-invariant CR holomorphic (resp. anti-holomorphic) vector fields Z^L_q (resp. $\bar{Z}^L_q$) are the horizontal lifts of the vector fields $\frac{\partial}{\partial z_q}$ (resp. $\frac{\partial}{\partial \bar{z}_q}$) with respect to α^L. They span the left-invariant CR structure of $\mathbf{H}^m_{\mathrm{red}}$ and the Z^L_q obviously have the form $Z^L_q = \frac{\partial}{\partial z_q} + A\frac{\partial}{\partial\theta}$ where the coefficient A is determined by the condition $\alpha^L(Z^L_q) = 0$. An easy calculation gives:

$$Z^L_q = \frac{\partial}{\partial z_q} + \frac{i}{2}\bar{z}_q\frac{\partial}{\partial\theta}, \qquad \bar{Z}^L_q = \frac{\partial}{\partial \bar{z}_q} - \frac{i}{2}z_q\frac{\partial}{\partial\theta}.$$

The vector fields $\{\frac{\partial}{\partial\theta}, Z^L_q, \bar{Z}^L_q\}$ span the Lie algebra of $\mathbf{H}^m_{\mathrm{red}}$ and satisfy the canonical commutation relations above.

We then define the Hardy space $\mathcal{H}^2(\mathbf{H}^m_{\mathrm{red}})$ of CR holomorphic functions, i.e. solutions of $\bar{Z}^L_q f = 0$, which lie in $\mathcal{L}^2(\mathbf{H}^m_{\mathrm{red}})$. We also put $\mathcal{H}^2_1 = \mathcal{V}_1 \cap \mathcal{H}^2(\mathbf{H}^m_{\mathrm{red}})$. The group $\mathbf{H}^m_{\mathrm{red}}$ acts by left translation on $\mathcal{H}^2_1$. The generators of this representation are the right-invariant vector fields $Z^R_q, \bar{Z}^R_q$ together with $\frac{\partial}{\partial\theta}$. They are horizontal with respect to the right-invariant contact form $\alpha^R = \sum_q (u_q dv_q - v_q du_q) - d\theta$ and are given by:

$$Z^R_q = \frac{\partial}{\partial z_q} - \frac{i}{2}\bar{z}_q\frac{\partial}{\partial\theta}, \qquad \bar{Z}^R_q = \frac{\partial}{\partial \bar{z}_q} + \frac{i}{2}z_q\frac{\partial}{\partial\theta}\,.$$

In physics terminology, Z^R_q is known as an annihilation operator and $\bar{Z}^R_q$ is a creation operator.

The representation $\mathcal{H}^2_1$ is irreducible and may be identified with the Bargmann-Fock space of entire holomorphic functions on $\mathbb{C}^m$ which are square integrable relative to $e^{-|z|^2}$ (or equivalently, holomorphic sections of the trivial line bundle $L_{\mathbf{H}} = \mathbb{C}^m \times \mathbb{C}$ mentioned above, with Hermitian metric $h_{\mathbf{H}} = e^{-|z|^2}$). The identification goes as follows: the function $\varphi_0(z, \theta) := e^{i\theta} e^{-|z|^2/2}$ is CR holomorphic and is also the ground state for the right invariant "annihilation operator;" i.e., it satisfies

$$\bar{Z}_q^L \varphi_0(z, \theta) = 0 = Z_q^R \varphi_0(z, \theta)\,.$$

Any element $F(z, \theta)$ of $\mathcal{H}_1^2$ may be written in the form $F(z, \theta) = f(z)\varphi_0$. Then $\bar{Z}_q^L F = (\frac{\partial}{\partial \bar{z}_q} f)\varphi_0$, so that F is CR if and only if f is holomorphic. Moreover, $F \in \mathcal{L}^2(\mathbf{H}^m_{\text{red}})$ if and only if f is square integrable relative to $e^{-|z|^2}$.

The Szegö kernel $\Pi_1^{\mathbf{H}}(z, \theta, w, \varphi)$ of $\mathbf{H}^m_{\text{red}}$ is by definition the orthogonal projection from $\mathcal{L}(\mathbf{H}^m_{\text{red}})$ to H_1^2. As will be seen below, $\Pi_1^{\mathbf{H}}(z, \theta, w, \varphi) = \frac{1}{\pi_m} e^{i(\theta-\varphi)} e^{(z\cdot\bar{w} - \frac{1}{2}|z|^2 - \frac{1}{2}|w|^2)}$, which is the left translate of φ_0 by $(-w, -\varphi)$. In the physics terminology it is the coherent state associated to the phase space point w.

So far we have set $N = 1$, but the story is very similar for any N. We define $\mathcal{H}_N^2$ as the space of square-integrable CR functions transforming by $e^{iN\theta}$ under the central S^1. By the Stone-von Neumann theorem there is a unique irreducible V_N with this central character. The main difference to the case $N = 1$ is that $\mathcal{H}_N^2$ is of multiplicity N^m. The Szegö kernel $\Pi_N^{\mathbf{H}}(x, y)$ is the orthogonal projection to $\mathcal{H}_N^2$ and is given by the dilate of $\Pi_1^{\mathbf{H}}$. Thus,

$$\Pi_N^{\mathbf{H}}(x, y) = \frac{1}{\pi^m} N^m e^{iN(\theta-\varphi)} e^{N\left(z\cdot\bar{w} - \frac{1}{2}|z|^2 - \frac{1}{2}|w|^2\right)}.$$

To prove these formulae for the Szegö kernels, we observe that the reduced Szegö kernels are obtained by projecting the Szegö kernel on $\mathbf{H}^m$ to $\mathbf{H}^m_{\text{red}}$ as an automorphic kernel, i.e.

$$\Pi^{\mathbf{H}}(x, y) = \sum_{n\in\mathbb{Z}} \Pi(x, y \cdot (0, 2\pi n)).$$

Let us write $x = (z, \theta)$, $y = (w, \varphi)$. Then the N-th Fourier component $\Pi_N^{\mathbf{H}}(x, y)$ of $\Pi^{\mathbf{H}}$, i.e. the projection onto square integrable holomorphic sections of L^N, is given by:

$$\Pi_N^{\mathbf{H}}(x, y) = \int_{\mathbb{R}} e^{-iNt} \Pi(e^{it}x, y)dt \;=\; \int_{\mathbb{R}} e^{-iNt} K(e^{it} y^{-1} x)dt$$
$$= \int_{\mathbb{R}} e^{-iNt} K(z - w, e^{i(\theta-\varphi+t+\Im(z\cdot\bar{w})})dt\,.$$

Here we abbreviated the element $(0, e^{it})$ by e^{it}. Change variables $t \mapsto t - \theta + \varphi - \Im(z \cdot \bar{w})$ to get

$$\begin{aligned}\Pi_N^{\mathbf{H}}(x, y) &= e^{iN(\theta-\varphi)} e^{iN\Im(z\cdot\bar{w})} \int_{\mathbb{R}} e^{-iNt} K(z - w, t)dt \\ &= e^{iN(\theta-\varphi)} e^{iN\Im(z\cdot\bar{w})} \hat{K}_t(z - w, N)\end{aligned}$$

where $\hat{K}_t$ is the Fourier transform of K with respect to the t variable. By [St, p. 585], the full $\mathbb{R}^{2m} \times \mathbb{R}$ Fourier transform of K is given by $\hat{K}(z, N) = C'_m e^{-|z|^2/2N}$, so by taking the inverse Fourier transform in the z variable we get the Fourier transform just in the t variable:

$$\Pi_N^{\mathbf{H}}(x, y) = \frac{1}{\pi^m} N^m e^{iN(\theta-\varphi)} e^{iN\Im(z\cdot\bar{w})} e^{-\frac{1}{2}N|z-w|^2}. \tag{25}$$

(Our constant factor $\frac{1}{\pi^m}$ in (25) is determined by the condition that $\Pi_N^{\mathbf{H}}$ is an orthogonal projection.)

In our study of the correlation functions, we will need explicit formulae for the horizontal derivatives of the Szegö kernel. The left-invariant derivatives are given by

$$\begin{aligned}N^{-m} Z_q^L \Pi_N^{\mathbf{H}}(z, \theta; w, \varphi) &= N(\bar{w}_q - \bar{z}_q)\Pi_N^{\mathbf{H}}(z, \theta; w, \varphi)\,, \\ N^{-m} Z_q^L \bar{W}_{q'}^L \Pi_N^{\mathbf{H}}(z, \theta; w, \varphi) &= N^2(z_{q'} - w_{q'})(\bar{w}_q - \bar{z}_q)\Pi_N^{\mathbf{H}}(z, \theta; w, \varphi) \\ &\quad + N\delta_{qq'} \Pi_N^{\mathbf{H}}(z, \theta; w, \varphi)\,.\end{aligned} \tag{26}$$

Comparing the definitions of the horizontal vector fields with (18), using $h_{\mathbf{H}} = e^{-|z|^2}$, we see that $d_{z_q}^H = Z_q^L$, as expected, since α^L agrees with the contact form α for $L_{\mathbf{H}}$ (as defined in §1.2). We will see later that our formulas for computing correlations are valid with any connection, and thus it is sometimes useful to also consider the right invariant derivatives:

$$\begin{aligned}N^{-m} Z_q^R \Pi_N^{\mathbf{H}}(z, \theta; w, \varphi) &= N\bar{w}_q \Pi_N^{\mathbf{H}}(z, \theta; w, \varphi)\,, \\ N^{-m} Z_q^R \bar{W}_{q'}^R \Pi_N^{\mathbf{H}}(z, \theta; w, \varphi) &= N^2 z_{q'} \bar{w}_q \Pi_N^{\mathbf{H}}(z, \theta; w, \varphi) \\ &\quad + N\delta_{qq'} \Pi_N^{\mathbf{H}}(z, \theta; w, \varphi)\,.\end{aligned} \tag{27}$$

Remark. Recall that the metric on $\mathcal{O}(N) \to \mathbb{CP}^m$ is given by $h^N(z) = (1 + |z|^2)^{-N}$ using the coordinates and local frame from Example 1.3.1. Since

$$h^N(z/\sqrt{N}) \to h_{\mathbf{H}}(z)\,,$$

the Heisenberg bundle can be regarded as the scaling limit of $\mathcal{O}(N)$. (Of course, in the same way $L_{\mathbf{H}}$ is the scaling limit of L^N, for any positive line bundle $L \to M$.)

2. Correlation functions

This section begins with a generalization to arbitrary dimension and codimension of a formula of [Han] and [BD] for the "correlation density function" in the one-dimensional case. In fact, our formula (Theorem 2.1) applies to general classes of probability spaces of sections of holomorphic vector bundles. (A further generalization to $\mathcal{C}^\infty$ real vector bundles is given in [BSZ2]; see also Theorem 2.2.) We then specialize to the case where the space of sections has a Gaussian measure. Finally, we show how the correlations of the zeros of k-tuples of sections of the N^{th} power of a holomorphic line bundle are given by a rational function in the Szegö kernel Π_N and its derivatives (Theorem 2.4).

2.1. General formula for zero correlations. For our general setting, we let (V, h) be a Hermitian holomorphic vector bundle on an m-dimensional Hermitian complex manifold (M, g). (Here, we make no curvature assumptions.) Suppose that $\mathcal{S}$ is a finite dimensional subspace of the space $H^0(M, V)$ of global holomorphic sections of V, and let $d\mu$ be a probability measure on $\mathcal{S}$ given by a semi-positive $\mathcal{C}^0$ volume form that is strictly positive in a neighborhood of $0 \in \mathcal{S}$. (We shall later apply our results to the case where $V = L^N \oplus \cdots \oplus L^N$, for a holomorphic line bundle L over a compact complex manifold M, and $\mathcal{S} = H^0(M, V)$ with a Gaussian measure $d\mu$. Our formulation involving general vector bundles allows us to reduce the study of n-point correlations to the case $n = 1$, i.e., to expected densities of zeros.)

