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Correlations between Zeros of Non-Gaussian Random Polynomials

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1 Introduction

Random polynomials, or more generally, linear combinations of functions with random coefficients serve as a basic model for eigenfunctions of chaotic quantum systems (see [1, 11, 12, 16, 19, 20, 23] and others). The geometric structure of random polynomials is, therefore, of significant interest for applications to quantum chaos. The basic questions are distribution and correlations between zeros of random polynomials, their critical points, distribution of values, nodal lines and surfaces, and so forth (see [1, 2, 3, 11, 12, 16, 19, 20, 22, 23] and others). The principal problem concerns the asymptotic behavior of typical, in the probabilistic sense, geometric structures as the degree of the polynomial, or the number of terms in a linear combination of functions, goes to infinity. The distribution of zeros of random polynomials is a classical question in probability theory and we refer to [4] and to [13, 14] for many earlier results in this area. The convergence of the distribution of zeros and their correlation functions in the scaling limit has been proved for *Gaussian* ensembles of random linear combinations of functions in a great generality (see [5, 6, 7, 8, 9, 10, 11, 12, 16, 21, 25, 26, 27] and others). In the present paper, we address the same question for *non-Gaussian* ensembles. In our approach, we will consider a Gaussian ensemble and assume that the random coefficients are no longer Gaussian but have a distribution of some class. Our goal will be to prove (or disprove) that the scaling limit of correlations between zeros is universal, that is, it is the same as for the Gaussian ensemble. Our main result will be that the universality of the scaling limit of

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correlations between zeros does hold away from the origin, while it fails at the origin. We will mostly consider the non-Gaussian SO(2) ensemble of random polynomials but our approach is quite general. In Section 9, we will discuss extensions to multivariate and complex ensembles of random polynomials.

Consider the real random polynomial of the form

$$f_n(x) = \sum_{k=0}^{n} \sqrt{\binom{n}{k}} c_k x^k, \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}, \tag{1.1}$$

where c_k are independent identically distributed random variables such that

$$\mathbf{E}c_k = 0, \qquad \mathbf{E}c_k^2 = 1. \tag{1.2}$$

When $c_k = N(0,1)$, the standard Gaussian random variable, the distribution of real zeros of $f_n(x)$ has two remarkable properties:

(1) the mathematical expectation of the number of real zeros of $f_n(x)$ is equal to $\sqrt{n}$:

$$\mathbf{E}\#\{k : f_n(x_k) = 0\} = \sqrt{n}; \tag{1.3}$$

(2) the normalized distribution of real zeros on the real line is the Cauchy distribution:

$$\frac{1}{\sqrt{n}}\mathbf{E}\#\{k : f_n(x_k) = 0,\ a \le x_k \le b\} = \int_a^b \frac{1}{\pi(x^2+1)}\,dx \tag{1.4}$$

(see [5, 13, 14]). Both properties are exact for any finite n. The second property can be reformulated as follows. Set

$$\theta_k = \arctan x_k, \tag{1.5}$$

so that θ_k is the stereographic projection of x_k. Then θ_k's are uniformly distributed on the circle

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \quad -\frac{\pi}{2} \equiv \frac{\pi}{2}. \tag{1.6}$$

For θ_k's, a stronger statement is valid: if $K_{nm}(s_1, \ldots, s_m)$ is the m-point correlation function of θ_k's, then for any a,

$$K_{nm}(s_1 + a, \ldots, s_m + a) = K_{nm}(s_1, \ldots, s_m), \tag{1.7}$$

the SO(2)-invariance [5]. As shown in [5], as $n \to \infty$, there exists the scaling limit of correlation functions

$$\lim_{n\to\infty} \frac{1}{n^{m/2}} K_{nm}\left(s + \frac{\tau_1}{\sqrt{n}}, \dots, s + \frac{\tau_m}{\sqrt{n}}\right) = K_m(\tau_1, \dots, \tau_m), \tag{1.8}$$

with explicit formulae for the limiting correlation functions.

In the present work, we will be interested in an extension of these results to non-Gaussian c_k's. When c_k is not Gaussian, the SO(2) symmetry is broken and properties (1) and (2) do not hold in general. The central question is whether properties (1) and (2) hold *asymptotically* as $n \to \infty$. We will answer this question under appropriate conditions on the probability distribution of c_k. We will assume that the probability distribution of c_k is absolutely continuous with respect to the Lebesgue measure, with a continuous density function $r(t)$. Conditions on $r(t)$ will be formulated below. In Sections 2, 3, and 4, we will show the universality of the Cauchy distribution as a limiting distribution of real zeros away from the origin. In Sections 5 and 6, we will derive a nonuniversal scaling behavior of the distribution of real zeros near the origin. In Sections 7 and 8, we will prove the universality of limiting correlation functions away from the origin. In the concluding Section 9, we will discuss extensions of our results to multivariate random polynomials.

2 Distribution of real zeros

We will be first interested in the distribution of real zeros of $f_n(x)$, and our calculations will be based on the Kac-Rice formula. The Kac-Rice formula [17, 24] expresses the density $p_n(x)$ of the distribution of real zeros of the polynomial $f_n(x)$ as

$$p_n(x) = \int_{-\infty}^{\infty} |\eta| D_n(0, \eta; x)\, d\eta, \tag{2.1}$$

where $D_n(\xi, \eta; x)$ is the joint distribution density of $f_n(x)$ and $f_n'(x)$. Since $f_n(0) = c_0$, $f_n'(0) = \sqrt{n} c_1$, we obtain that

$$p_n(0) = \sqrt{n}\, r(0) \int_{-\infty}^{\infty} |t| r(t)\, dt. \tag{2.2}$$

In particular, if $c_k = N(0, 1)$,

$$p_n(0) = \frac{\sqrt{n}}{\pi}. \tag{2.3}$$

Assume now that $x \neq 0$.

From (1.2),

$$\mathbf{E} f_n(x) = 0, \qquad \mathbf{E} f_n'(x) = 0, \tag{2.4}$$

and

$$\begin{aligned} &\mathbf{E} f_n^2(x) = \left(1+x^2\right)^n \equiv \sigma_n^2(x), \\ &\mathbf{E} f_n(x) f_n'(x) = nx\left(1+x^2\right)^{n-1}, \\ &\mathbf{E}\left(f_n'(x)\right)^2 = n\left(1+nx^2\right)\left(1+x^2\right)^{n-2} \equiv \zeta_n^2(x). \end{aligned} \tag{2.5}$$

To study the limit as $n \to \infty$, it is useful to rescale $f_n(x)$, $f_n'(x)$. Let

$$g_n(x) \equiv \frac{f_n(x)}{\sigma_n(x)} = \sum_{k=0}^n \mu_k(x) c_k, \qquad \widetilde{g}_n(x) \equiv \frac{f_n'(x)}{\zeta_n(x)} = \sum_{k=0}^n \nu_k(x) c_k, \tag{2.6}$$

where

$$\mu_k(x) = \frac{x^k}{\sigma_n(x)} \binom{n}{k}^{1/2}, \quad \nu_k(x) = \frac{k x^{k-1}}{\zeta_n(x)} \binom{n}{k}^{1/2}, \quad k = 0, 1, \ldots, n, \tag{2.7}$$

are the weights. Let $\widetilde{D}_n(\xi, \eta; x)$ be the joint distribution density of $g_n(x)$ and $\widetilde{g}_n(x)$. Then

$$D_n(\xi, \eta; x) = \frac{1}{\sigma_n(x)\zeta_n(x)} \widetilde{D}_n\left(\frac{\xi}{\sigma_n(x)}, \frac{\eta}{\zeta_n(x)}; x\right); \tag{2.8}$$

hence equation (2.1) reduces to

$$p_n(x) = \frac{\zeta_n(x)}{\sigma_n(x)} \int_{-\infty}^{\infty} |\eta| \widetilde{D}_n(0, \eta; x) \, d\eta. \tag{2.9}$$

From (2.7),

$$\mu_k(x) = \frac{x^k}{\left(1+x^2\right)^{n/2}} \binom{n}{k}^{1/2}, \qquad \nu_k(x) = \mu_k(x) \frac{k\left(1+x^2\right)}{\sqrt{n} x \sqrt{1+nx^2}}, \tag{2.10}$$

and

$$\sum_{k=0}^n \mu_k(x)^2 = 1, \qquad \sum_{k=0}^n \nu_k(x)^2 = 1, \qquad \sum_{k=0}^n \mu_k(x)\nu_k(x) = \frac{x\sqrt{n}}{\sqrt{1+nx^2}}. \tag{2.11}$$