As in the introduction, we introduce the punctured product

$$M_n = \{(z^1, \dots, z^n) \in \underbrace{M \times \cdots \times M}_{n} : z^p \neq z^q \ \text{ for } p \neq q\},$$

and we write

$$s(z) = (s(z^1), \dots, s(z^n)), \ \nabla s(z) = (\nabla s(z^1), \dots, \nabla s(z^n)),$$
$$z = (z^1, \dots, z^n) \in M_n,$$

where $\nabla s(\zeta) \in T^*_\zeta \otimes V_\zeta$ is the covariant derivative with respect to the Hermitian connection on V. We define the map

$$\mathcal{J} : M_n \times \mathcal{S} \to \left[\left(\mathbb{C} \oplus T^*_M\right) \otimes V\right]^n, \quad \mathcal{J}(z, s) = (s(z), \nabla s(z));$$

i.e., $\mathcal{J}(z, s)$ is the 1-jet of s at $z \in M_n$.

We write $g = \Re \sum g_{qq'} dz_q \otimes d\bar{z}_{q'}$, $h_{jj'} = h(e_j, e_{j'})$, where $\{z_1, \dots, z_m\}$ are local coordinates in M and $\{e_1, \dots, e_k\}$ is a local frame in V ($m = \dim M$, $k = \operatorname{rank} V$). We let $G = \det(g_{qq'})$, $H = \det(h_{jj'})$. We let

$$d\zeta = \frac{1}{m!}\omega_\zeta^m = G(\zeta) \prod_{j=1}^m d\Re\zeta_j d\Im\zeta_j, \quad \zeta \in M$$

denote Riemannian volume in M, and we write

$$x^p = \sum_j x^p_j e_j(z^p), \quad dx^p = H(z^p) \prod_j d\Re x^p_j d\Im x^p_j \quad x^p \in V_{z^p}\,, \tag{28}$$

$$\xi^p = \sum_{j,q} \xi^p_{jq} dz_q \otimes e_j|_{z^p}, \quad d\xi^p = G(z^p)^{-k} H(z^p)^m \prod_{j,q} d\Re\xi^p_{jq} d\Im\xi^p_{jq}$$
$$\xi_j \in (T^*_M \otimes V)_{z^p}\,.$$

The quantities dx^p, $d\xi^p$ are the intrinsic volume measures on V_{z^p} and $(T^*_M \otimes V)_{z^p}$, respectively, induced by the metrics g, h. We further write

$$x = (x^1, \dots, x^n) \in V_{z^1} \times \dots \times V_{z^n}\,,$$
$$\xi = (\xi^1, \dots, \xi^n) \in (T^*_M \otimes V)_{z^1} \times \dots \times (T^*_M \otimes V)_{z^n}\,.$$

Definition. Suppose that $\mathcal{J}$ is surjective. We define the *n-point density* $D_n(x, \xi, z) dx d\xi dz$ of μ by

$$\mathcal{J}_*(dz \times d\mu) = D_n(x, \xi, z) dx d\xi dz\,, \tag{29}$$

where

$$dx = dx^1 \cdots dx^n\,, \quad d\xi = d\xi^1 \cdots d\xi^n\,, \quad dz = dz^1 \cdots dz^n\,.$$

In particular, for each $z \in M_n$, the (vector-valued) random variable $(s(z), \nabla s(z))$ has (joint) probability distribution $D_n(x, \xi, z) dx d\xi$.

Remark. If we let $n = 1$ and fix a point $z \in M$, then the measure $D(x, \xi, z) dx d\xi$ is intrinsically defined as a measure on the space $J^1_z(M, V)$ of 1-jets of sections of V at z. Taking a section to its 1-jet at z defines a map $\mathcal{J}_z : \mathcal{S} \to J^1_z(M, V)$ and hence induces a measure $\mathcal{J}_{z*}\mu$ on $J^1_z(M, V)$ independently of any choices of connections, coordinates or metrics. Similarly for $n > 1$, $D(x, \xi, z) dx d\xi$ is an intrinsic measure on $\prod_{p=1}^n J^1_{z_p}(M, V)$.

For a vector-valued 1-form $\xi \in T^*_{M,z} \otimes V_z = \mathrm{Hom}(T_{M,z}, V_z)$, we let $\xi^* \in \mathrm{Hom}(V_z, T_{M,z})$ denote the adjoint to ξ (i.e., $\langle \xi^* v, t\rangle = \langle v, \xi t\rangle$), and we consider the endomorphism $\xi\xi^* \in \mathrm{Hom}(V_z, V_z)$. In terms of local frames, if

$$\xi = \sum_j \xi_j \otimes e_j = \sum_{j,q} \xi_{jq} dz_j \otimes e_j\,,$$

then

$$\xi^* = \sum_{j,q} \xi^*_{jq} \frac{\partial}{\partial z_q} \otimes e^*_j\,, \qquad \xi^*_{jq} = \sum_{j',q'} h_{jj'} \gamma_{q'q} \bar{\xi}_{j'q'}\,,$$

where $\left(\gamma_{qq'}\right) = \left(g_{qq'}\right)^{-1}$; hence we have

$$\xi\xi^* = \sum_{j,j',j'',q,q'} h_{j'j''} \xi_{jq} \gamma_{q'q} \bar{\xi}_{j''q'}\, e_j \otimes e^*_{j'}\,. \tag{30}$$

Its determinant is given by

$$\det(\xi\xi^*) = H \det\left(\sum_{q,q'} \xi_{jq}\gamma_{q'q}\bar{\xi}_{j'q'}\right)_{1\le j,j'\le k} = H \det\langle \xi_j, \xi_{j'}\rangle \qquad (31)$$
$$= H\|\xi_1 \wedge \cdots \wedge \xi_k\|^2 .$$

Remark. The measure $\det(\xi\xi^*)D(0,\xi,z)d\xi dz$ will play a fundamental role in our study of correlation functions. We observe here that it depends only on the metric ω on M, and in the case where the zero sets are points ($k = m$), it is independent of the choice of metric on M as well. Indeed, as mentioned in the previous remark, $D(x,\xi,z)dxd\xi$ is well-defined on $J^1_z(M,V)$. The conditional density $D(0,\xi,z)d\xi$ equals $\mathcal{J}_{z*}\mu/dx|_{x=0}$ and thus depends only on the choice of volume forms dx^p on V_{z^p}. Since dz/dx transforms in the opposite way to $\det\xi\xi^*$ it follows that $\det(\xi\xi^*)D(0,\xi,z)d\xi dz$ is an invariantly defined measure on $(T^*_M \otimes V)^n$.

Recall that for $s \in \mathcal{S}$ such that $\operatorname{codim} Z_s = k$, we let $|Z_s|$ denote Riemannian $(2m-2k)$-volume along the regular points of Z_s, regarded as a measure on M.

Definition. For $s \in \mathcal{S}$ such that $\operatorname{codim} Z_s = k$, we consider the product measure on M_n,

$$|Z_s|^n = \big(\underbrace{|Z_s| \times \cdots \times |Z_s|}_{n}\big).$$

Its expectation $\mathbf{E}\,|Z_s|^n$ is called the *n-point zero correlation measure.*

We shall use the following general formula to compute the correlations of zeros and to show universality of the scaling limit:

Theorem 2.1. *Let M, V, $\mathcal{S}$, $d\mu$ be as above, and suppose that $\mathcal{J}$ is surjective and the volumes $|Z_s|$ are locally uniformly bounded above. Then*

$$\mathbf{E}\,|Z_s|^n = K_n(z)dz\,, \quad K_n(z) = \int d\xi\; D_n(0,\xi;z)\prod_{p=1}^n \det\left(\xi^p\xi^{p*}\right) . \qquad (32)$$

The function $K_n(z^1,\ldots,z^n)$, which is continuous on M_n is called the *n-point zero correlation function*. For $k < m$, (32) holds on all of the n-fold product $M \times \cdots \times M$, including the diagonal, and K_n is locally integrable on $M \times \cdots \times M$ (and is infinite on the diagonal). In the case $k = m$, when the zero sets are discrete, the zero correlation measure on $M \times \cdots \times M$ is the sum of the absolutely continuous measure $K_n(z)dz$ plus a measure supported on the diagonal.

Proof of Theorem 2.1. Consider the Hermitian vector bundle $V_n = \bigoplus_{p=1}^n \pi_p^* V \longrightarrow M_n$, where $\pi_p : M_n \to M$ denotes the projection onto the p-th factor. By replacing $V \to M$ with $V_n \to M_n$ and $s \in H^0(M, V)$ with

$$\tilde{s}(z^1, \dots, z^n) = (s(z^1), \dots, s(z^n)) \in H^0(M_n, V_n)\,,$$

and noting that $T_{M_n,z} = \prod_p T_{M,z^p}$ and $|Z_s|^n = |Z_{\tilde{s}}|$, we can assume without loss of generality that $n = 1$.

It follows from the above remarks that $D(0, \xi; z)$ does not depend on the choice of connection on V. We can also verify this in terms of local coordinates: write $s = \sum x_j e_j$, $\nabla s = \sum \xi_{jq} dz_q \otimes e_j$ as in (29); we have $\xi_{jq} = \frac{\partial x_j}{\partial z_q} + \sum_k x_k \theta^k_{jq}$. Then if we write $\xi^0_{jq} = \frac{\partial x_j}{\partial z_q}$, we have

$$\frac{\partial(\xi_{jq}, x_j)}{\partial(\xi^0_{jq}, x_j)} = 1\,.$$

Hence $D(0, \xi; z)$ is unchanged if we substitute the (local) flat connection given by ξ^0_{jq}.

We now restrict to a coordinate neighborhood $U \subset M$ where V has a local frame $\{e_j\}$. By hypothesis, we can suppose that the e_j are restrictions of sections in $\mathcal{S}$. We write $s = \sum s_j e_j$, and by the above we may assume that $\nabla s = \sum ds_j \otimes e_j$. We use the notation

$$|\!|\!|\xi|\!|\!| = \sqrt{\det(\xi\xi^*)}\,, \quad \text{for } \xi \in T^*_{M,z} \otimes V_z = \mathrm{Hom}(T_{M,z}, V_z)\,.$$

Then by (31),

$$|\!|\!|\nabla s|\!|\!|^2 = H\|ds_1 \wedge \cdots \wedge ds_k\|^2 = \|\Psi\|\,,$$

where Ψ is the (k, k)-form on U given by:

$$\Psi = H\left(\frac{i}{2} ds_1 \wedge \overline{ds_1}\right) \wedge \cdots \wedge \left(\frac{i}{2} ds_k \wedge \overline{ds_k}\right)\,.$$

Thus, by the Leray formula,

$$|Z_s| = \|ds_1 \wedge \cdots \wedge ds_k\|^2 \left.\frac{dz}{\frac{i}{2} ds_1 \wedge d\bar{s}_1 \cdots \wedge \frac{i}{2} ds_k \wedge d\bar{s}_k}\right|_{Z_s} = |\!|\!|\nabla s|\!|\!|^2 \left.\frac{dz}{\Psi}\right|_{Z_s}\,, \tag{33}$$

Define the measure λ on $M \times \mathcal{S}$ by

$$(\lambda, \varphi) = \int_{\mathcal{S}} (|Z_s|, \varphi(z, s))\, d\mu(s)\,. \tag{34}$$

Then

$$\pi_* \lambda = \mathbf{E}\, |Z_s|^n\,,$$

where $\pi : M \times \mathcal{S} \to M$ is the projection. Hence,

$$\lambda = \int_{\mathcal{S}} d\mu(s)\ |Z_s| = \int_{\mathcal{S}} d\mu(s)\ \left(\|\nabla s\|^2 \frac{dz}{\Psi} \right)\Big|_{Z_s} . \tag{35}$$

For (almost all) $x \in \mathbb{C}^k$, let $I(s, x)$ be the measure on U given by

$$\begin{aligned}(I(s,x),\varphi) &= \int_{s(z)=\sum x_j e_j(z)} \varphi(z) d\mathrm{Vol}_{(2m-2k)n}(z) \\ &= \int_{s(z)=\sum x_j e_j(z)} \|\nabla s\|^2 \frac{dz}{\Psi}\varphi(z)\,,\ \varphi \in \mathcal{C}^0(U)\,,\end{aligned}$$

where the second equality is by (33) applied to $s - \sum x_j e_j(z)$. Then

$$\int I(s,x)dx = \|\nabla s(z)\|^2 dz\,. \tag{36}$$

Now let λ_x be the measure on U given by

$$(\lambda_x, \varphi) = \int_{\mathcal{S}} (I(s,x),\varphi) d\mu(s)\,.$$

Claim. The map $x \mapsto (\lambda_x, \varphi)$ is continuous.