It is convenient to orthogonalize the pair $g_n(x)$, $\tilde{g}_n(x)$. To that end, define

$$h_n(x) \equiv \frac{\tilde{g}_n(x) - \left(\nu(x), \mu(x)\right) g_n(x)}{\tau_n(x)} = \sum_{k=0}^n \lambda_k(x) c_k, \tag{2.12}$$

where

$$\mu(x) = \big(\mu_0(x), \ldots, \mu_n(x)\big), \qquad \nu(x) = \big(\nu_0(x), \ldots, \nu_n(x)\big),$$
$$\big(\nu(x), \mu(x)\big) = \sum_{k=0}^{n} \nu_k(x)\mu_k(x), \tag{2.13}$$

and

$$\lambda(x) = \big(\lambda_0(x), \ldots, \lambda_n(x)\big) = \frac{\nu(x) - \big(\nu(x), \mu(x)\big)\mu(x)}{\tau_n(x)},$$
$$\tau_n(x) = \big\|\nu(x) - \big(\nu(x), \mu(x)\big)\mu(x)\big\| = \left(\sum_{k=0}^{n} \big[\nu_k(x) - \big(\nu(x), \mu(x)\big)\mu_k(x)\big]^2\right)^{1/2}. \tag{2.14}$$

Observe that by (2.11) and (2.14),

$$\sum_{k=0}^{n} \mu_k(x)^2 = 1, \qquad \sum_{k=0}^{n} \lambda_k(x)^2 = 1, \qquad \sum_{k=0}^{n} \lambda_k(x)\mu_k(x) = 0. \tag{2.15}$$

From (2.10),

$$\nu_k(x) = \mu_k(x)\frac{u\big(1+x^2\big)\sqrt{n}}{x\sqrt{1+nx^2}} = \mu_k(x)\frac{u\sqrt{n}x}{u_0\sqrt{1+nx^2}}, \quad u \equiv \frac{k}{n},\ u_0 \equiv \frac{x^2}{1+x^2}; \tag{2.16}$$

hence

$$\lambda_k(x) = \frac{\mu_k(x)\sqrt{n}\big(u-u_0\big)\big(1+x^2\big)}{x}. \tag{2.17}$$

Let $\widehat{D}_n(\xi, \eta; x)$ be the joint distribution density of $g_n(x)$ and $h_n(x)$. Then

$$\widetilde{D}_n(\xi, \eta; x) = \frac{1}{\tau_n(x)}\widehat{D}_n\left(\xi, \frac{\eta - \big(\nu(x), \mu(x)\big)\xi}{\tau_n(x)}; x\right); \tag{2.18}$$

hence equation (2.9) reduces to

$$p_n(x) = \frac{\zeta_n(x)\tau_n(x)}{\sigma_n(x)} \int_{-\infty}^{\infty} |\eta|\widehat{D}_n(0, \eta; x)d\eta. \tag{2.19}$$

From (2.11) and (2.14),

$$\big(\nu(x), \mu(x)\big) = \frac{\sqrt{n}x}{\sqrt{1+nx^2}}, \qquad \tau_n(x)^2 = 1 - \big(\nu(x), \mu(x)\big)^2 = \frac{1}{1+nx^2}. \tag{2.20}$$

Hence (2.19) implies that

$$p_n(x) = \frac{\sqrt{n}}{(1+x^2)} \int_{-\infty}^{\infty} |\eta| \widehat{D}_n(0,\eta;x)\,d\eta. \tag{2.21}$$

By (1.2) and (2.15), for all real x,

$$\mathbf{E} g_n(x) = \mathbf{E} h_n(x) = 0, \qquad \mathbf{E} g_n(x)^2 = \mathbf{E} h_n(x)^2 = 1, \qquad \mathbf{E} g_n(x) h_n(x) = 0. \tag{2.22}$$

If c_k's are Gaussian, then $g_n(x)$ and $h_n(x)$ are Gaussian as well and equation (2.21) reduces to

$$p_n(x) = \frac{\sqrt{n}}{\pi(1+x^2)}. \tag{2.23}$$

We calculate the asymptotics of the weights $\mu_k(x)$ as $n \to \infty$. By the Stirling formula,

$$\binom{n}{k} = \frac{1}{\sqrt{2\pi n u(1-u)}} e^{-n\Theta(u)} \left(1 + O(\varepsilon_k)\right), \quad u = \frac{k}{n}, \tag{2.24}$$

where

$$\Theta(u) \equiv u \ln u + (1-u) \ln(1-u), \qquad \varepsilon_k \equiv (k+1)^{-1} + (n+1-k)^{-1}. \tag{2.25}$$

Therefore,

$$\mu_k(x)^2 = \frac{1}{\sqrt{2\pi n u(1-u)}} e^{-n\Theta(u;x)} \left(1 + O(\varepsilon_k)\right), \tag{2.26}$$

where

$$\Theta(u;x) \equiv u \ln u + (1-u) \ln(1-u) + \ln\left(1+x^2\right) - u \ln x^2. \tag{2.27}$$

The minimum of $\Theta(u;x)$ in u is attained at

$$u_0 \equiv \frac{x^2}{1+x^2}, \tag{2.28}$$

and

$$\Theta(u_0;x) = 0, \qquad \Theta''(u_0;x) = \frac{1}{u_0(1-u_0)} = \frac{\left(1+x^2\right)^2}{x^2} > 0. \tag{2.29}$$

Observe that

$$\Theta''(u;x) = \frac{1}{u(1-u)} > 0; \tag{2.30}$$

hence $\Theta(u;x)$ is a convex function on the interval $0 < u < 1$. Therefore, we obtain from (2.26) that there exists $C > 0$ such that

$$\max_{0\le k\le n} |\mu_k(x)| \le \frac{C}{\left[nu_0(1-u_0)\right]^{1/4}} = \frac{C\left(1+x^2\right)^{1/2}}{n^{1/4}|x|^{1/2}}. \tag{2.31}$$

By (2.17), (2.26), and (2.29),

$$\lambda_k(x)^2 = \frac{1}{\sqrt{2\pi n u(1-u)}} n\Theta''(u_0;x)(u-u_0)^2 e^{-n\Theta(u;x)}(1+O(\varepsilon_k)). \tag{2.32}$$

Simple estimates show that, similarly to (2.31), there exists $C > 0$ such that

$$\max_{0\le k\le n} |\lambda_k(x)| \le \frac{C\left(1+x^2\right)^{1/2}}{n^{1/4}|x|^{1/2}}. \tag{2.33}$$

The main results of this section are summarized as follows.

Proposition 2.1. For $x \neq 0$, the density $p_n(x)$ of the zeros distribution of random polynomial (1.1) is given by formula (2.21), where $\widehat{D}_n(\xi,\eta)$ is the joint distribution density of the random variables

$$g_n(x) = \sum_{k=0}^{n} \mu_k(x)c_k, \qquad h_n(x) = \sum_{k=0}^{n} \lambda_k(x)c_k, \tag{2.34}$$

and

$$\mu_k(x) = \frac{x^k}{(1+x^2)^{n/2}}\binom{n}{k}^{1/2}, \qquad \lambda_k(x) = \frac{\mu_k(x)\sqrt{n}(u-u_0)(1+x^2)}{x},$$
$$u = \frac{k}{n}, \qquad u_0 = \frac{x^2}{1+x^2}. \tag{2.35}$$

For every $x \neq 0$, the vectors $\mu(x) = (\mu_0(x), \ldots, \mu_n(x))$ and $\lambda(x) = (\lambda_0(x), \ldots, \lambda_n(x))$ are orthonormal (cf. (2.15) and (2.22)). In addition, there exists $C > 0$ such that for all $x \neq 0$,

$$\max_{0\le k\le n} |\mu_k(x)| \le \frac{C\left(1+x^2\right)^{1/2}}{n^{1/4}|x|^{1/2}}, \qquad \max_{0\le k\le n} |\lambda_k(x)| \le \frac{C\left(1+x^2\right)^{1/2}}{n^{1/4}|x|^{1/2}}. \tag{2.36}$$

For $x = 0$, the density $p_n(x)$ is given by formula (2.2). □

Estimates (2.36) provide us with a Lindeberg-type condition (see, e.g., [15]): they show that the random variables $g_n(x)$ and $h_n(x)$ are represented, when $x \neq 0$, as a sum of a big number of small independent random variables. Therefore, we can expect the central limit theorem for the random vector $(g_n(x), h_n(x))$, that is, as $n \to \infty$, the joint distribution of $g_n(x)$ and $h_n(x)$ converges to the joint distribution of two independent standard Gaussian random variables. Below we will prove that under appropriate conditions on the distribution of c_k, this convergence does take place, and it is uniform in x on any compact set away from the origin. To apply this result further to the convergence of $p_n(x)$, by using the Kac-Rice formula (2.1), we will, in fact, prove a local central limit theorem, together with some estimates on the tail of the distribution of $(g_n(x), h_n(x))$.