To prove this claim, we first note that the hypothesis that $|Z_s|$ is locally uniformly bounded implies that $(I(s, x), \varphi) \le C < +\infty$ uniformly in s, x. Thus we can assume without loss of generality that μ has compact support in $\mathcal{S}$. By hypothesis, the map

$$\sigma : U \times \mathcal{S} \to \mathbb{C}^k, \qquad \sigma(z,s) = (s_1(z), \ldots, s_k(z))$$

is a submersion. We can now write λ_x as a fiber integral of a compactly supported $\mathcal{C}^0$ form:

$$\lambda_x = \frac{1}{(m-k)!} \int_{\sigma^{-1}(x)} \varphi(z) \omega^{m-k}(z) \wedge d\mu(s)\,,$$

and thus λ_x is continuous, verifying the claim.

We note that $\lambda_0 = \lambda|_U$. Hence, to complete the proof, we must show that

$$\pi_* \lambda_0 = K_1(z) dz|_U\,.$$

By (29) and (36), for a test function $\varphi(x, \xi, z)$,

$$\begin{aligned}\int \varphi(x,\xi,z)\|\xi\|^2 D_1(x,\xi,z) dx d\xi dz &= \int d\mu(s) \int \varphi(\mathcal{J}(z,s))\|\nabla s(z)\|^2 dz \\ &= \int dx \int (I(s,x), \varphi\circ\mathcal{J}) d\mu(s) \\ &= \int (\lambda_x, \varphi \circ \mathcal{J}) dx\,.\end{aligned}$$

By choosing $\varphi(x, \xi, z) = \rho_\varepsilon(x)\psi(z)$, where ρ_ε is an approximate identity, and letting $\varepsilon \to 0$, we conclude that

$$\int \psi(z) K_1(z) dz = \int \psi(z) \|\xi\|^2 D_1(0, \xi, z) d\xi dz = (\lambda_0, \psi(z)) .$$

□

The proof of Theorem 2.1 easily carries over to the case of real vector bundles over $\mathcal{C}^\infty$ manifolds, where the Leray formula (33) becomes

$$|Z_s| = \|ds_1 \wedge \cdots \wedge ds_k\| \left. \frac{d\zeta}{ds_1 \wedge \cdots \wedge ds_k} \right|_{Z_s} , \tag{37}$$

giving us the following analogous result for real manifolds:

Theorem 2.2. *Let V be a $\mathcal{C}^\infty$ real vector bundle over a $\mathcal{C}^\infty$ Riemannian manifold M, and let μ be a probability measure on a finite dimensional vector space $\mathcal{S}$ of $\mathcal{C}^\infty$ sections of V given by a semi-positive volume form that is strictly positive at 0. Suppose that the volumes $|Z_s|$ are locally uniformly bounded above and that the real n-point jet map is surjective. Let $D_n(x, \xi, z) dx d\xi dz$ denote the n-point density of μ. Then*

$$\mathbf{E}\, |Z_s|^n = K_n(z) dz\, , \quad K_n(z) = \int d\xi\; D_k(0, \xi, z) \prod_{p=1}^{n} \sqrt{\det(\xi^p \xi^{p*})}\, . \tag{38}$$

In fact, we show in a forthcoming article [BSZ2] that (38) holds under weaker hypotheses.

2.2. Formula for Gaussian densities. We now specialize our formula from Theorem 2.1 to the case where μ is a Gaussian measure. Fix $z = (z^1, \dots, z^n) \in M_n$ and choose local coordinates $\{z^p_q\}$ and local frames $\{e^p_j\}$ near z^p, $p = 1, \dots, n$. We consider the random variables x^p_j, ξ^p_{jq} given by

$$s(z^p) = \sum_{j=1}^{k} x^p_j e^p_j, \quad \nabla s(z^p) = \sum_{j=1}^{k} \sum_{q=1}^{m} \xi^p_{jq} dz^p_q \otimes e^p_j, \qquad p = 1, \dots, n. \tag{39}$$

By (8)–(9) and (28)–(29) the n-point density is given by:

$$D_n(x, \xi; z) = \frac{\exp\langle -\Delta_n^{-1} v, v\rangle}{\pi^{kn(1+m)} \det \Delta_n}\, , \qquad v = \begin{pmatrix} x \\ \xi \end{pmatrix} , \tag{40}$$

where

$$\Delta_n = \begin{pmatrix} A_n & B_n \\ B_n^* & C_n \end{pmatrix} \tag{41}$$

$$A_n = \left(A^{jp}_{j'p'}\right) = \left(\mathbf{E}\, x^p_j \bar{x}^{p'}_{j'}\right), \quad B_n = \left(B^{jp}_{j'p'q'}\right) = \left(\mathbf{E}\, x^p_j \bar{\xi}^{p'}_{j'q'}\right),$$
$$C_n = \left(C^{jpq}_{j'p'q'}\right) = \left(\mathbf{E}\, \xi^p_{jq} \bar{\xi}^{p'}_{j'q'}\right);$$
$$j, j' = 1, \ldots, k; \quad p, p' = 1, \ldots, n; \quad q, q' = 1, \ldots, m.$$

(We note that $A_n,\ B_n,\ C_n$ are $kn \times kn,\ kn \times knm,\ knm \times knm$ matrices, respectively; j, p, q index the rows, and j', p', q' index the columns.)

The function $D_n(0, \xi; z)$ is a Gaussian function, but it is not normalized as a probability density. It can be represented as

$$D_n(0, \xi; z) = Z_n(z) D_{\Lambda_n}(\xi; z), \tag{42}$$

where

$$D_{\Lambda_n}(\xi; z) = \frac{1}{\pi^{knm} \det \Lambda_n} \exp\left(-\langle \Lambda_n^{-1}\xi, \xi\rangle\right) \tag{43}$$

is the Gaussian density with covariance matrix

$$\Lambda_n = C_n - B_n^* A_n^{-1} B_n = \left(C^{jpq}_{j'p'q'} - \sum_{j_1, p_1, j_2, p_2} \bar{B}^{j_1p_1}_{jpq} \Gamma^{j_1p_1}_{j_2p_2} B^{j_2p_2}_{j'p'q'}\right) \quad \left(\Gamma = A_n^{-1}\right) \tag{44}$$

and

$$Z_n(z) = \frac{\det \Lambda_n}{\pi^{kn} \det \Delta_n} = \frac{1}{\pi^{kn} \det A_n}\,. \tag{45}$$

This reduces formula (32) to

$$K_n(z) = \frac{1}{\pi^{kn} \det A_n} \left\langle \prod_{p=1}^{n} \det\left(\xi^{p*} \gamma^p \xi^p\right) \right\rangle_{\Lambda_n} \tag{46}$$

where $\langle\cdot\rangle_{\Lambda_n}$ stands for averaging with respect to the Gaussian density $D_{\Lambda_n}(\xi; z)$, and $\gamma^p = (\gamma^p_{qq'}) = (g_{qq'}(z^p))^{-1}$.

Remark. In order to use (46) to compute the correlation function $K_n(z)$, we will apply the Wick formula ([Si, I.13]), which expresses the higher moments of a Gaussian random vector as a sum of products of its second moments. Since our Gaussians are complex, by (10) we need to consider only second moments of the form $\langle \xi^p_{jq} \bar{\xi}^{p'}_{j'q'} \rangle$. Indeed by (10), these second moments are simply the coefficients of the matrix Λ_n. Furthermore, these second moments vanish if $j \neq j'$, since the sections $s_1, \ldots, s_k$ are independent.

2.3. Densities and the Szegö kernel. We return to our positive Hermitian line bundle (L, h) on a compact complex manifold M with Kähler form $\omega = \frac{i}{2}\Theta_h$. We now apply formulas (41)–(46) to the vector bundle

$$V = \underbrace{L^N \oplus \cdots \oplus L^N}_{k}$$

and space of sections

$$\mathcal{S} = H^0(M, V) = H^0(M, L^N)^k$$

with the Gaussian measure $\mu = \nu_N \times \cdots \times \nu_N$, where ν_N is the standard Gaussian measure on $H^0(M, L^N)$ given by (8). We denote the resulting n-point density by D^N_{nk}, and we also write $\Delta_n = \Delta^N_{nk}$, $A_n = A^N_{nk}$, etc.

As above, we fix $z = (z^1, \dots, z^n) \in M_n$ and choose local coordinates $\{z^p_q\}$ near z^p, $p = 1, \dots, n$. We also choose local holomorphic frames $\{e^p_L\}$ for L near the points z^p so that

$$\left\| e^p_L(z^p) \right\|_h = 1\,.$$

For $s \in \mathcal{S}$, we write

$$s(z^p) = \begin{pmatrix} s_1(z^p) \\ \vdots \\ s_k(z^p) \end{pmatrix} = \begin{pmatrix} x^p_1 \\ \vdots \\ x^p_k \end{pmatrix} \left(e^p_L(z^p)\right)^{\otimes N}, \tag{47}$$

$$\nabla_N s_j(z^p) = \sum_{q=1}^{m} \xi^p_{jq} dz^p_q \otimes \left(e^p_L(z^p)\right)^{\otimes N}. \tag{48}$$

Since the s_j are independent and have identical distributions, we have

$$\begin{gathered} A^N_{nk} = \left(A^{jp}_{j'p'}\right) = \left(\delta_{jj'}\mathbf{E}\left(x^p_1 \bar{x}^{p'}_1\right)\right), \quad B^N_{nk} = \left(B^{jp}_{j'p'q'}\right) = \left(\delta_{jj'}\mathbf{E}\left(x^p_1 \bar{\xi}^{p'}_{1q'}\right)\right), \\ C^N_{nk} = \left(C^{jpq}_{j'p'q'}\right) = \left(\delta_{jj'}\mathbf{E}\left(\xi^p_{1q} \bar{\xi}^{p'}_{1q'}\right)\right). \end{gathered} \tag{49}$$

We write

$$s_1 = \sum_{\alpha=1}^{d_N} c_\alpha S^N_\alpha = \left(\sum_{\alpha=1}^{d_N} c_\alpha f^p_\alpha\right) \left(e^p_L\right)^{\otimes N},$$

where $\{S^N_\alpha\}$ is an orthonormal basis for $H^0(M, L^N)$. Using the local coordinates (z^p, θ) in X as described in §1.1, we have by (49) and (17) (noting that $h(z^p) = 0$ by the above choice of local frames),

$$\begin{aligned} A^{jp}_{j'p'} &= \delta_{jj'} \sum_{\alpha,\beta=1}^{d_N} \mathbf{E}\left(c_\alpha \bar{c}_\beta\right) f^p_\alpha(z^p) \overline{f^{p'}_\beta(z^{p'})} = \delta_{jj'} \sum_{\alpha=1}^{d_N} f^p_\alpha(z^p) \overline{f^{p'}_\alpha(z^{p'})} \qquad (50) \\ &= \delta_{jj'} \Pi_N(z^p, 0; z^{p'}, 0)\,. \end{aligned}$$

Similarly,

$$B^{jp}_{j'p'q'} = \delta_{jj'} \sum_{\alpha=1}^{d_N} f^p_\alpha(z^p)\,\overline{f^{p'}_{\alpha;q'}(z^{p'})} = \delta_{jj'} d^H_{\bar{w}_{q'}} \Pi_N(z^p, 0; z^{p'}, 0)\,, \tag{51}$$

$$C^{jpq}_{j'p'q'} = \delta_{jj'} \sum_{\alpha=1}^{d_N} f^p_{\alpha;q}(z^p)\,\overline{f^{p'}_{\alpha;q'}(z^{p'})} = \delta_{jj'} d^H_{z_q} d^H_{\bar{w}_{q'}} \Pi_N(z^p, 0; z^{p'}, 0)\,. \tag{52}$$

Lemma 2.3. *There is a positive integer $N_0 = N_0(M, n)$ such that*

$$\det\left(\Pi_N(z^p, 0; z^{p'}, 0)\right)_{1 \le p, p' \le n} \ne 0\,,$$

for distinct points $z^1, \dots, z^n$ of M and for all $N \ge N_0$.

Proof. It is a well-known consequence of the Kodaira Vanishing Theorem (see for example, [GH]) that we can find N_0 such that if $N \ge N_0$ and $x_1, \dots, x_n \in M$ with $x_p \ne x_1$ for $2 \le p \le n$, then there is a section $s \in H^0(M, L^N)$ with $s(x_1) \ne 0$ and $s(x_p) = 0$ for $2 \le p \le n$.

We write $\widetilde{A}_{pp'} = \Pi_N(z^p, 0; z^{p'}, 0)$. Suppose on the contrary that $\det(\widetilde{A}_{pp'}) = 0$, and choose a nonzero vector $v = (v_1, \dots, v_n)$ such that $\sum_p v_p \widetilde{A}_{pp'} = 0$. Then recalling (15), we have

$$0 = \sum_{p,p'} v_p \widetilde{A}_{pp'} \bar{v}_{p'} = \sum_{p,p',\alpha} v_p \widehat{S}^N_\alpha(z^p, 0)\overline{\widehat{S}^N_\alpha(z^{p'}, 0) v_{p'}} = \sum_{\alpha=1}^{d_N} |x_\alpha|^2\,, \tag{53}$$

where $x_\alpha = \sum_p v_p \widehat{S}^N_\alpha(z^p, 0)$. Since the S^N_α span $H^0(M, L^N)$, it follows that for all $s \in H^0(M, L^N)$, we have $\sum_p v_p \hat{s}(z^p) = 0$. But this contradicts the fact that, choosing p_0 with $v_{p_0} \ne 0$, we can find a section $s \in H^0(M, L^N)$ with $s(z^{p_0}) \ne 0$ and $s(z^p) = 0$ for $p \ne p_0$. □

Thus we see that the n-point correlation functions depend only on the Szegö kernel, as follows:

Theorem 2.4. *Let (L, h) be a positive Hermitian line bundle on an m-dimensional compact complex manifold M with Kähler form $\omega = \frac{i}{2}\Theta_h$, let $\mathcal{S} = H^0(M, L^N)^k$ ($k \ge 1$), and give $\mathcal{S}$ the standard Gaussian measure μ described above. Let $n \ge 1$ and suppose that N is sufficiently large so that $\mathcal{J}$ is surjective. Let $z = (z^1, \dots, z^n) \in M_n$ and choose local coordinates $(\zeta_1, \dots, \zeta_m)$ at each point z^p such that $\Theta_h(z^p) = \sum_q d\zeta_q \wedge d\bar{\zeta}_q$, $1 \le p \le n$. Then the n-point correlation $K^N_{nk}(z)$ is given by a universal rational function, homogeneous of degree 0, in the values of Π_N and its first and second derivatives at the points $(z^p, z^{p'})$. Specifically,*

$$K_{nk}^N(z) = \frac{\mathcal{P}_{nkm}\Big(\Pi_N(z^p, z^{p'}), d^H_{\bar{w}_q}\Pi_N(z^p, z^{p'}), d^H_{z_q}\Pi_N(z^p, z^{p'}), d^H_{z_q}d^H_{\bar{w}_{q'}}\Pi_N(z^p, z^{p'})\Big)}{\pi^{kn}\left[\det\left(\Pi_N(z^p, z^{p'})\right)_{1\le p,p'\le n}\right]^{k(n+1)}} \tag{54}$$

($1 \le p, p' \le n,\ 1 \le q, q' \le m$), where $\mathcal{P}_{nkm}$ is a universal homogeneous polynomial of degree $kn(n+1)$ with integer coefficients depending only on n, k, m.

Proof. The n-point zero correlation $K_{nk}^N(z)$ is given by equation (46) with $\gamma_{qq'}^p = \delta_{qq'}$. By the Wick formula (see the remark at the end of §2.2), the expectation

$$\left\langle \prod_{p=1}^n \det\left(\xi^{p*}\xi^p\right)\right\rangle_{\Lambda_n}$$

in (46) is a homogeneous polynomial (over $\mathbb{Z}$) of degree kn in the coefficients of Λ_n. By (44) and (50), the coefficients of $\det\left(\Pi_N(z^p, z^{p'})\right)\Lambda_n$ are homogeneous polynomials of degree $n+1$ in the coefficients of A_n, B_n, C_n. The conclusion then follows from (50)–(52). □

Remark. In the statement of Theorem 2.4, we wrote $\Pi_N(z, w)$ for $\Pi_N(z, \theta; w, \varphi)$. Since the expression is homogeneous of degree 0, it is independent of θ and φ. Alternately, we could regard $\Pi_N(z, w)$ as functions on $M \times M$ having values in $L_z \otimes \overline{L_w}$ (replacing the horizontal derivatives with the corresponding covariant derivatives); again the degree 0 homogeneity makes the expression a scalar. Furthermore, since Theorem 2.1 is valid for all connections, we can replace the horizontal derivatives in (54) with the derivatives with respect to an arbitrary connection.

2.4. Zero correlation for SU$(m+1)$**-polynomials.** In this section, we use our methods to describe the zero correlation functions for SU$(m+1)$-polynomials. We do not carry out the computations in complete detail, since we are primarily interested in the scaling limits, which we shall compute in §4.

The SU$(m+1)$-polynomials are random homogeneous polynomials of degree $N > 0$ on $\mathbb{C}^{m+1}$,

$$s(z) = s(z_0, z_1, \ldots, z_m) = \sum_{|J|=N} \sqrt{N!/J!}\, c_J z^J, \tag{55}$$

$$z^J = z_0^{j_0}\cdots z_m^{j_m}, \qquad J! = j_0!\cdots j_m!,$$

where the coefficients c_J are complex independent Gaussian random variables with mean 0 and variance 1:

$$\mathbf{E}\, c_J = 0;\ \mathbf{E}\, c_J\overline{c_K} = \delta_{JK},\ \delta_{JK} = \delta_{j_0k_0}\ldots\delta_{j_mk_m};\ \mathbf{E}\, c_Jc_K = 0. \tag{56}$$

Then $s(z)$ is a Gaussian random polynomial on $\mathbb{C}^{N+1}$ with first and second moments given by

$$\mathbf{E}\, s(z) = 0;\ \mathbf{E}\, s(z)\overline{s(w)} = \langle z, w\rangle^N = \left(\sum_{q=0}^{m} z_q \overline{w_q}\right)^N;\ \mathbf{E}\, s(z)s(w) = 0. \tag{57}$$

This implies that the probability distribution of $s(z)$ is invariant with respect to the map $s(z) \to s(Uz)$ for all $U \in \mathrm{SU}(m+1)$.

Let $(\mathcal{S}_N, \mu_N)$ denote the Gaussian probability space of independent k-tuples ($k \leq m$) of SU($m+1$)-polynomials of degree N. For $s = (s_1, \ldots, s_k) \in \mathcal{S}_N$, the zero set

$$Z_s = \{z : s_1(z) = \cdots = s_k(z) = 0\}\,.$$

is an algebraic variety in the complex projective space $\mathbb{CP}^m$. We will assume that $\mathbb{CP}^m$ is supplied with the Fubini-Study Hermitian metric ω, which is SU($m+1$)-invariant. In the affine coordinates $z = (1, z_1, \ldots, z_m)$,

$$\begin{aligned}\omega &= \frac{\sqrt{-1}}{2}\partial\bar\partial \log\left(1 + \sum |z_q|^2\right) \qquad (58)\\ &= \frac{\sqrt{-1}}{2}\left[\frac{\sum dz_q \wedge \overline{dz_q}}{1+\sum |z_q|^2} - \frac{\left(\sum \overline{z_q}dz_q\right)\wedge\left(\sum z_q\overline{dz_q}\right)}{\left(1+\sum |z_q|^2\right)^2}\right];\end{aligned}$$

i.e.,

$$\omega = \frac{\sqrt{-1}}{2}\sum g_{qq'}dz_q \wedge dz_{q'}\,, \quad g_{qq'} = \frac{(1+|z|^2)\delta_{qq'} - \bar z_q z_{q'}}{(1+|z|^2)^2}\,. \tag{59}$$

To simplify our computations, we consider only points z^p with finite affine coordinates, $z^p = (1, z_1^p, \ldots, z_m^p)$, $p = 1, \ldots, n$, and we regard the SU($m+1$)-polynomials s_j as polynomials of degree $\leq N$ on $\mathbb{C}^m$; i.e., we regard the s_j as sections of the trivial line bundle on $\mathbb{C}^m$ with the flat metric $h = 1$ (so that the covariant derivatives coincide with the usual derivatives of functions).

As above, we consider the random variables

$$x_j^p = s_j(z^p),\ \xi_{jq}^p = \frac{\partial s_j}{\partial z_q}(z^p)\,,$$

and we denote their joint distribution by

$$\begin{aligned}&D_{nk}^N(x, \xi; z)dx\, d\xi,\ x = \left(x^1, \ldots, x^n\right),\ x^p = \left(x_j^p\right)_{j=1,\ldots,k};\\ &\qquad \xi = \left(\xi^1, \ldots, \xi^n\right),\ \xi^p = \left(\xi_{jq}^p\right)_{j=1,\ldots,k;\ q=1,\ldots,m}\,.\end{aligned} \tag{60}$$

(Here, the n-point density is with respect to Lebesgue measure $dx = \prod d\Re x_j^p d\Im x_j^p$, $d\xi = \prod d\Re \xi_{jq}^p d\Im \xi_{jq}^p$.) We assume that $N > nm$ to ensure that μ_N possesses a continuous n-point density. Since μ_N is Gaussian, the density $D_{nk}^N(x, \xi; z)$ is Gaussian as well, and it is described by the covariance matrix

$$\Delta_{nk}^N = \begin{pmatrix} A_{nk}^N & B_{nk}^N \\ B_{nk}^{N*} & C_{nk}^N \end{pmatrix} \tag{61}$$

where

$$\begin{aligned} A_{nk}^N &= \left(\mathbf{E}\, s_j(z^p)\overline{s_{j'}(z^{p'})} \right), \\ B_{nk}^N &= \left(\mathbf{E}\, s_j(z^p)\overline{\tfrac{\partial s_{j'}}{\partial z_{q'}}(z^{p'})} \right), \\ C_{nk}^N &= \left(\mathbf{E}\, \tfrac{\partial s_j}{\partial z_q}(z^p)\overline{\tfrac{\partial s_{j'}}{\partial z_{q'}}(z^{p'})} \right); \\ & j, j' = 1, \dots, k; \quad p, p' = 1, \dots, n; \quad q, q' = 1, \dots, m. \end{aligned} \tag{62}$$

By (62) and (57),

$$\begin{aligned} A_{nk}^N &= \left(\delta_{jj'} S_N(z^p, z^{p'})\right), \quad S_N(z, w) = \left(1 + \sum_{r=1}^m z_r \overline{w_r}\right)^N, \\ B_{nk}^N &= \left(\delta_{jj'} S_{Nq'}(z^p, z^{p'})\right), \quad S_{Nq'}(z, w) = N z_{q'} \left(1 + \sum_{r=1}^m z_r \overline{w_r}\right)^{N-1}, \\ C_{nk}^N &= \left(\delta_{jj'} S_{Nqq'}(z^p, z^{p'})\right), \\ S_{Nqq'}(z, w) &= N(N-1)\overline{w_q} z_{q'} \left(1 + \sum_{r=1}^m z_r \overline{w_r}\right)^{N-2} \\ &\qquad + \delta_{qq'} N \left(1 + \sum_{r=1}^m z_r \overline{w_r}\right)^{N-1} \end{aligned} \tag{63}$$

The n-point zero correlation functions K_{nk}^N for the SU$(m+1)$-polynomial k-tuples $\mathcal{S}_N$ can be computed by substituting (63) into formulas (44) and (46). (Alternately, we can compute the zero correlation functions with respect to the Euclidean volume on $\mathbb{C}^m$ by setting $\gamma^p =$Id in (46).)