3 Universality of the limiting distribution of real zeros

Let $\varphi(s)$ be the characteristic function of c_k:

$$\varphi(s) = \int_{-\infty}^{\infty} r(t) e^{its}\, dt. \tag{3.1}$$

Observe that by (1.2),

$$\varphi(0) = 1, \qquad \varphi'(0) = 0, \qquad \varphi''(0) = -1; \qquad |\varphi(s)| \leq 1, \quad s \in \mathbb{R}. \tag{3.2}$$

We will assume that $\varphi(s)$ satisfies the following estimate: for some $a, q > 0$,

$$|\varphi(s)| \leq \frac{1}{(1 + as^2)^q}, \quad s \in \mathbb{R}. \tag{3.3}$$

In addition, we will assume that $\varphi(s)$ is a three-times differentiable function and there exist $c_2, c_3 > 0$ such that

$$\sup_{-\infty < s < \infty} \left| \frac{d^j \varphi(s)}{ds^j} \right| \leq c_j, \quad j = 2, 3. \tag{3.4}$$

Since $\varphi'(0) = 0$, this implies that for real s,

$$\left| \frac{d\varphi(s)}{ds} \right| \leq c_2 |s|. \tag{3.5}$$

Conditions (3.3), (3.4) are fulfilled for any density of the form

$$r(t) = e^{-V(t)}, \tag{3.6}$$

where $V(t)$ is a polynomial of even degree with a positive leading coefficient such that

$$\int_{-\infty}^{\infty} te^{-V(t)}dt = 0, \qquad \int_{-\infty}^{\infty} t^2e^{-V(t)}dt = 1. \tag{3.7}$$

More generally, introduce the following class of densities.

Class $\mathcal{D}_n$ of densities. A probability density function $r(t)$ belongs to the class $\mathcal{D}_n$ if

$$\int_{-\infty}^{\infty} tr(t)dt = 0, \qquad \int_{-\infty}^{\infty} t^2r(t)dt = 1, \tag{3.8}$$

$r(t)$ is C^2-smooth, and for any $j = 0, \dots, n$, there exists $C_j > 0$ such that for all $t \in \mathbb{R}$,

$$\big|r(t)\big| + \big|r''(t)\big| \le \frac{C_j}{1+|t|^j}. \tag{3.9}$$

We will denote $\mathcal{D}_\infty = \cap_{n=0}^{\infty}\mathcal{D}_n$.

Conditions (3.3), (3.4) are fulfilled for any density from the class $\mathcal{D}_5$.

Theorem 3.1. Let $f_n(x)$ be a random polynomial of degree n, as defined in (1.1). Let $p_n(x)$ be the distribution density function of real zeros of $f_n(x)$, and let $\varphi(s)$ be the characteristic function of c_k. If $\varphi(s)$ satisfies conditions (3.3), (3.4), then for all $\delta > 0$,

$$\lim_{n\to\infty} \frac{p_n(x)}{\sqrt{n}} = \frac{1}{\pi\big(1+x^2\big)}, \tag{3.10}$$

uniformly for all x such that $\delta^{-1} \ge |x| \ge \delta$. This means that the normalized distribution density function of real zeros of $f_n(x)$ has a universal limit of the Cauchy distribution if $x \neq 0$. □

Remark 3.2. Observe that by (2.2),

$$\frac{p_n(0)}{\sqrt{n}} = r(0)\int_{-\infty}^{\infty} |t|r(t)dt. \tag{3.11}$$

This shows that at $x = 0$, the universal limit (3.10) does not hold in general. It is possible to derive a scaling formula for $p_n(x)$ in a vicinity of $x = 0$, which interpolates between (3.11) and (3.10); see Section 5.

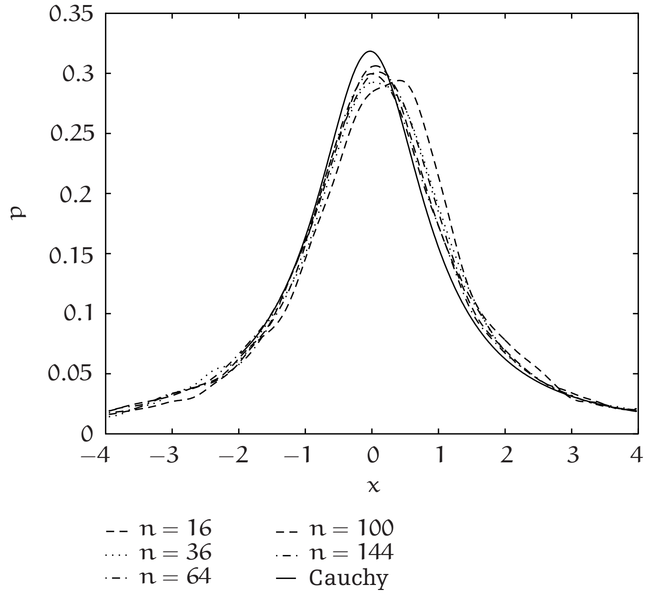

Figure 3.1 Empirical distributions of zeros of random polynomials of different degrees $n = 16, 36, 64, 100, 144$ when c_k has a centered Gamma distribution.

We will prove Theorem 3.1 in Section 4. In fact, we will prove that for any $\delta > 0$ and any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that for all x such that $\delta < |x| < \delta^{-1}$,

$$\left| \frac{p_n(x)}{\sqrt{n}} - \frac{1}{\pi(1+x^2)} \right| \le C_\epsilon n^{-1/12+\epsilon}. \tag{3.12}$$

This estimates the rate of convergence of $p_n(x)/\sqrt{n}$ to the Cauchy distribution density $1/\pi(1+x^2)$. In subsequent sections, we will prove the convergence of correlation functions. We will assume a stronger condition on the characteristic function $\varphi(s)$: it is C^∞ smooth and for any $j \ge 2$, there is $c_j > 0$ such that

$$\left| \frac{d^j \varphi(s)}{ds^j} \right| \le c_j. \tag{3.13}$$

Under the stronger condition, it will follow the better rate of convergence for the correlation functions, and, in particular, for the density

$$\left| \frac{p_n(x)}{\sqrt{n}} - \frac{1}{\pi(1+x^2)} \right| \le C_\epsilon n^{-1/4+\epsilon}. \tag{3.14}$$

Figure 3.1 illustrates the empirical distributions of zeros of random polynomials of different degrees $n = 16, 36, 64, 100, 144$ when c_k has a centered Gamma distribution.

4 Proof of Theorem 3.1

Let $\widehat{D}_n(\xi,\eta;x)$ be the joint distribution density of $g_n(x)$, $h_n(x)$ and $\Phi_n(\gamma)=\Phi_n(\gamma;x)$, the corresponding characteristic function,

$$\Phi_n(\gamma)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\widehat{D}_n(\xi,\eta;x)e^{i\alpha\xi+i\beta\eta}d\xi\,d\eta, \tag{4.1}$$

where $\gamma=(\alpha,\beta)$. Our strategy will be to prove that $\widehat{D}_n(\xi,\eta;x)$ converges, in an appropriate sense, to the Gaussian density $(1/2\pi)e^{-(1/2)(\xi^2+\eta^2)}$. This will be a Lindeberg-type local central limit theorem for vector random variables, with an additional estimate of the tail of the density $\widehat{D}_n(\xi,\eta;x)$. First we will prove that the characteristic function $\Phi_n(\gamma)$ converges to $e^{-(1/2)|\gamma|^2}$. From (2.6) and (2.12), we have that

$$\Phi_n(\gamma)=\prod_{k=0}^{n}\varphi(\omega_k), \tag{4.2}$$

where φ is the characteristic function of c_k and

$$\omega_k=\mu_k(x)\alpha+\lambda_k(x)\beta. \tag{4.3}$$

Lemma 4.1. If $\varphi(s)$ satisfies (3.3), then for any $L>0$ there exist $a_0>0$ and $N_0>0$ such that for all $n\geq N_0$,

$$\left|\Phi_n(\gamma)\right|\leq\frac{1}{\left(1+a_0|\gamma|^2\right)^L}. \tag{4.4}$$

□

Proof. From (4.2) and (3.3),

$$\left|\Phi_n(\gamma)\right|\leq\prod_{k=0}^{n}\frac{1}{\left(1+a\omega_k^2\right)^q}. \tag{4.5}$$

We have that

$$\sum_{k=0}^{n}\omega_k^2=\alpha^2\sum_{k=0}^{n}\mu_k^2+2\alpha\beta\sum_{k=0}^{n}\mu_k\lambda_k+\beta^2\sum_{k=0}^{n}\lambda_k^2=\alpha^2+\beta^2=|\gamma|^2. \tag{4.6}$$

By (2.36), $\mu_k,\lambda_k=O(n^{-1/4})$; hence

$$\omega_k^2=O\left(n^{-1/2}|\gamma|^2\right). \tag{4.7}$$

Partition all k's into T groups M_j so that, for each group,

$$\sum_{k\in M_j} \omega_k^2 \geq \frac{1}{2T}|\gamma|^2. \tag{4.8}$$

Then from (4.5),

$$|\Phi_n(\gamma)| \leq \prod_{j=1}^{T} \prod_{k\in M_j} \frac{1}{(1+a\omega_k^2)^q} \leq \frac{1}{(1+a_0\omega_k^2)^{Tq}}, \quad a_0 = \frac{a}{2T}. \tag{4.9}$$