Remark. Note that the one-point correlation function, or the zero-density function, is constant, since it is invariant with respect to the group SU$(m+1)$. Indeed, by Bézout's theorem and (11),

$$|Z_s|(1) = \mathrm{Vol}(V_s) = \int_{V_s} \frac{1}{(m-k)!}\omega^{m-k} = \Omega_{2m-2k} \deg V_s = \Omega_{2m-2k} N^k, \tag{64}$$

where

$$\Omega_{2\ell} = \mathrm{Vol}\,\mathbb{CP}^\ell = \frac{\pi^\ell}{\ell!}\,. \tag{65}$$

Hence,

$$K_{1k}^N(z) = \frac{\mathrm{Vol}\,Z_s}{\mathrm{Vol}\,\mathbb{CP}^m} = \frac{N^k\Omega_{2m-2k}}{\Omega_{2m}} = \frac{N^k m!}{(m-k)!\pi^k}\,. \tag{66}$$

We can also use our formulas to compute K_{1k}^N directly: By (63),

$$A_{1k}^N = \left(\delta_{jj'}(1+|z|^2)^N\right), \quad B_{1k}^N = \left(\delta_{jj'} N z_{q'}(1+|z|^2)^{N-1}\right), \tag{67}$$

$$C_{1k}^N = \left(\delta_{jj'} N\big[(N-1)\bar z_q z_{q'} + (1+|z|^2)\delta_{qq'}\big](1+|z|^2)^{N-2}\right).$$

Hence by (44),

$$\begin{aligned}\Lambda_{1k}^N &= \left(\delta_{jj'} N[(1+|z|^2)\delta_{qq'} - \bar z_q z_{q'}](1+|z|^2)^{N-2}\right) \\ &= \left(\delta_{jj'} N(1+|z|^2)^N g_{qq'}(z)\right).\end{aligned} \tag{68}$$

In the hypersurface case ($k = 1$), we compute

$$K_{11}^N = \frac{1}{\pi(1+|z|^2)^N}\left\langle \sum_{q,q'=1}^m \bar\xi_{1q}\gamma_{qq'}\xi_{1q'}\right\rangle\Bigg|_{\Lambda_{11}^N} = \frac{N}{\pi}\sum_{q,q'=1}^m \gamma_{qq'} g_{q'q} = \frac{Nm}{\pi}\,,$$

as expected. For $k > 1$, we have $\Lambda_{1k}^N(0) = NI$ where I is the unit matrix, and (46) yields

$$K_{1k}^N(0) = \frac{N^k}{\pi^k}\left\langle \det\left(\sum_{q=1}^m \bar\xi_{jq}\xi_{j'q}\right)_{j,j'=1,\dots,k}\right\rangle_I = \frac{N^k m!}{(m-k)!\pi^k}\,,$$

which agrees with (66).

3. Universality and scaling

Our goal is to derive scaling limits of the n-point correlations between the zeros of *random* k-tuples of sections of powers of a positive line bundle over a complex manifold. We expect the scaling limits to exist and to be universal in the sense that they should depend only on the dimensions of the algebraic variety of zeros and the manifold. Our plan is the following. We first describe scaling in the Heisenberg model, which we use to provide the universal scaling limit for the Szegö kernel (Theorem 3.1). Together with Theorem 2.4, this demonstrates the universality of the scaling-limit zero correlation in the case of powers of any positive line bundle on any complex manifold.

3.1. Scaling of the Szegö kernel in the Heisenberg group. Our model for scaling is the Szegö kernel for the reduced Heisenberg group described in §1.3.2. Recall that for the simply-connected Heisenberg group $\mathbf{H}^m$, the scaling operators (or Heisenberg dilations)

$$\delta_r(\zeta, t) = (r\zeta, r^2 t)\,, \quad r \in \mathbb{R}^+$$

are automorphisms of $\mathbf{H}^m$ ([Fo] [St]). Since the Szegö kernel Π of $\mathbf{H}^m$ is the unique self-adjoint holomorphic reproducing kernel, it follows that it must be invariant (up to a multiple) under these automorphisms. In fact, one has ([St, p. 538]):

$$\Pi(\delta_r x, \delta_r y) = r^{-2m-2}\Pi(x, y). \tag{69}$$

The condition for a dilation δ_r to descend to the quotient group $\mathbf{H}^m_{\text{red}}$ is that $r^2\mathbb{Z} \subset \mathbb{Z}$, or equivalently, $r = \sqrt{N}$ with $N \in \mathbb{Z}^+$. Note however that $\delta_{\sqrt{N}}$ is not an automorphism of $\mathbf{H}^m_{\text{red}}$ and there is no well-defined dilation by $\sqrt{N}^{-1}$.

The scaling identity (69) descends to $\mathbf{H}^m_{\text{red}}$ in the form

$$\Pi^{\mathbf{H}}_N(x, y) = N^m\,\Pi^{\mathbf{H}}_1(\delta_{\sqrt{N}}\, x, \delta_{\sqrt{N}}\, y) \tag{70}$$

with

$$\Pi^{\mathbf{H}}_1(x, y) = \frac{1}{\pi^m} e^{i(\theta-\varphi)} e^{i\Im(z\cdot\bar{w})} e^{-\frac{1}{2}|z-w|^2}\,, \quad x = (z, \theta)\,,\ y = (w, \varphi)\,. \tag{71}$$

(Recall (25).) Informally, we may say that the scaling limit of $\Pi^{\mathbf{H}}_N$ equals $\Pi^{\mathbf{H}}_1$. Since scaling by $\sqrt{N}^{-1}$ is not well-defined on $\mathbf{H}^m_{\text{red}}$ it is more correct to say that $\Pi^{\mathbf{H}}_N$ is the $\sqrt{N}$ scaling of the scaling limit kernel.

3.2. Scaling limit of a general Szegö kernel. We now show that $\Pi^{\mathbf{H}}_1$ is the scaling limit of the N-th Szegö kernel Π_N of an arbitrary positive line bundle $L \to M$ in the sense of the following "near-diagonal asymptotic estimate for the Szegö kernel".

Theorem 3.1. *Let $z_0 \in M$ and choose local coordinates $\{z_j\}$ and a local holomorphic frame e_L in a neighborhood of z_0 so that $\Theta_h(z_0) = \sum dz_j \wedge d\bar{z}_j$ and $\frac{\partial h}{\partial z_j}(z_0) = \frac{\partial^2 h}{\partial z_j \partial z_k}(z_0) = 0$ $(1 \le j, k \le m)$, where $h = \|e_L\|^2$. Then*

$$\begin{aligned} N^{-m}\Pi_N\left(z_0 + \frac{u}{\sqrt{N}}, \frac{\theta}{N}; z_0 + \frac{v}{\sqrt{N}}, \frac{\varphi}{N}\right) \\ &= \frac{1}{\pi^m} e^{i(\theta-\varphi)+i\Im(u\cdot\bar{v})-\frac{1}{2}|u-v|^2} + O(N^{-1/2}) \\ &= \Pi^{\mathbf{H}}_1(u, \theta; v, \varphi) + O(N^{-1/2})\,. \end{aligned}$$

Note that a local frame satisfying the hypothesis of the theorem can always be constructed, simply by multiplying an arbitrary frame by a holomorphic function with prescribed 2-jet. For a geometric interpretation of this condition, see [SZ2], where the result is generalized to symplectic manifolds and more precise asymptotics are provided.

To prove Theorem 3.1, we need to recall the Boutet de Monvel-Sjöstrand parametrix construction:

Theorem 3.2. [BS, Th. 1.5 and §2.c] *Let $\Pi(x, y)$ be the Szegö kernel of the boundary X of a bounded strictly pseudo-convex domain Ω in a complex manifold. Then there exists a symbol $s \in S^n(X \times X \times \mathbb{R}^+)$ of the type*

$$s(x, y, t) \sim \sum_{k=0}^{\infty} t^{n-k} s_k(x, y)$$

so that

$$\Pi(x, y) = \int_0^\infty e^{it\psi(x,y)} s(x, y, t) dt$$

where the phase $\psi \in C^\infty(X \times X)$ is determined by the following properties:

- $\psi(x, x) = \frac{1}{i}\rho(x)$ *where ρ is the defining function of X.*
- $\bar\partial_x \psi$ *and* $\partial_y \psi$ *vanish to infinite order along the diagonal.*
- $\psi(x, y) = -\bar\psi(y, x)$.

The integral is defined as a complex oscillatory integral and is regularized by taking the principal value (see [BS]). The phase is determined only up to a function which vanishes to infinite order at $x = y$. (The phase ψ is given by (75) and (77) below.)

We now prove Theorem 3.1, using the Boutet de Monvel-Sjöstrand parametrix. The Szegö kernels Π_N are Fourier coefficients of Π and hence may be expressed as:

$$\Pi_N(x, y) = \int_0^\infty \int_0^{2\pi} e^{-iN\theta} e^{it\psi(r_\theta x, y)} s(r_\theta x, y, t) d\theta dt \tag{72}$$

where r_θ denotes the S^1 action on X. Changing variables $t \mapsto Nt$ gives

$$\Pi_N(x, y) = N \int_0^\infty \int_0^{2\pi} e^{iN(-\theta + t\psi(r_\theta x, y))} s(r_\theta x, y, tN) d\theta dt\,. \tag{73}$$

We now fix z_0 and consider the asymptotics of

$$\begin{aligned}\Pi_N\Big(z_0 + \frac{u}{\sqrt{N}}, 0; z_0 + \frac{v}{\sqrt{N}}, 0\Big) & \\ = N \int_0^\infty \int_0^{2\pi} & e^{iN\left(-\theta + t\psi\left(z_0 + \frac{u}{\sqrt{N}}, \theta; z_0 + \frac{v}{\sqrt{N}}; 0\right)\right)} \\ & \times s\Big(z_0 + \frac{u}{\sqrt{N}}, \theta; z_0 + \frac{v}{\sqrt{N}}, 0; tN\Big) d\theta dt\,.\end{aligned} \tag{74}$$

In our setting the phase takes the following concrete form: We write $a(z) = \|e_L^*(z)\|^2 = h(z)^{-1}$, and we let $a(z, w)$ be the almost analytic function on $M \times M$ satisfying $a(z, z) = a(z)$. The function $a(z, w)$ is determined up to infinite order along the diagonal $z = w$ and has the Taylor expansion

$$a(z_0 + u, z_0 + v) \sim \sum \frac{\partial^{\alpha+\beta} a}{\partial z^\alpha \partial \bar{z}^\beta}(z_0) \frac{u^\alpha}{\alpha!} \frac{\bar{v}^\beta}{\beta!}. \tag{75}$$

Note that since $a(z, z)$ is real, we may assume that $a(z, w) = \bar{a}(w, z)$. (In the case where $a(z)$ is real-analytic, then the right side of (75) gives the unique $a(z, w)$ that is holomorphic in z and anti-holomorphic in w near the diagonal.)

We consider the complex manifold $Y = L^*$ and we let (z, λ) denote the coordinates of $\xi \in Y$ given by $\xi = \lambda e_L^*(z)$. In the associated coordinates $(x, y) = (z, \lambda, w, \mu)$ on $Y \times Y$, we have:

$$\rho(z, \lambda) = 1 - a(z, z)|\lambda|^2, \qquad \psi(z, \lambda, w, \mu) = \frac{1}{i}(1 - a(z, w)\lambda\bar{\mu}) \; . \tag{76}$$

We consider $\Omega = \{\rho < 0\}$ and $X = \partial\Omega = \{\rho = 0\}$. On X we have $a(z, z))|\lambda|^2 = 1$ so we may write $\lambda = a(z, z))^{-\frac{1}{2}} e^{i\varphi}$, and similarly for μ. So for $(x, y) = (z, \varphi, w, \varphi') \in X \times X$ we have

$$\psi(z, \varphi, w, \varphi') = \frac{1}{i}\left[1 - \frac{a(z, w)}{\sqrt{a(z, z)}\sqrt{a(w, w)}} e^{i(\varphi - \varphi')}\right] . \tag{77}$$

It follows that

$$\psi\left(z_0 + \frac{u}{\sqrt{N}}, \theta; z_0 + \frac{v}{\sqrt{N}}, 0\right)$$
$$= \frac{1}{i}\left[1 - \frac{a\left(z_0 + \frac{u}{\sqrt{N}}, z_0 + \frac{v}{\sqrt{N}}\right)}{\sqrt{a\left(z_0 + \frac{u}{\sqrt{N}}, z_0 + \frac{u}{\sqrt{N}}\right)}\sqrt{a\left(z_0 + \frac{v}{\sqrt{N}}, z_0 + \frac{v}{\sqrt{N}}\right)}} e^{i\theta}\right] . \tag{78}$$

Our hypothesis on the coordinates $\{z_j\}$ says that

$$\frac{\partial^2 a}{\partial z^\alpha \partial \bar{z}^\beta}(z_0) = \delta_{\alpha\beta} \; . \tag{79}$$

Combining (79) with our condition on e_L gives us the second order approximation

$$a(z_0 + u) = 1 + |u|^2 + O(|u|^3) \; . \tag{80}$$

Hence by (75),

$$a\left(z_0 + \frac{u}{\sqrt{N}}, z_0 + \frac{v}{\sqrt{N}}\right) = 1 + \frac{1}{N} u \cdot \bar{v} + O(N^{-3/2}) \; . \tag{81}$$

Now let us return to the phase. It is given by

$$t\left[1-\frac{a\left(z_0+\frac{u}{\sqrt{N}},z_0+\frac{v}{\sqrt{N}}\right)}{a\left(z_0+\frac{u}{\sqrt{N}},z_0+\frac{u}{\sqrt{N}}\right)^{\frac{1}{2}}a\left(z_0+\frac{v}{\sqrt{N}},z_0+\frac{v}{\sqrt{N}}\right)^{\frac{1}{2}}}e^{i\theta}\right]-i\theta. \quad (82)$$

By (81), the phase (82) has the form:

$$(t[1-e^{i\theta}]-i\theta)+\frac{t}{N}\left[u\cdot\bar{v}-\frac{1}{2}|u|^2-\frac{1}{2}|v|^2\right]e^{i\theta}+O(N^{-3/2}). \quad (83)$$

It is now evident that $\Pi_N(z_0+\frac{u}{\sqrt{N}},0;z_0+\frac{v}{\sqrt{N}},0)$ is given by an oscillatory integral with phase $(t[1-e^{i\theta}]-i\theta)$; the latter two terms can be absorbed into the amplitude.

Thus we have:

$$\begin{aligned}\Pi_N&\left(z_0+\frac{u}{\sqrt{N}},0;z_0+\frac{v}{\sqrt{N}},0\right)\\ &=N\int_0^\infty\int_0^{2\pi}e^{iN(t[1-e^{i\theta}]-i\theta)}e^{t\left[u\cdot\bar{v}-\frac{1}{2}|u|^2-\frac{1}{2}|v|^2\right]+O(N^{-1/2})}\\ &\quad\times s\left(z_0+\frac{u}{\sqrt{N}},\theta;z_0+\frac{v}{\sqrt{N}},0;tN\right)d\theta dt.\end{aligned} \quad (84)$$

We may then evaluate the integral asymptotically by the stationary phase method as in [Ze]. The phase is precisely the same as occurs in $\Pi_N(x,x)$, and as discussed in [Ze], the single critical point occurs at $t=1,\theta=0$. We may also Taylor-expand the amplitude to determine its contribution to the asymptote. Precisely as in [Ze], we get:

$$\Pi_N\left(z_0+\frac{u}{\sqrt{N}},0;z_0+\frac{v}{\sqrt{N}},0\right)=\frac{N^m}{\pi^m}e^{u\cdot\bar{v}-\frac{1}{2}|u|^2-\frac{1}{2}|v|^2}+O(N^{m-\frac{1}{2}}). \quad (85)$$

(See also [SZ2, §2] for a detailed derivation of (85) by the method of stationary phase.) Finally, we note that

$$u\cdot\bar{v}-\frac{1}{2}|u|^2-\frac{1}{2}|v|^2=-\frac{1}{2}|u-v|^2+i\Im(u\cdot\bar{v}),$$

which completes the proof of Theorem 3.1. □

3.3. Universality of the scaling limit of correlations of zeros. We are now ready to pass to the scaling limit as $N \to \infty$ of the correlation functions of sections of powers L^N of our line bundle. As we indicated in the introduction, we must expand our neighborhood (or contract our "yardstick") by a factor of $\sqrt{N}$. To be precise, let $z^0 \in M$ and choose a coordinate neighborhood $U \in M$ with coordinates $\{z_q\}$ for which $z^0 = 0$ and $\omega(z^0) = \frac{i}{2} \sum_q dz_q \wedge d\bar{z}_q$. We define the *n-point scaling limit zero correlation function*

$$K^\infty_{nkm}(z) = \lim_{N\to\infty} \frac{1}{N^{nk}} K^N_{nk}\left(\frac{z}{\sqrt{N}}\right), \qquad z = (z^1, \dots, z^n) \in (\mathbb{C}^m)_n \, .$$

We show below (Theorem 3.6) that this limit exists and is universal by passing to the limit in Theorem 2.4, using Theorem 3.1. We shall need the following fact:

Lemma 3.3. *Let $z^1, \dots, z^n$ be distinct points of $\mathbb{C}^m$. Then*

$$\det\left(\Pi^{\mathbf{H}}_1(z^p, 0; z^{p'}, 0)\right) = \frac{1}{\pi^{mn}} e^{-\sum |z^p|^2} \det\left(e^{z^p \cdot \bar{z}^{p'}}\right) \neq 0 \, .$$

Lemma 3.3 is a consequence of the following elementary fact:

Lemma 3.4. *If $A = (a_{jk})$ is a semi-positive Hermitian matrix, then $(e^{a_{jk}})$ is also semi-positive. If in addition no two rows of A are identical, then $(e^{a_{jk}})$ is strictly positive definite.*

Proof. This is the complex form of VII.36 in [PS]. The proof of the first conclusion is exactly as in [PS]. Now assume that the rows of A are distinct. As in [PS], we reduce to the case where $a_{jk} = \gamma_j \bar{\gamma}_k$, with distinct γ_j. Then for any nonzero vector (x_j), we have

$$\sum_{j,k} e^{a_{jk}} x_j \bar{x}_k = \sum_{q=1}^\infty \frac{1}{q!} \sum_{j,k} \gamma_j^q \bar{\gamma}_k^q x_j \bar{x}_k = \sum_{q=1}^\infty \frac{1}{q!} |y_q|^2 \, , \qquad (86)$$

where $y_q = \sum_j \gamma_j^q x_j$. But the y_q cannot all vanish (since the Vandermonde matrix is nonsingular), and hence the sum in (86) is positive. □

Corollary 3.5. *If $z^1, \dots, z^n$ are distinct points of $\mathbb{C}^m$, then*

$$\left(e^{z^p \cdot \bar{z}^{p'}}\right)_{1 \le p, p' \le n} > 0 \, .$$

Proof. By Lemma 3.4, it suffices to show that the rows of the matrix $(z^p \cdot \bar{z}^{p'})$ are distinct. Suppose on the contrary that $z^1 \cdot \bar{z}^p = z^2 \cdot \bar{z}^p$, for $1 \le p \le n$. Then

$$|z^1 - z^2|^2 = (z^1 - z^2) \cdot \bar{z}^1 - (z^1 - z^2) \cdot \bar{z}^2 = 0 \, ,$$

contradicting the assumption that $z^1 \neq z^2$. □

A more geometrical way to understand Lemma 3.3 is as follows: Identify the Szegö projector $\Pi_1^{\mathbf{H}} : \mathcal{L}^2(\mathbf{H}^m_{\rm red}) \to \mathcal{H}^2_1(\mathbf{H}^m_{\rm red})$ on the reduced Heisenberg group with the Szegö projector Π_1^{BF} of the equivalent Bargmann-Fock representation of $\mathbf{H}^m_{\rm red}$ on the space $\mathcal{L}^2(\mathbb{C}^m, e^{-|z|^2}) \cap \mathcal{O}(\mathbb{C}^m)$ of entire holomorphic functions on $\mathbb{C}^m$ that are square-integrable with respect to the Gaussian measure. (As discussed at the end of §1.3.2, the isomorphism between the representations is to take the coefficient of the ground state φ_0.) It follows that the Bargmann-Fock Szegö kernel is given by:

$$\Pi_1^{BF}(z, w) = \frac{1}{\pi^m} e^{z\cdot\bar{w}} = \sum_{\alpha=1}^{\infty} f_\alpha(z)\,\overline{f_\alpha(w)}\,, \tag{87}$$

where the f_α form a complete orthonormal basis for $\mathcal{L}^2(\mathbb{C}^m, e^{-|z|^2}) \cap \mathcal{O}(\mathbb{C}^m)$; e.g., $\{f_\alpha\}$ can be taken to be the set of normalized monomials in the coordinates $z_1, \dots, z_m$. (In fact, Π_1^{BF} is just a "weighted Bergman kernel" on $\mathbb{C}^m$.) Suppose on the contrary that the determinant vanishes. We now mimic the proof of Lemma 2.3, except this time we have an infinite sum over the index α, and obtain a nonzero vector $(v_1, \dots, v_n) \in \mathbb{C}^n$ such that $\sum_p v_p f_\alpha(z^p) = 0$ for all α. (The infinite sum converges uniformly on bounded sets in $\mathbb{C}^n \times \mathbb{C}^n$, since the sup norm over a bounded set is dominated by the Gaussian-weighted $\mathcal{L}^2$ norm as in the case of the ordinary Bergman kernel on a bounded domain.) But then $\sum_p v_p f(z^p) = 0$ for all polynomials f on $\mathbb{C}^m$, a contradiction.

We can now show the universality of the scaling limit of the zero correlation functions:

Theorem 3.6. *Let (L, h) be a positive Hermitian line bundle on an m-dimensional compact complex manifold M with Kähler form $\omega = \frac{i}{2}\Theta_h$, let $\mathcal{S} = H^0(M, L^N)^k$ $(k \geq 1)$, and give $\mathcal{S}$ the standard Gaussian measure μ. Then*

$$\frac{1}{N^{nk}} K^N_{nk}\left(\frac{z^1}{\sqrt{N}}, \dots, \frac{z^n}{\sqrt{N}}\right) = K^\infty_{nkm}(z^1, \dots, z^n) + O\left(\frac{1}{\sqrt{N}}\right),$$

where $K^\infty_{nkm}(z^1, \dots, z^n)$ is given by a universal rational function in the quantities $z^p_q, \bar{z}^p_q, e^{z^p\cdot\bar{z}^{p'}}$, and the error term has ℓ^{th} order derivatives $\leq \frac{C_{S,\ell}}{\sqrt{N}}$ on each compact subset $S \subset (\mathbb{C}^m)_n$, for all $\ell \geq 0$.

Proof. By taking the scaling limit of (54), we obtain

$$K^\infty_{nkm}(z) = \frac{\mathcal{P}_{nkm}\left(\Pi_1^{\mathbf{H}}(z^p, z^{p'}), \overline{W}^L_q \Pi_1^{\mathbf{H}}(z^p, z^{p'}), Z^L_q \Pi_1^{\mathbf{H}}(z^p, z^{p'}), Z^L_q \overline{W}^L_{q'} \Pi_1^{\mathbf{H}}(z^p, z^{p'})\right)}{\pi^{kn}\left[\det\left(\Pi_1^{\mathbf{H}}(z^p, z^{p'})\right)_{1\leq p, p'\leq n}\right]^{k(n+1)}}. \tag{88}$$

Indeed, since the coefficients of Λ_n are either of degree 1 in the coefficients of C_n or of degree 2 in the coefficients of B_n, we see by the proof of Theorem 2.4, using (27), (51)–(52) and Theorem 3.1, that the leading term of the asymptotic expansion of K^N_{nk} is N^{nk} times the right side of (88). The bound on the error term follows from Theorem 3.1 and Lemma 3.3.