Take $T = L/q$, then (4.1) follows. Lemma 4.1 is proved. ∎

Lemma 4.2. If $\varphi(s)$ satisfies (3.3), (3.4), then for any $L > 0$, there exist $a_0, C > 0$ and $N_0 > 0$ such that for all $n \geq N_0$,

$$\left|\frac{\partial^j \Phi_n(\gamma)}{\partial \beta^j}\right| \leq \frac{C}{(1+a_0|\gamma|^2)^L}, \quad j = 1, 2, 3. \tag{4.10}$$

□

Proof. Consider first $j = 1$. From (4.2),

$$\left|\frac{\partial \Phi_n(\gamma)}{\partial \beta}\right| = \left|\sum_{k=0}^{n} \lambda_k \varphi'(\omega_k) \prod_{l\neq k} \varphi(\omega_k)\right|. \tag{4.11}$$

By repeating the proof of Lemma 4.1, we obtain that

$$\left|\prod_{l\neq k} \varphi(\omega_k)\right| \leq \frac{1}{(1+a_0|\gamma|^2)^L}. \tag{4.12}$$

By (3.5),

$$|\varphi'(\omega_k)| \leq c_2|\omega_k|; \tag{4.13}$$

hence

$$\sum_{k=0}^{n} |\lambda_k \varphi'(\omega_k)| \leq c_2 \left(\sum_{k=0}^{n} \lambda_k^2\right)^{1/2} \left(\sum_{k=0}^{n} \omega_k^2\right)^{1/2} = c_2|\gamma|. \tag{4.14}$$

Thus,

$$\left|\frac{\partial \Phi_n(\gamma)}{\partial \beta}\right| \le \frac{c_2|\gamma|}{\left(1+a_0|\gamma|^2\right)^L}, \tag{4.15}$$

which implies (4.10) for $j = 1$.

Consider $j = 2$. From (4.2),

$$\frac{\partial^2 \Phi_n(\gamma)}{\partial \beta^2} = \sum_{k=0}^{n} \sum_{i\neq k} \lambda_i \lambda_k \varphi'(\omega_i)\varphi'(\omega_k) \prod_{l\neq i,k} \varphi(\omega_k) + \sum_{k=0}^{n} \lambda_k^2 \varphi''(\omega_k) \prod_{l\neq k} \varphi(\omega_k). \tag{4.16}$$

By repeating the proof of Lemma 4.1, we obtain that

$$\left|\prod_{l\neq i,k} \varphi(\omega_k)\right| \le \frac{1}{\left(1+a_0|\gamma|^2\right)^L}. \tag{4.17}$$

In addition, from (4.14), we obtain that

$$\left|\sum_{k=0}^{n} \sum_{i\neq k} \lambda_i \lambda_k \varphi'(\omega_i)\varphi'(\omega_k)\right| \le \left(\sum_{k=0}^{n} |\lambda_k \varphi'(\omega_k)|\right)^2 \le c_2^2|\gamma|^2. \tag{4.18}$$

Therefore,

$$\left|\sum_{k=0}^{n} \sum_{i\neq k} \lambda_i \lambda_k \varphi'(\omega_i)\varphi'(\omega_k) \prod_{l\neq i,k} \varphi(\omega_k)\right| \le \frac{c_2^2|\gamma|^2}{\left(1+a_0|\gamma|^2\right)^L}. \tag{4.19}$$

By (3.4) and (4.12),

$$\left|\sum_{k=0}^{n} \lambda_k^2 \varphi''(\omega_k) \prod_{l\neq k} \varphi(\omega_k)\right| \le \frac{c_2}{\left(1+a_0|\gamma|^2\right)^L} \sum_{k=0}^{n} \lambda_k^2 = \frac{c_2}{\left(1+a_0|\gamma|^2\right)^L}. \tag{4.20}$$

Equation (4.16) and estimates (4.19), (4.20) imply (4.10) for $j = 2$. The case $j = 3$ is considered in the same way. Lemma 4.2 is proved. ■

Lemma 4.3. If $\varphi(s)$ satisfies (3.3), (3.4), then there exist $C > 0$ and $N_0 > 0$ such that for all $n \ge N_0$,

$$\left|\widehat{D}_n(0,\eta;x)\right| \le \frac{C}{\left(1+|\eta|\right)^3}. \tag{4.21}$$

□

Proof. Observe that by (3.1),

$$\eta^k \widehat{D}_n(0,\eta;x) = \frac{(-i)^k}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-i\beta\eta} \frac{\partial^k \Phi_n(\gamma)}{\partial \beta^k}\, d\gamma; \tag{4.22}$$

hence

$$\left(1+|\eta|^3\right)\left|\widehat{D}_n(0,\eta;x)\right| \le \int_{\mathbb{R}^2} \left|\Phi_n(\gamma)\right| d\gamma + \int_{\mathbb{R}^2} \left|\frac{\partial^3 \Phi_n(\gamma)}{\partial \beta^3}\right| d\gamma. \tag{4.23}$$

From (4.4) and (4.10), we obtain that

$$\left(1+|\eta|^3\right)\left|\widehat{D}_n(0,\eta;x)\right| \le \int_{\mathbb{R}^2} \frac{C}{\left(1+\varepsilon_0|\gamma|^2\right)^L} d\gamma \le C_0, \tag{4.24}$$

if L is taken greater than 1. This implies (4.21). Lemma 4.3 is proved. ■

Let $\kappa > 0$ be a fixed small number:

$$\kappa < \frac{1}{12}. \tag{4.25}$$

Set

$$\Lambda_n = \left\{\gamma : |\gamma| \le n^{\kappa}\right\}. \tag{4.26}$$

Lemma 4.4. If $\varphi(s)$ satisfies (3.3), (3.4), then for any $L > 0$, there exist $a_0 > 0$, $C > 0$, and $N_0 > 0$ such that for all $n \ge N_0$,

$$\sup_{\gamma \in \Lambda_n} \left|\Phi_n(\gamma) - e^{-(1/2)|\gamma|^2}\right| \le \frac{C n^{-(1/4)+\kappa_0}}{\left(1+a_0|\gamma|^2\right)^L}, \quad \kappa_0 = 3\kappa. \tag{4.27}$$

□

Proof. Observe that

$$\sum_{k=0}^{n} \omega_k^2 = |\gamma|^2, \tag{4.28}$$

and if $\gamma \in \Lambda_n$, then

$$\omega_k = O\left(n^{-(1/4)+\kappa}\right). \tag{4.29}$$

To prove (4.27), we write that

$$\Phi_n(\gamma) - e^{-(1/2)\sum_{k=0}^{n}\omega_k^2} = \prod_{k=0}^{n}\varphi(\omega_k) - \prod_{k=0}^{n} e^{-(1/2)\omega_k^2}$$
$$= \sum_{j=0}^{n}\left(\prod_{k=0}^{j-1}\varphi(\omega_k)\right)\left(\varphi(\omega_j) - e^{-(1/2)\omega_j^2}\right)\prod_{k=j+1}^{n} e^{-(1/2)\omega_k^2}. \tag{4.30}$$

We have the estimate

$$e^{-(1/2)x^2} \le \frac{1}{1+\frac{1}{2}x^2}; \tag{4.31}$$

hence similar to Lemma 4.1, we obtain that

$$\left(\prod_{k=0}^{j-1}\varphi(\omega_k)\right)\prod_{k=j+1}^{n} e^{-(1/2)\omega_k^2} \le \frac{1}{\left(1+a_0|\gamma|^2\right)^L}. \tag{4.32}$$

Due to (3.2), we have that as $s \to 0$, $\varphi(s) = 1 - (1/2)s^2 + O(|s|^3)$; hence there exists some constant $C_0 > 0$ such that

$$\left|\varphi(\omega_j) - e^{-(1/2)\omega_j^2}\right| \le C_0\left|\omega_j\right|^3, \quad \gamma \in \Lambda_n. \tag{4.33}$$

Thus, from (4.30), we obtain that

$$\left|\Phi_n(\gamma) - e^{-(1/2)\sum_{k=0}^{n}\omega_k^2}\right| \le \frac{C_0}{\left(1+a_0|\gamma|^2\right)^L}\sum_{k=0}^{n}\left|\omega_k\right|^3$$
$$\le \frac{C_0}{\left(1+a_0|\gamma|^2\right)^L}\left(\sup_k\left|\omega_k\right|\right)|\gamma|^2. \tag{4.34}$$

Hence,

$$\left|\Phi_n(\gamma) - e^{-(1/2)|\gamma|^2}\right| \le \frac{Cn^{-(1/4)+\kappa_0}}{\left(1+a_0|\gamma|^2\right)^L}. \tag{4.35}$$

Lemma 4.4 is proved. ■

Lemma 4.5. If $\varphi(s)$ satisfies (3.3), (3.4), then there exist $C > 0$ and $N_0 > 0$ such that for all $n \ge N_0$,

$$\sup_{\xi,\eta}\left|\widehat{D}_n(\xi,\eta;x) - \frac{1}{2\pi}e^{-(1/2)(\xi^2+\eta^2)}\right| \le Cn^{-(1/4)+\kappa_0}, \quad \kappa_0 = 3\kappa. \tag{4.36}$$