Substituting into (88) the values of $\Pi_1^{\mathbf{H}}(z^p, z^{p'})$ and its horizontal derivatives obtained from (26) (with $N = 1$) and (71) and canceling out common factors of $e^{-|z^p|^2/2}$ and π, we can replace $\Pi_1^{\mathbf{H}}$ with $\pi^m \Pi_1^{BF}$ (see (87)) and obtain

$$\begin{aligned} K^\infty_{nkm}(z) &= \frac{\mathcal{P}_{nkm}\left(e^{z^p\cdot\bar z^{p'}}, \left(z^p_q - z^{p'}_q\right)e^{z^p\cdot\bar z^{p'}}, \left(\bar z^{p'}_q - \bar z^p_q\right)e^{z^p\cdot\bar z^{p'}}, \left[\left(z^p_{q'} - z^{p'}_{q'}\right)\left(\bar z^{p'}_q - \bar z^p_q\right) + \delta_{qq'}\right]e^{z^p\cdot\bar z^{p'}}\right)}{\pi^{kn}\left[\det\left(e^{z^p\cdot\bar z^{p'}}\right)_{1\le p,p'\le n}\right]^{k(n+1)}} \\ &= \frac{\mathcal{Q}_{nkm}\left(z^p_q, \bar z^p_q, e^{z^p\cdot\bar z^{p'}}\right)}{\pi^{kn}\left[\det\left(e^{z^p\cdot\bar z^{p'}}\right)\right]^{k(n+1)}}, \end{aligned} \tag{89}$$

where $\mathcal{Q}_{nkm}$ is a universal polynomial (with integer coefficients) and is homogeneous of degree $k(n+1)$ in each of the variables $e^{z^p\cdot\bar z^{p'}}$. □

Remark. As we remarked previously, formula (88) is valid for any connection, so we can replace the left invariant vector fields with their right-invariant counterparts to obtain

$$K^\infty_{nkm}(z) = \frac{\mathcal{P}_{nkm}\left(e^{z^p\cdot\bar z^{p'}}, z^p_q e^{z^p\cdot\bar z^{p'}}, \bar z^{p'}_q e^{z^p\cdot\bar z^{p'}}, \left(z^p_{q'}\bar z^{p'}_q + \delta_{qq'}\right)e^{z^p\cdot\bar z^{p'}}\right)}{\pi^{kn}\left[\det\left(e^{z^p\cdot\bar z^{p'}}\right)_{1\le p,p'\le n}\right]^{k(n+1)}}. \tag{90}$$

4. Formulas for the scaling limit zero correlation function

We now apply the formulas from §§2.2–2.3 to transform (90) into explicit formulas for K^∞_{nkm}. We use the right-invariant connection α^R so that $d^H_{z_q} = Z^R_q$. Indeed, by the proofs of Theorems 2.4 and 3.6 (which use formulas (44), (46), (50)–(52)), formula (90) becomes

$$K^\infty_{nkm}(z^1, \dots, z^n) = \frac{1}{\pi^{kn}\det A_{nkm}}\left\langle \prod_{p=1}^n \det(\xi^p\xi^{p*})\right\rangle_{\Lambda_{nkm}}, \tag{91}$$

where

$$\Lambda_{nkm} = C_{nkm} - B^*_{nkm}A^{-1}_{nkm}B_{nkm} \tag{92}$$

with

$$\begin{aligned}
A_{nkm} &= \left(\delta_{jj'} S(z^p, z^{p'})\right), \quad S(z,w) = \exp\left(\sum_{r=1}^m z_r \overline{w_r}\right), \\
B_{nkm} &= \left(\delta_{jj'} S_{q'}(z^p, z^{p'})\right), \quad S_{q'}(z,w) = z_{q'} \exp\left(\sum_{r=1}^m z_r \overline{w_r}\right), \qquad (93)\\
C_{nkm} &= \left(\delta_{jj'} S_{qq'}(z^p, z^{p'})\right), \quad S_{qq'}(z,w) = (\delta_{qq'} + \overline{w_q} z_{q'}) \exp\left(\sum_{r=1}^m z_r \overline{w_r}\right) \\
& j, j' = 1, \dots, k; \; p, p' = 1, \dots, n; \; q, q' = 1, \dots, m.
\end{aligned}$$

(The metric tensor g^p in (46) becomes a unit tensor in the scaling limit, so there is no γ^p on the right in (91).)

Because $\Pi_1^{\mathbf{H}}$ is invariant with respect to unitary transformations and equivariant with respect to translations (i.e., $\Pi_1^{\mathbf{H}}(z+u, w+u) = e^{i\Im(z\cdot\bar{u})} e^{-i\Im(w\cdot\bar{u})} \Pi_1^{\mathbf{H}}(z,w)$), the scaling limit zero correlation K_{nkm}^∞ is invariant with respect to the group of isometric transformations – unitary transformations and translations – of $\mathbb{C}^m$.

In particular, the limit one-point zero correlation, or the zero-density function, is constant, since it is invariant under translation. Indeed by (93), $A_{1km} = e^{|z|^2} I_k$ and $\Lambda_{1km} = e^{|z|^2} I_{km}$, where I_k, resp. I_{km}, denotes the unit $k \times k$, resp. $(km) \times (km)$, matrix. Thus by (91) and the Wick formula,

$$K_{1km}^\infty(z) = \frac{1}{\pi^k e^{k|z|^2}} \left\langle \det \left(\sum_{q=1}^m \bar{\xi}_{jq} \xi_{j'q} \right)_{j,j'=1,\dots,k} \right\rangle_{e^{|z|^2} I_{km}} = \frac{m!}{\pi^k (m-k)!}. \qquad (94)$$

Thus we define the *normalized n-point scaling limit zero correlation function*

$$\widetilde{K}_{nkm}^\infty(z) = (K_{1km}^\infty)^{-n} K_{nkm}^\infty(z) = \left(\frac{\pi^k (m-k)!}{m!}\right)^n K_{nkm}^\infty(z). \qquad (95)$$

Remark. These formulas also follow from §2.4. For example, equation (94) is a consequence of (66) since

$$K_{1km}^\infty(z) = \frac{1}{N^k} K_{1k}^N(z).$$

Furthermore, using the notation of §2.4, we observe that

$$\begin{aligned}
\lim_{N\to\infty} S_N\left(\frac{z}{\sqrt{N}}, \frac{w}{\sqrt{N}}\right) &= \lim_{N\to\infty} \left(1 + N^{-1} \sum_{r=1}^m z_r \overline{w_r}\right)^N = S(z,w), \\
\lim_{N\to\infty} N^{-1/2} S_{Nq'}\left(\frac{z}{\sqrt{N}}, \frac{w}{\sqrt{N}}\right) &= S_{q'}(z,w), \qquad (96)\\
\lim_{N\to\infty} N^{-1} S_{Nqq'}\left(\frac{z}{\sqrt{N}}, \frac{w}{\sqrt{N}}\right) &= S_{qq'}(z,w).
\end{aligned}$$

Equations (96) provide an alternate derivation of (93).

4.1. Decay of correlations. Explicit formulas for the correlation functions $\widetilde{K}^{\infty}_{nkm}$ can be obtained from (91), (93) and the Wick formula. We shall illustrate these computations for the cases $n = 2,\ k = 1, 2$ in §§4.2–4.3 below. We now note that the limit correlations are "short range" in the following sense:

Theorem 4.1. *The correlation functions satisfy the estimate*

$$\widetilde{K}^{\infty}_{nkm}(z^1, \dots, z^n) = 1 + O(r^4 e^{-r^2}) \quad \text{as } r \to \infty\,, \quad r = \min_{p \neq p'} |z^p - z^{p'}|\,.$$

Proof. To obtain this estimate, we apply (91)–(92) using $\Pi_1^{\mathbf{H}}$ and the left-invariant vector fields instead of Π_1^{BF} and the right-invariant vector fields we used above. Recalling (26), we have:

$$\begin{aligned} A^{jp}_{j'p'} &= \delta_{jj'} A^p_{p'}\,, \qquad A^p_{p'} = \pi^m \Pi_1^{\mathbf{H}}(z^p, 0; z^{p'}, 0)\,, \\ B^{jp}_{j'p'q'} &= \delta_{jj'}\big(z^p_{q'} - z^{p'}_{q'}\big) A^p_{p'}\,, \\ C^{jpq}_{j'p'q'} &= \delta_{jj'}\big(\delta_{qq'} + \big(\bar z^{p'}_q - \bar z^p_q\big)\big(z^p_{q'} - z^{p'}_{q'}\big)\big) A^p_{p'}\,. \end{aligned} \tag{97}$$

By (71),

$$\begin{aligned} A^p_{p'} &= \begin{cases} 1 & p = p' \\ O(e^{-r^2/2}) & p \neq p' \end{cases}\,, \\ B &= O(re^{-r^2/2})\,, \\ C &= I + O(r^2 e^{-r^2/2})\,, \qquad C^{jpq}_{jpq} = 1\,. \end{aligned}$$

Recalling (44), we have

$$\Lambda = I + O(r^2 e^{-r^2/2})\,, \qquad \Lambda^{jpq}_{jpq} = 1 + O(r^2 e^{-r^2})\,. \tag{98}$$

We now apply formula (91), using the Wick formula ([Si, I.13]) to evaluate the right side. Note that the Wick formula involves terms that are products of diagonal elements of Λ, and products that contain at least two off-diagonal elements of Λ. The former terms are of the form $1 + O(r^2 e^{-r^2})$, and the latter are $O(r^4 e^{-r^2})$. Similarly, $\det A = 1 + O(r^4 e^{-r^2})$. The desired estimate then follows from (91). □

We shall see from our computations of the pair correlation below that Theorem 4.1 is sharp. In [BSZ2], we give decay estimates for the "connected" correlation functions.

4.2. Hypersurface pair correlation. We now give an explicit formula (107) for pair correlations in codimension 1 ($k = 1,\ n = 2$). The case

$m = 1$ of this formula coincides, as it must, with the formula given by [Han] and [BBL] for the universal scaling limit pair correlation for SU(2) polynomials. In another paper [BSZ1], we gave a different proof of (107) using the Poincaré-Lelong formula.

Since the scaling-limit pair correlation function $K^{\infty}_{2km}(z^1, z^2)$ is invariant with respect to the group of isometries of $\mathbb{C}^m$, it depends only on the distance $r = |z^1 - z^2|$, so we can set $z^1 = 0$ and $z^2 = (r, 0, \dots, 0)$. To simplify notation, we shall henceforth write $A = A_{2km}$, $B = B_{2km}$, $C = C_{2km}$, $\Lambda = \Lambda_{2km}$.

In this case, (93) reduces to

$$A = \begin{pmatrix} 1 & 1 \\ 1 & e^{r^2} \end{pmatrix};$$

$$B = \left(B^p_{p'q}\right); \quad \left(B^p_{p'1}\right) = \begin{pmatrix} 0 & 0 \\ r & re^{r^2} \end{pmatrix}; \quad \left(B^p_{p'q}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, q \geq 2;$$

$$C = \left(C^{pq}_{p'q'}\right); \quad \left(C^{p1}_{p'1}\right) = \begin{pmatrix} 1 & 1 \\ 1 & (1+r^2)e^{r^2} \end{pmatrix};$$

$$\left(C^{pq}_{p'q'}\right) = \delta_{qq'} \begin{pmatrix} 1 & 1 \\ 1 & e^{r^2} \end{pmatrix}, q, q' \geq 2. \tag{99}$$

The matrix

$$\Lambda = \left(\Lambda^{pq}_{p'q'}\right) = C - B^* A^{-1} B \tag{100}$$

is given by

$$\Lambda^{p1}_{p'1} = \begin{pmatrix} \dfrac{e^u - 1 - u}{e^u - 1} & \dfrac{e^u - 1 - ue^u}{e^u - 1} \\ \dfrac{e^u - 1 - ue^u}{e^u - 1} & \dfrac{e^{2u} - e^u - ue^u}{e^u - 1} \end{pmatrix}; \tag{101}$$

$$\Lambda^{pq}_{p'q'} = \delta_{qq'} \begin{pmatrix} 1 & 1 \\ 1 & e^u \end{pmatrix}, q, q' \geq 2,$$

where $u = r^2 = |z^1 - z^2|^2$. By (91), (95) and the formula for A in (99), we have

$$\widetilde{K}^{\infty}_{21m}(z^1, z^2) = \frac{1}{m^2(e^u - 1)} \left\langle \left(\sum_{q=1}^{m} \xi^1_q \overline{\xi^1_q} \right) \left(\sum_{q'=1}^{m} \xi^2_{q'} \overline{\xi^2_{q'}} \right) \right\rangle_{\Lambda} \tag{102}$$

By the Wick formula (see for example, [Si, (I.13)]),

$$\widetilde{K}^{\infty}_{21m}(z^1,z^2) = \frac{1}{m^2(e^u-1)}\left[\left(\sum_{q=1}^{m}\left\langle \xi^1_q\overline{\xi^1_q}\right\rangle_{\Lambda_2}\right)\left(\sum_{q'=1}^{m}\left\langle \xi^2_{q'}\overline{\xi^2_{q'}}\right\rangle_{\Lambda_2}\right) + \sum_{q,q'=1}^{m}\left\langle \xi^1_q\overline{\xi^2_{q'}}\right\rangle_{\Lambda_2}\left\langle \overline{\xi^1_q}\xi^2_{q'}\right\rangle_{\Lambda_2}\right]$$

$$= \frac{1}{m^2(e^u-1)}\left[\left(\sum_{q=1}^{m}\Lambda^{1q}_{1q}\right)\left(\sum_{q'=1}^{m}\Lambda^{2q'}_{2q'}\right) + \sum_{q,q'=1}^{m}\Lambda^{1q}_{2q'}\Lambda^{2q'}_{1q}\right]. \qquad (103)$$

Substituting the values of $\Lambda^{pq}_{p'q'}$ given by (101), we obtain

$$\widetilde{K}^{\infty}_{21m}(z^1,z^2) = \frac{1}{m^2(e^u-1)}\left[\left(\frac{e^u-1-u}{e^u-1}+m-1\right)\right.$$
$$\times\left(\frac{e^{2u}-e^u-ue^u}{e^u-1}+(m-1)e^u\right)+\left.\left(\frac{e^u-1-ue^u}{e^u-1}\right)^2+(m-1)\right],$$
$$u=|z^1-z^2|^2\,. \qquad (104)$$

After simplification,

$$\widetilde{K}^{\infty}_{21m}(z^1,z^2) = \frac{u^2(e^{2u}+e^u)-2u(e^{2u}-e^u)+m^2(e^u-1)^2e^u+m(e^u-1)^2}{m^2(e^u-1)^3}. \qquad (105)$$

Putting $u=2t$ and writing

$$\widetilde{K}^{\infty}_{21m}(z^1,z^2) = \kappa_{1m}(|z^1-z^2|)\,, \qquad (106)$$

we then obtain

$$\kappa_{1m}(r) = \frac{\left[\frac{1}{2}(m^2+m)\sinh^2 t + t^2\right]\cosh t - (m+1)t\sinh t}{m^2\sinh^3 t} + \frac{(m-1)}{2m}, \quad t=\frac{r^2}{2}\,. \qquad (107)$$

The case $m=1$ of formula (107) was obtained by Hannay [Han].