□

Proof. From (4.1),

$$\widehat{D}_n(\xi,\eta;x) - \frac{1}{2\pi}e^{-(1/2)(\xi^2+\eta^2)} = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\Phi(\gamma) - e^{-(1/2)|\gamma|^2}\right)e^{-i\alpha\xi - i\beta\eta}\,d\xi\,d\eta; \tag{4.37}$$

hence (4.36) follows from (4.27). Lemma 4.5 is proved. ∎

Proof of Theorem 3.1. By (2.21), we have that

$$\frac{p_n(x)}{\sqrt{n}} - \frac{1}{\pi(1+x^2)} = \frac{1}{1+x^2}\int_{-\infty}^{\infty}|\eta|\left(\widehat{D}_n(0,\eta;x) - \frac{1}{2\pi}e^{-(1/2)\eta^2}\right)d\eta. \tag{4.38}$$

Let $\tau > 0$ be an arbitrary number. By Lemma 4.5,

$$\int_{-n^\tau}^{n^\tau}|\eta|\left|\widehat{D}_n(0,\eta;x) - \frac{1}{2\pi}e^{-(1/2)\eta^2}\right|d\eta \le Cn^{2\tau-(1/4)+\kappa_0}. \tag{4.39}$$

By Lemma 4.3,

$$\int_{|\eta|>n^\tau}|\eta|\left|\widehat{D}_n(0,\eta;x)\right|d\eta \le C\int_{|\eta|>n^\tau}\frac{|\eta|\,d\eta}{(1+|\eta|)^3} \le 2Cn^{-\tau}. \tag{4.40}$$

Also,

$$\int_{|\eta|>n^\tau}|\eta|e^{-(1/2)\eta^2}\,d\eta = 2e^{-(1/2)n^{2\tau}}. \tag{4.41}$$

Take $\tau = 1/12 - \kappa_0/3$. Then, combining the last three estimates, we obtain that there exists $C > 0$ such that

$$\left|\frac{p_n(x)}{\sqrt{n}} - \frac{1}{\pi(1+x^2)}\right| \le \frac{Cn^{-1/12+\kappa_0/3}}{1+x^2} = \frac{Cn^{-1/12+\kappa}}{1+x^2}, \tag{4.42}$$

which implies (3.10). Theorem 3.1 is proved. ∎

5 Scaling near zero

In this section, we will describe a crossover asymptotics from (3.11) to (3.10). The crossover takes place on a small scale of the order of $n^{-1/2}$. Define the scaled variable y as

$$y = n^{1/2}x. \tag{5.1}$$

Consider in Proposition 2.1 the random variables

$$g_n\left(\frac{y}{\sqrt{n}}\right) = \sum_{k=0}^{n} \mu_k\left(\frac{y}{\sqrt{n}}\right)c_k, \qquad h_n\left(\frac{y}{\sqrt{n}}\right) = \sum_{k=0}^{n} \lambda_k\left(\frac{y}{\sqrt{n}}\right)c_k. \tag{5.2}$$

By (2.35),

$$\mu_k\left(\frac{y}{\sqrt{n}}\right) = \frac{y^k}{n^{k/2}\left(1+\frac{y^2}{n}\right)^{n/2}}\binom{n}{k}^{1/2}, \qquad \lambda_k\left(\frac{y}{\sqrt{n}}\right) = \mu_k\left(\frac{y}{\sqrt{n}}\right)(u-u_0)\frac{n+y^2}{y},$$
$$u = \frac{k}{n}, \qquad u_0 \equiv \frac{y^2}{n+y^2}, \tag{5.3}$$

which gives that

$$\lambda_k\left(\frac{y}{\sqrt{n}}\right) = \mu_k\left(\frac{y}{\sqrt{n}}\right)\left(\frac{k-y^2}{y} + \frac{ky}{n}\right). \tag{5.4}$$

In particular,

$$\mu_0\left(\frac{y}{\sqrt{n}}\right) = \left(1+\frac{y^2}{n}\right)^{-n/2}, \qquad \mu_1\left(\frac{y}{\sqrt{n}}\right) = y\left(1+\frac{y^2}{n}\right)^{-n/2},$$
$$\lambda_0\left(\frac{y}{\sqrt{n}}\right) = -y\left(1+\frac{y^2}{n}\right)^{-n/2}, \qquad \lambda_1\left(\frac{y}{\sqrt{n}}\right) = \left(1-y^2\frac{n-1}{n}\right)\left(1+\frac{y^2}{n}\right)^{-n/2}. \tag{5.5}$$

As $n \to \infty$, we have the limits

$$\lim_{n\to\infty} \mu_k\left(\frac{y}{\sqrt{n}}\right) = \frac{1}{\sqrt{k!}}y^k e^{-y^2/2} \equiv m_k(y),$$
$$\lim_{n\to\infty} \lambda_k\left(\frac{y}{\sqrt{n}}\right) = \frac{(k-y^2)}{\sqrt{k!}}y^{k-1}e^{-y^2/2} \equiv l_k(y) = m_k(y)\frac{k-y^2}{y}. \tag{5.6}$$

Moreover, we have the following estimate of the error term.

Lemma 5.1. There is $C > 0$ such that if $|y| \le n^{1/8}$ and $k \le n^{1/4}$, then

$$\left|\mu_k\left(\frac{y}{\sqrt{n}}\right) - m_k(y)\right| \le \begin{cases} C\dfrac{y^4}{n}\big|m_k(y)\big|, & k = 0, 1, \\ C\dfrac{k^2 + y^4}{n}\big|m_k(y)\big|, & k \ge 2, \end{cases} \tag{5.7}$$

$$\left|\lambda_k\left(\frac{y}{\sqrt{n}}\right) - l_k(y)\right| \le \begin{cases} C\dfrac{|y| + |y|^5}{n}\big|m_k(y)\big|, & k = 0, 1, \\ C\dfrac{k^3 + y^6}{n|y|}\big|m_k(y)\big|, & k \ge 2. \end{cases} \tag{5.8}$$

In addition, for $|y| \le n^{1/8}$ and all k,

$$\begin{aligned} \left|\mu_k\left(\frac{y}{\sqrt{n}}\right)\right| &\le C\frac{|y|^k}{\sqrt{k!}}e^{-y^2/2}, \\ \left|\lambda_k\left(\frac{y}{\sqrt{n}}\right)\right| &\le \frac{|y|^k}{\sqrt{k!}}e^{-y^2/2}\left[k\left(\frac{1}{|y|} + \frac{|y|}{n}\right) + 1\right]. \end{aligned} \tag{5.9}$$

□

Proof. From (5.3), (5.6), we have that

$$\frac{\mu_k\left(\frac{y}{\sqrt{n}}\right)}{m_k(y)} = \left[\prod_{j=0}^{k-1}\left(1 - \frac{j}{n}\right)\right]^{1/2} \frac{e^{y^2/2}}{\left(1 + \frac{y^2}{n}\right)^{n/2}}; \tag{5.10}$$

hence

$$\begin{aligned} \left|\ln \frac{\mu_k\left(\frac{y}{\sqrt{n}}\right)}{m_k(y)}\right| &= \left|\frac{1}{2}\left[\sum_{j=0}^{k-1}\ln\left(1 - \frac{j}{n}\right)\right] + \frac{y^2}{2} - \frac{n}{2}\ln\left(1 + \frac{y^2}{n}\right)\right| \\ &\le C_0\left(\frac{k(k-1)}{n} + \frac{y^4}{n}\right), \end{aligned} \tag{5.11}$$

which gives (5.7). From (5.4),

$$\begin{aligned} \frac{\lambda_k\left(\frac{y}{\sqrt{n}}\right)}{l_k(y)} &= \frac{\mu_k\left(\frac{y}{\sqrt{n}}\right)}{m_k(y)}\left(1 + \frac{y^2}{n(k - y^2)}\right) \\ &= \left[1 + O\left(\frac{k(k-1)}{n} + \frac{y^4}{n}\right)\right]\left(1 + \frac{y^2}{n(k - y^2)}\right), \end{aligned} \tag{5.12}$$

which implies (5.8). To prove (5.9), observe that if $|y| \le n^{1/8}$, then

$$\left(1+\frac{y^2}{n}\right)^{-n/2} \le Ce^{-y^2/2}. \tag{5.13}$$

Indeed,

$$\frac{n}{2}\ln\left(1+\frac{y^2}{n}\right) = \frac{y^2}{2} + O\left(\frac{y^4}{n}\right); \tag{5.14}$$

hence (5.13) follows. From (5.3) and (5.13), we obtain (5.9). ■

Observe that

$$\sum_{k=0}^{\infty} m_k(y)^2 = \sum_{k=0}^{\infty} l_k(y)^2 = 1, \qquad \sum_{k=0}^{\infty} m_k(y) l_k(y) = 0. \tag{5.15}$$

Consider the random variables

$$\begin{aligned} g(y) &= \sum_{k=0}^{\infty} m_k(y) c_k = e^{-y^2/2} \sum_{k=0}^{\infty} \frac{1}{\sqrt{k!}} y^k c_k, \\ h(y) &= \sum_{k=0}^{\infty} l_k(y) c_k = e^{-y^2/2} \sum_{k=0}^{\infty} \frac{(k-y^2)}{\sqrt{k!}} y^{k-1} c_k. \end{aligned} \tag{5.16}$$

Let $D(\xi, \eta; y)$ be the joint distribution density of $g(y)$ and $h(y)$. By the Kac-Rice formula, the density $\widehat{p}(y)$ of the distribution of zeros of $g(y)$ is equal to

$$\widehat{p}(y) = \int_{-\infty}^{\infty} |\eta| D(0, \eta; y)\, d\eta. \tag{5.17}$$

We will prove the following result.