As $r\to\infty$,

$$\kappa_{1m}(r) = 1+\frac{r^4-2(m^2+1)r^2+m(3m+1)}{m^2}e^{-r^2}+O(r^4e^{-2r^2})\,. \qquad (108)$$

The following expansion of the correlation function was obtained from (107) using Maple™:

$$\kappa_{1m}=\frac{m-1}{2m}t^{-1}+\frac{m-1}{2m}+\frac{1}{6}\,\frac{(m+2)\,(m+1)}{m^2}t-\frac{1}{90}\,\frac{(m+4)\,(m+3)}{m^2}t^3$$
$$+\frac{1}{945}\,\frac{(m+6)\,(m+5)}{m^2}t^5-\frac{1}{9450}\,\frac{(m+8)\,(m+7)}{m^2}t^7$$
$$+\frac{1}{93555}\,\frac{(m+10)\,(m+9)}{m^2}t^9-\frac{691}{638512875}\,\frac{(m+12)\,(m+11)}{m^2}t^{11}$$
$$+\frac{2}{18243225}\,\frac{(m+14)\,(m+13)}{m^2}t^{13}-\cdots\,.$$

In particular, in the one-dimensional case we have

$$\kappa_{11}(r)=\frac{1}{2}r^2-\frac{1}{36}r^6+\frac{1}{720}r^{10}-\frac{1}{16800}r^{14}+\cdots\,. \tag{109}$$

4.3. Pair correlation in higher codimension. Next we compute the two-point correlation functions for the case $k=2$. For $k>1$, we have

$$A=\left(A^{jp}_{j'p'}\right)=\left(\delta_{jj'}A^{p}_{p'}\right),\quad B=\left(B^{jp}_{j'p'q'}\right)=\left(\delta_{jj'}B^{p}_{p'q'}\right), \tag{110}$$
$$C=\left(C^{jpq}_{j'p'q'}\right)=\left(\delta_{jj'}C^{pq}_{p'q'}\right),$$

where $A^{p}_{p'}$, $B^{p}_{p'q'}$, $C^{pq}_{p'q'}$ are given by (99). It follows that

$$\Lambda=\left(\Lambda^{jpq}_{j'p'q'}\right)=\left(\delta_{jj'}\Lambda^{pq}_{p'q'}\right), \tag{111}$$

where $\Lambda^{pq}_{p'q'}$ is given by (101).

By (91),

$$K^{\infty}_{2km}(z^1,z^2)=\frac{1}{\pi^{2k}(e^u-1)^k}\left\langle \det\left|\xi^1_j\overline{\xi^1_{j'}}\right|_{j,j'=1,\dots,k}\det\left|\xi^2_j\overline{\xi^2_{j'}}\right|_{j,j'=1,\dots,k}\right\rangle_{\Lambda},$$
$$\xi^p_j\overline{\xi^p_{j'}}=\sum_{q=1}^{m}\xi^p_{jq}\overline{\xi^p_{j'q}}, \tag{112}$$

where $u=r^2=|z^1-z^2|^2$ as before. Observe that the random variables ξ^p_{jq} and $\overline{\xi^{p'}_{j'q'}}$ are independent if either $j\neq j'$ or $q\neq q'$.

Recalling (95), we write

$$\widetilde{K}^{\infty}_{2km}(z^1,z^2)=\kappa_{km}(|z^1-z^2|)\,. \tag{113}$$

When $k = 2$, (112) reduces to the following

$$\kappa_{2m}(r) \tag{114}$$
$$= \frac{\left\langle \left[\left(\xi_1^1 \overline{\xi_1^1}\right)\left(\xi_2^1 \overline{\xi_2^1}\right) - \left(\xi_1^1 \overline{\xi_2^1}\right)\left(\xi_2^1 \overline{\xi_1^1}\right) \right] \left[\left(\xi_1^2 \overline{\xi_1^2}\right)\left(\xi_2^2 \overline{\xi_2^2}\right) - \left(\xi_1^2 \overline{\xi_2^2}\right)\left(\xi_2^2 \overline{\xi_1^2}\right) \right] \right\rangle_\Lambda}{m^2(m-1)^2(e^u-1)^2}.$$

By the Wick formula,

$$\kappa_{2m}(r) = \frac{d_{11} - d_{21} - d_{12} + d_{22}}{m^2(m-1)^2(e^u-1)^2}, \tag{115}$$

where

$$\begin{aligned} d_{11} &= \left\langle \left(\xi_1^1 \overline{\xi_1^1}\right)\left(\xi_2^1 \overline{\xi_2^1}\right)\left(\xi_1^2 \overline{\xi_1^2}\right)\left(\xi_2^2 \overline{\xi_2^2}\right) \right\rangle_\Lambda \\ &= \textstyle\sum_{\alpha,\beta,\gamma,\delta} \left\langle \xi_{1\alpha}^1 \overline{\xi_{1\alpha}^1} \xi_{2\beta}^1 \overline{\xi_{2\beta}^1} \xi_{1\gamma}^2 \overline{\xi_{1\gamma}^2} \xi_{2\gamma}^2 \overline{\xi_{2\gamma}^2} \right\rangle_\Lambda \\ &= \sum_{\alpha,\beta,\gamma,\delta} \Lambda_{1\alpha}^{1\alpha} \Lambda_{1\beta}^{1\beta} \Lambda_{2\gamma}^{2\gamma} \Lambda_{2\delta}^{2\delta} + 2 \sum_{\alpha,\beta,\gamma} \Lambda_{1\alpha}^{1\alpha} \Lambda_{2\beta}^{1\beta} \Lambda_{1\beta}^{2\beta} \Lambda_{2\gamma}^{2\gamma} \\ &\quad + \textstyle\sum_{\alpha,\beta} \Lambda_{2\alpha}^{1\alpha} \Lambda_{1\alpha}^{2\alpha} \Lambda_{2\beta}^{1\beta} \Lambda_{1\beta}^{2\beta} \\ &= \left[\left(\textstyle\sum_q \Lambda_{1q}^{1q}\right)\left(\textstyle\sum_q \Lambda_{2q}^{2q}\right) + \textstyle\sum_q \Lambda_{2q}^{1q} \Lambda_{1q}^{2q} \right]^2; \end{aligned} \tag{116}$$

similarly,

$$\begin{aligned} d_{12} &= \left\langle \left(\xi_1^1 \overline{\xi_1^1}\right)\left(\xi_2^1 \overline{\xi_2^1}\right)\left(\xi_1^2 \overline{\xi_2^2}\right)\left(\xi_2^2 \overline{\xi_1^2}\right) \right\rangle_\Lambda \\ &= \left(\textstyle\sum_q \left[\Lambda_{2q}^{2q}\right]^2\right)\left(\textstyle\sum_q \Lambda_{1q}^{1q}\right)^2 + 2\left(\textstyle\sum_q \Lambda_{2q}^{2q} \Lambda_{1q}^{2q} \Lambda_{2q}^{1q}\right)\left(\textstyle\sum_q \Lambda_{2q}^{1q}\right) \\ &\quad + \textstyle\sum_q \left[\Lambda_{1q}^{2q} \Lambda_{1q}^{1q}\right]^2, \end{aligned} \tag{117}$$

$$\begin{aligned} d_{21} &= \left\langle \left(\xi_1^1 \overline{\xi_2^1}\right)\left(\xi_2^1 \overline{\xi_1^1}\right)\left(\xi_1^2 \overline{\xi_1^2}\right)\left(\xi_2^2 \overline{\xi_2^2}\right) \right\rangle_\Lambda \\ &= \left(\textstyle\sum_q \left[\Lambda_{1q}^{1q}\right]^2\right)\left(\textstyle\sum_q \Lambda_{2q}^{2q}\right)^2 + 2\left(\textstyle\sum_q \Lambda_{1q}^{1q} \Lambda_{1q}^{2q} \Lambda_{2q}^{1q}\right)\left(\textstyle\sum_q \Lambda_{2q}^{2q}\right) \\ &\quad + \textstyle\sum_q \left[\Lambda_{1q}^{2q} \Lambda_{2q}^{1q}\right]^2, \end{aligned} \tag{118}$$

$$\begin{aligned} d_{22} &= \left\langle \left(\xi_1^1 \overline{\xi_2^1}\right)\left(\xi_2^1 \overline{\xi_1^1}\right)\left(\xi_1^2 \overline{\xi_2^2}\right)\left(\xi_2^2 \overline{\xi_1^2}\right) \right\rangle_\Lambda \\ &= \left(\textstyle\sum_q \left[\Lambda_{1q}^{1q}\right]^2\right)\left(\textstyle\sum_q \left[\Lambda_{2q}^{2q}\right]^2\right) + 2\textstyle\sum_q \Lambda_{1q}^{1q} \Lambda_{2q}^{2q} \Lambda_{1q}^{2q} \Lambda_{2q}^{1q} \\ &\quad + \left(\textstyle\sum_q \Lambda_{1q}^{2q} \Lambda_{2q}^{1q}\right)^2. \end{aligned} \tag{119}$$

Substituting the values of the matrix elements of Λ we then obtain

$$\kappa_{2m}(r) = \frac{(m^2-m)e^{2u} + 2(m-1)e^u + 2}{(e^u-1)^2 m(m-1)} - \frac{4ue^u[(m-1)e^u+1](m+1)}{(e^u-1)^3(m-1)m^2} + \frac{2u^2e^u[(m-1)e^{2u}+2me^u+1]}{(e^u-1)^4(m-1)m^2}, \qquad u = r^2. \tag{120}$$

As $r \to \infty$,

$$\kappa_{2m} = 1 + \frac{2[r^4 - 2(m+1)r^2 + m(m+1)]e^{-r^2}}{m^2} + O(r^4 e^{-2r^2}). \tag{121}$$

As $r \to 0$,

$$\kappa_{2m}(r) = \frac{m-2}{m} r^{-4} + \frac{m-2}{m} r^{-2} + \frac{5m^2 - 7m + 12}{12(m-1)m} + \frac{(m-2)(m+2)(m+1)}{12(m-1)m^2} r^2 + \frac{(m+3)(m+2)}{240(m-1)m} r^4 - \frac{(m-2)(m+4)(m+3)}{720(m-1)m^2} r^6 + \dots \tag{122}$$

When $m = 2$ the asymptotics reduce to

$$\kappa_{22}(r) = \frac{3}{4} + \frac{r^4}{24} - \frac{r^8}{288} + \frac{r^{12}}{4800} - \frac{r^{16}}{96768} + \dots, \tag{123}$$

and in this case κ_{22} is a series in r^4.

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