Theorem 5.2. Let $f_n(x)$ be a random polynomial of degree n, as defined in (1.1). Let $p_n(x)$ be the distribution density function of real zeros of $f_n(x)$, and let $\varphi(s)$ be the characteristic function of c_k. Assume that for some $a, A > 0$,

$$\left|\varphi(s)\right| \le \frac{1}{(1+a|s|)^6}, \qquad \left|\frac{d^j \varphi(s)}{ds^j}\right| \le \frac{A}{(1+a|s|)^6}, \qquad j = 1, 2, 3;\ s \in \mathbb{R}. \tag{5.18}$$

Then

$$\lim_{n\to\infty} \frac{p_n\left(\frac{y}{\sqrt{n}}\right)}{\sqrt{n}} = \widehat{p}(y), \tag{5.19}$$

uniformly in y in the interval $|y| \le n^{1/8}$. In addition, (3.10) holds uniformly in x in the set $\{x : n^{-3/8} \le |x| \le 1\}$. □

As a corollary of Theorems 3.1 and 5.2, we will prove the following result.

Theorem 5.3. Under the assumptions of Theorem 5.2, the average number of zeros of random polynomial (1.1) is asymptotically equal to $\sqrt{n}$ so that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} p_n(x)dx = 1. \tag{5.20}$$

□

6 Proof of Theorems 5.2 and 5.3

By (2.21),

$$\frac{p_n\left(\frac{y}{\sqrt{n}}\right)}{\sqrt{n}} = \frac{1}{1+\frac{y^2}{n}} \int_{-\infty}^{\infty} |\eta| D_n(0,\eta;y)d\eta, \tag{6.1}$$

where $D_n(\xi,\eta;y)$ is the joint distribution density of $g_n(y/\sqrt{n})$ and $h_n(y/\sqrt{n})$. Therefore, Theorem 5.2 will be proven if we prove that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |\eta| D_n(0,\eta;y)d\eta = \int_{-\infty}^{\infty} |\eta| D(0,\eta;y)d\eta. \tag{6.2}$$

Let $\Phi_n(\gamma;y)$, where $\gamma = (\alpha,\beta)$, be the joint characteristic function of $g_n(y/\sqrt{n})$ and $h_n(y/\sqrt{n})$. Then, by (5.2),

$$\Phi_n(\gamma;y) = \prod_{k=0}^{n} \varphi\big(\omega_{kn}(y)\big), \tag{6.3}$$

where φ is the characteristic function of c_k and

$$\omega_{kn}(y) = \mu_k\left(\frac{y}{\sqrt{n}}\right)\alpha + \lambda_k\left(\frac{y}{\sqrt{n}}\right)\beta. \tag{6.4}$$

Let $\Phi(\gamma;y)$ be the joint characteristic function of $g(y/\sqrt{n})$ and $h(y/\sqrt{n})$. Then, by (5.16),

$$\Phi(\gamma;y) = \prod_{k=0}^{\infty} \varphi\big(w_k(y)\big), \tag{6.5}$$

where

$$w_k(y) = m_k(y)\alpha + l_k(y)\beta. \tag{6.6}$$

Observe that

$$\sum_{k=0}^{n} \omega_{kn}(y)^2 = \sum_{k=0}^{\infty} w_k(y)^2 = |\gamma|^2. \tag{6.7}$$

Lemma 6.1. There exists $C > 0$ such that for all y in the interval $|y| \le n^{1/8}$,

$$\sum_{k=0}^{\infty} \left|\omega_{kn}(y) - w_k(y)\right| \le C\frac{|\gamma|}{n}, \tag{6.8}$$

where we set $\omega_{kn}(y) = 0$ for $k > n$. □

Proof. By Lemma 5.1, if $k \le n^{1/4}$, then for some $C_0 > 0$,

$$\left|\omega_{kn}(y) - w_k(y)\right| \le \begin{cases} C_0\dfrac{|y|+|y|^5}{n}|y|^k e^{-y^2/2}|\gamma|, & k = 0, 1, \\ C_0\dfrac{k^3+y^6}{n\sqrt{k!}}|y|^{k-1} e^{-y^2/2}|\gamma|, & k \ge 2, \end{cases} \tag{6.9}$$

and for $k > n^{1/4}$,

$$\left|\omega_{kn}(y) - w_k(y)\right| \le \frac{|y|^k}{\sqrt{k!}} e^{-y^2/2}\left[k\left(\frac{1}{|y|} + \frac{|y|}{n}\right) + 1\right]|\gamma|. \tag{6.10}$$

By summing up these inequalities over $k = 0, 1, \ldots,$ we obtain (6.8). ■

Lemma 6.2. If $|y| \le n^{1/8}$, then for some $A_0 > 0$,

$$\left|\frac{\partial^j \Phi_n(\gamma; y)}{\partial \beta^j}\right| \le \frac{A_0}{(1+a|\gamma|)^{6-j}}, \quad n = 1, 2, \ldots;\ j = 0, 1, 2, 3. \tag{6.11}$$

□

Proof. Consider $j = 0$. From (6.3),

$$\begin{aligned} \left|\Phi_n(\gamma; y)\right| &\le \prod_{k=0}^{n} \left|\varphi\left(\omega_{kn}(y)\right)\right| \\ &\le \prod_{k=0}^{n} \frac{1}{\left(1 + a\left|\omega_{kn}(y)\right|^2\right)^3} \\ &\le \frac{1}{\left(1 + a\sum_{k=0}^{n}\left|\omega_{kn}(y)\right|^2\right)^3} \\ &\le \frac{1}{(1+a|\gamma|^2)^3}, \end{aligned} \tag{6.12}$$

which implies (6.11) for $j = 0$. Let now $j = 1$. Since $\varphi(0) = 0$, we obtain from (5.18) that

$$\left|\frac{d\varphi(s)}{ds}\right| \le \frac{A|s|}{\left(1 + a|s|\right)^6}. \tag{6.13}$$

From (6.3),

$$\frac{\partial \Phi_n(\gamma)}{\partial \beta} = \sum_{k=0}^{n} \lambda_k \varphi'(\omega_{kn}) \prod_{l\neq k} \varphi(\omega_{kn}), \tag{6.14}$$

hence there exists $C > 0$ such that

$$\begin{aligned}\left|\frac{\partial \Phi_n(\gamma)}{\partial \beta}\right| &\le C \sum_{k=0}^{n} |\lambda_k||\omega_{kn}| \prod_{l=0}^{n} \frac{1}{\left(1 + a\left|\omega_{ln}(y)\right|^2\right)^3} \\ &\le \frac{C}{(1 + a|\gamma|^2)^3} \sum_{k=0}^{n} |\lambda_k||\omega_{kn}| \\ &\le \frac{C}{(1 + a|\gamma|^2)^3} \left(\sum_{k=0}^{n} |\lambda_k|^2\right)^{1/2} \left(\sum_{k=0}^{n} |\omega_{kn}|^2\right)^{1/2} \\ &= \frac{C|\gamma|}{(1 + a|\gamma|^2)^3},\end{aligned} \tag{6.15}$$

which implies (6.11) for $j = 1$. The cases $j = 2, 3$ are dealt similarly. ■

By the same argument, we prove similar estimates for $\Phi(\gamma; y)$.

Lemma 6.3. If $|y| \le n^{1/8}$, then for some $A_0 > 0$,

$$\left|\frac{\partial^j \Phi(\gamma; y)}{\partial \beta^j}\right| \le \frac{A_0}{\left(1 + a|\gamma|\right)^{6-j}}, \quad n = 1, 2, \ldots; \ j = 0, 1, 2, 3. \tag{6.16}$$

□

Now we estimate the difference of $\Phi_n(\gamma; y)$ and $\Phi(\gamma; y)$.

Lemma 6.4. There exist $C > 0$ such that for all $|y| \le n^{1/8}$,

$$\left|\Phi_n(\gamma; y) - \Phi(\gamma; y)\right| \le \frac{C|\gamma|}{n\left(1 + a|\gamma|\right)^6}. \tag{6.17}$$

□

Proof. Observe that

$$\begin{aligned}\Phi_n(\gamma;y)-\Phi(\gamma;y)&\\&=\prod_{k=0}^{\infty}\varphi\big(\omega_k(y)\big)-\prod_{k=0}^{\infty}\varphi\big(w_k(y)\big)\\&=\sum_{j=0}^{\infty}\left(\prod_{k=0}^{j-1}\varphi\big(\omega_k(y)\big)\right)\big(\varphi\big(\omega_j(y)\big)-\varphi\big(w_j(y)\big)\big)\left(\prod_{k=j+1}^{\infty}\varphi\big(w_k(y)\big)\right).\end{aligned}\tag{6.18}$$

Therefore, for some $C>0$,

$$\big|\Phi_n(\gamma;y)-\Phi(\gamma;y)\big|\le\frac{C}{\big(1+a|\gamma|\big)^6}\sum_{j=0}^{\infty}\big|\omega_{jn}(y)-w_j(y)\big|.\tag{6.19}$$

By Lemma 6.1, this implies that

$$\big|\Phi_n(\gamma;y)-\Phi(\gamma;y)\big|\le\frac{C_0|\gamma|}{n\big(1+a|\gamma|\big)^6}.\tag{6.20}$$

■

We apply Lemmas 6.2 and 6.3 to estimate the tail of $D_n(\xi,\eta;y)$ and $D(\xi,\eta;y)$.

Lemma 6.5. There exists $C>0$ such that for all $|y|\le n^{1/8}$,

$$\big|D_n(0,\eta;y)\big|,\big|D(0,\eta;y)\big|\le\frac{C}{\big(1+|\eta|\big)^3},\quad \eta\in\mathbb{R}.\tag{6.21}$$

□

From Lemma 6.4, we obtain the estimate of the difference of $D_n(\xi,\eta;y)$ and $D(\xi,\eta;y)$.

Lemma 6.6. There exists $C>0$ such that for all $|y|\le n^{1/8}$,

$$\sup_{\eta\in\mathbb{R}}\big|D_n(0,\eta;y)-D(0,\eta;y)\big|\le\frac{C}{n}.\tag{6.22}$$

□

Proof of Theorem 5.2. By (6.1) and (5.17),

$$\left|\left(1+\frac{y^2}{n}\right)\frac{p_n\left(\frac{y}{\sqrt{n}}\right)}{\sqrt{n}}-\widehat{p}(y)\right|\le\int_{-\infty}^{\infty}\big|D_n(0,\eta;y)-D(0,\eta;y)\big|dy.\tag{6.23}$$

Set $R=n^{1/3}$. Then by Lemma 6.6,

$$\int_{|\eta|\le R}\big|D_n(0,\eta;y)-D(0,\eta;y)\big|dy\le\frac{2CR}{n}=2Cn^{-2/3},\tag{6.24}$$

and by Lemma 6.5,

$$\int_{|\eta|\geq R} \left|D_n(0,\eta;y) - D(0,\eta;y)\right| dy \leq CR^{-2} = Cn^{-2/3}. \tag{6.25}$$

Thus, if $|y| \leq n^{1/8}$, then

$$\left| \left(1+\frac{y^2}{n}\right) \frac{p_n\left(\frac{y}{\sqrt{n}}\right)}{\sqrt{n}} - \widehat{p}(y) \right| \leq 3Cn^{-2/3}. \tag{6.26}$$

This proves (5.19).

To prove (3.10), observe that if $n^{-3/8} \leq |x| \leq 1$, then by (2.36), there exists $C > 0$ such that

$$\max_{0\leq k\leq n} \left|\mu_k(x)\right| \leq \frac{C}{n^{1/16}}, \qquad \max_{0\leq k\leq n} \left|\lambda_k(x)\right| \leq \frac{C}{n^{1/16}}; \tag{6.27}$$

hence

$$\lim_{n\to\infty} \max_{0\leq k\leq n} \left|\mu_k(x)\right| = \lim_{n\to\infty} \max_{0\leq k\leq n} \left|\lambda_k(x)\right| = 0. \tag{6.28}$$

Therefore, in this case, the proof of Theorem 3.1 is applicable and (3.10) follows. ■

Proof of Theorem 5.3. By comparing (5.19) with (3.10), we obtain that

$$\lim_{y\to\infty} \widehat{p}(y) = \frac{1}{\pi}; \tag{6.29}$$

hence by (5.19),

$$\begin{aligned} \lim_{n\to\infty} n^{3/8} \int_{|x|\leq n^{-3/8}} p_n(x)dx &= \lim_{n\to\infty} n^{3/8} \int_{|y|\leq n^{1/8}} p_n\left(\frac{y}{\sqrt{n}}\right) \frac{dy}{\sqrt{n}} \\ &= \lim_{n\to\infty} \frac{1}{n^{1/8}} \int_{|y|\leq n^{1/8}} \widehat{p}(y)dy \\ &= \frac{2}{\pi}. \end{aligned} \tag{6.30}$$

In addition, from (3.10), we obtain that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \int_{n^{-3/8}\leq|x|\leq 1} p_n(x)dx = \int_{|x|\leq 1} \frac{1}{\pi(1+x^2)} dx = \frac{1}{2}. \tag{6.31}$$

By combining these two relations, we obtain that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \int_{|x|\le 1} p_n(x)dx = \frac{1}{2}. \tag{6.32}$$

Observe that the probability distribution of zeros $p_n(x)dx$ is invariant with respect to the transformation $x \to x^{-1}$. Indeed, the probability distribution of the polynomial

$$x^n f_n(x^{-1}) = \sum_{k=0}^{n} \sqrt{\binom{n}{k}} c_{n-k} x^k \tag{6.33}$$

coincides with the one of $f_n(x)$ because c_k's are identically distributed. Hence the distribution of zeros of $x^n f_n(x^{-1})$ coincides with the one of $f_n(x)$ so that it is invariant with respect to the transformation $x \to x^{-1}$. Thus,

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \int_{|x|\ge 1} p_n(x)dx = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \int_{|x|\le 1} p_n(x)dx = \frac{1}{2}, \tag{6.34}$$

and (5.20) follows. ■

7 Existence and universality of limiting correlation functions

Let $f_n(x)$ be a random polynomial of degree n, as defined in (1.1), and let $(x_1, \ldots, x_m)$ be m distinct points. We will assume that all $x_i \neq 0$. To evaluate the m-point correlation function of zeros, we will use the following extension of the Kac-Rice formula (see [5, 8]):

$$K_{nm}(x_1, \ldots, x_m) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |\eta_1 \cdots \eta_m| D_{nm}(0, \eta; x_1, \ldots, x_m) d\eta_1 \cdots d\eta_m, \tag{7.1}$$

where $D_{nm}(\xi, \eta; x_1, \ldots, x_m)$ is the joint distribution density function of the random vectors $\xi = (f_n(x_1), \ldots, f_n(x_m))$ and $\eta = (f_n'(x_1), \ldots, f_n'(x_m))$. By a change of variables, formula (7.1) is first reduced to

$$\begin{aligned} &K_{nm}(x_1, \ldots, x_m) \\ &\quad = \prod_{i=1}^{m} \left[\frac{\zeta_n(x_i)}{\sigma_n(x_i)}\right] \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |\eta_1 \cdots \eta_m| \widetilde{D}_{nm}(0, \eta; x_1, \ldots, x_m) d\eta_1 \cdots d\eta_m, \end{aligned} \tag{7.2}$$

where $\widetilde{D}_{nm}(0, \eta; x_1, \ldots, x_m)$ is the joint distribution density function of the random vectors

$$\xi = (g_n(x_1), \ldots, g_n(x_m)), \qquad \eta = (\widetilde{g}_n(x_1), \ldots, \widetilde{g}_n(x_m)) \tag{7.3}$$

(cf. (2.9)), and then it is reduced to

$$K_{nm}(x_1,\dots,x_m) = \prod_{i=1}^{m}\left[\frac{\sqrt{n}}{(1+x_i^2)}\right]\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|\eta_1\cdots\eta_m|\widehat{D}_{nm}(0,\eta;x_1,\dots,x_m)\,d\eta_1\cdots d\eta_m, \tag{7.4}$$

where $\widehat{D}_{nm}(\xi,\eta;x_1,\dots,x_m)$ is the joint distribution density function of the random vectors

$$\xi = (g_n(x_1),\dots,g_n(x_m)), \qquad \eta = (h_n(x_1),\dots,h_n(x_m)) \tag{7.5}$$

(cf. (2.21)). We find now that

$$\mathbf{E}g_n(x_i) = \mathbf{E}\left[\sum_{k=0}^{n}\mu_k(x_i)c_k\right] = 0, \qquad \mathbf{E}h_n(x_i) = \mathbf{E}\left[\sum_{k=0}^{n}\lambda_k(x_i)c_k\right] = 0, \tag{7.6}$$

and from (2.7), that

$$\mathbf{E}g_n(x_i)g_n(x_j) = \sum_{k=0}^{n}\mu_k(x_i)\mu_k(x_j) = \left[\frac{1+x_ix_j}{(1+x_i^2)^{1/2}(1+x_j^2)^{1/2}}\right]^n \equiv \alpha_n(x_i,x_j). \tag{7.7}$$

Next, by (2.14) and (2.20),

$$\lambda_k(x) = \frac{\nu_k(x)-(\nu(x),\mu(x))\mu_k(x)}{\sigma(x)}, \quad (\nu(x),\mu(x)) = \frac{\sqrt{n}x}{\sqrt{1+nx^2}}, \ \sigma(x) = \frac{1}{\sqrt{1+nx^2}}, \tag{7.8}$$

and by (2.7),

$$\begin{aligned}\sum_{k=0}^{n}\mu_k(x_i)\nu_k(x_j) &= \frac{\sqrt{n}(1+x_j^2)x_i}{\sqrt{1+nx_j^2}(1+x_ix_j)}\alpha_n(x_i,x_j),\\ \sum_{k=0}^{n}\nu_k(x_i)\nu_k(x_j) &= \frac{(1+nx_ix_j)(1+x_i^2)(1+x_j^2)}{\sqrt{1+nx_i^2}\sqrt{1+nx_j^2}(1+x_ix_j)^2}\alpha_n(x_i,x_j).\end{aligned} \tag{7.9}$$

This gives that

$$\begin{aligned}\mathbf{E}g_n(x_i)h_n(x_j) &= \sum_{k=0}^{n}\mu_k(x_i)\lambda_k(x_j) = \sqrt{n}\frac{x_i-x_j}{1+x_ix_j}\alpha_n(x_i,x_j),\\ \mathbf{E}h_n(x_i)h_n(x_j) &= \sum_{k=0}^{n}\lambda_k(x_i)\lambda_k(x_j) = \frac{(1+x_i^2)(1+x_j^2)-n(x_i-x_j)^2}{(1+x_ix_j)^2}\alpha_n(x_i,x_j).\end{aligned} \tag{7.10}$$

We make the change of variable

$$\theta = \arctan x. \tag{7.11}$$

Then formulae (7.7) and (7.10) simplify to

$$\begin{aligned}
\mathbf{E} g_n(\theta_i) g_n(\theta_j) &= \cos^n(\theta_i - \theta_j), \\
\mathbf{E} g_n(\theta_i) h_n(\theta_j) &= \sqrt{n} \tan(\theta_i - \theta_j) \cos^n(\theta_i - \theta_j), \\
\mathbf{E} h_n(\theta_i) h_n(\theta_j) &= \left[\frac{1}{\cos^2(\theta_i - \theta_j)} - n \tan^2(\theta_i - \theta_j) \right] \cos^n(\theta_i - \theta_j).
\end{aligned} \tag{7.12}$$

To get a proper scaling, we fix a θ^0, the reference point, and set

$$\theta = \theta^0 + \frac{y}{\sqrt{n}}, \tag{7.13}$$

where y is a scaled variable. Then (7.12) reduces to

$$\begin{aligned}
\mathbf{E} g_n\left(\theta^0 + \frac{y_i}{\sqrt{n}}\right) g_n\left(\theta^0 + \frac{y_j}{\sqrt{n}}\right) &= \cos^n \frac{y_i - y_j}{\sqrt{n}} \equiv a_n(y_i, y_j), \\
\mathbf{E} g_n\left(\theta^0 + \frac{y_i}{\sqrt{n}}\right) h_n\left(\theta^0 + \frac{y_j}{\sqrt{n}}\right) &= \sqrt{n} \tan \frac{y_i - y_j}{\sqrt{n}} \cos^n \frac{y_i - y_j}{\sqrt{n}} \equiv b_n(y_i, y_j), \\
\mathbf{E} h_n\left(\theta^0 + \frac{y_i}{\sqrt{n}}\right) h_n\left(\theta^0 + \frac{y_j}{\sqrt{n}}\right) &= \left[\frac{1}{\cos^2 \frac{y_i - y_j}{\sqrt{n}}} - n \tan^2 \frac{y_i - y_j}{\sqrt{n}} \right] \cos^n \frac{y_i - y_j}{\sqrt{n}} \\
&\equiv c_n(y_i, y_j).
\end{aligned} \tag{7.14}$$

As $n \to \infty$,

$$\begin{aligned}
\lim_{n \to \infty} a_n(y_i, y_j) &= e^{-(y_i - y_j)^2/2} \equiv a(y_i, y_j), \\
\lim_{n \to \infty} b_n(y_i, y_j) &= (y_i - y_j) e^{-(y_i - y_j)^2/2} \equiv b(y_i, y_j), \\
\lim_{n \to \infty} c_n(y_i, y_j) &= \left[1 - (y_i - y_j)^2\right] e^{-(y_i - y_j)^2/2} \equiv c(y_i, y_j).
\end{aligned} \tag{7.15}$$

More precisely, it follows from (7.12) that as $n \to \infty$,

$$\begin{aligned}
a_n(y_i, y_j) &= a(y_i, y_j) + O(n^{-1}), \\
b_n(y_i, y_j) &= b(y_i, y_j) + O(n^{-1}), \\
c_n(y_i, y_j) &= c(y_i, y_j) + O(n^{-1}).
\end{aligned} \tag{7.16}$$

From (2.35), we obtain that

$$\mu_k(\tan\theta) = \frac{\tan^k\theta}{(1+\tan^2\theta)^{n/2}}\binom{n}{k}^{1/2} = \sin^k\theta\cos^{n-k}\theta\binom{n}{k}^{1/2},$$
$$\lambda_k(\tan\theta) = \mu_k(\tan\theta)\frac{\sqrt{n}(u-u_0)}{\sin\theta\cos\theta}; \tag{7.17}$$

hence formula (7.4) reduces, under the change of variable (7.11), to the following.

Proposition 7.1. The m-point correlation function of the scaled zeros $\tau_j \equiv \sqrt{n}(\arctan x_j - \theta^0)$ is given by the formula

$$K_{nm}(y_1,\ldots,y_m) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|\eta_1\cdots\eta_m|D_{nm}(0,\eta;y_1,\ldots,y_m)\,d\eta_1\cdots d\eta_m, \tag{7.18}$$

where $D_{nm}(\xi,\eta;y_1,\ldots,y_m)$ is the density of the joint distribution of the random vectors

$$\xi = (g_n(y_1),\ldots,g_n(y_m)), \qquad \eta = (h_n(y_1),\ldots,h_n(y_m)), \tag{7.19}$$

where

$$g_n(y) = \sum_{k=0}^{n}\mu_k(y)c_k, \qquad h_n(y) = \sum_{k=0}^{n}\lambda_k(y)c_k, \tag{7.20}$$

and

$$\mu_k(y) = \sin^k\left(\theta^0+\frac{y}{\sqrt{n}}\right)\cos^{n-k}\left(\theta^0+\frac{y}{\sqrt{n}}\right)\binom{n}{k}^{1/2},$$
$$\lambda_k(y) = \frac{\mu_k(y)\sqrt{n}(u-u_0)}{\sin\left(\theta^0+\frac{y}{\sqrt{n}}\right)\cos\left(\theta^0+\frac{y}{\sqrt{n}}\right)}, \quad u=\frac{k}{n},\ u_0=\sin^2\left(\theta^0+\frac{y}{\sqrt{n}}\right). \tag{7.21}$$

□

We can formulate now our main result. Consider a Gaussian vector random field $(g(y), h(y))$ on the line such that

$$\begin{aligned}
&\mathbf{E}g(y) = \mathbf{E}h(y) = 0,\\
&\mathbf{E}g(y_i)g(y_j) = e^{-(y_i-y_j)^2/2} \equiv a(y_i,y_j),\\
&\mathbf{E}g(y_i)h(y_j) = (y_i-y_j)e^{-(y_i-y_j)^2/2} \equiv b(y_i,y_j),\\
&\mathbf{E}h(y_i)h(y_j) = \left[1-(y_i-y_j)^2\right]e^{-(y_i-y_j)^2/2} \equiv c(y_i,y_j).
\end{aligned} \tag{7.22}$$

It is realized as

$$g(y) = e^{-y^2/2} \sum_{k=0}^{\infty} \frac{1}{\sqrt{k!}} c_k y^k, \qquad h(y) = g'(y), \tag{7.23}$$

where c_k are independent standard Gaussian random variables. Observe that the random series in (7.23) converges a.s., and it defines $g(y)$ as an entire function. Let $D_m(\xi, \eta; y_1, \ldots, y_m)$ be the (Gaussian) joint distribution density of the vectors

$$\xi = \big(g(y_1), \ldots, g(y_m)\big), \qquad \eta = \big(h(y_1), \ldots, h(y_m)\big). \tag{7.24}$$

Theorem 7.2. Assume that $\varphi(s)$, the characteristic function of c_k, is an infinitely differentiable function such that for some $a, q > 0$,

$$|\varphi(s)| \le \frac{1}{\left(1 + as^2\right)^q}, \tag{7.25}$$

and for any $j \ge 2$, there exists $c_j > 0$ such that

$$\left| \frac{d^j \varphi(s)}{ds^j} \right| < c_j. \tag{7.26}$$

Assume that the reference point $\theta^0 \neq 0$ and $y_i \neq y_j$ for $i \neq j$. Then for every $m \ge 1$, there exists the limit

$$\lim_{n \to \infty} K_{nm}(y_1, \ldots, y_m) = K_m(y_1, \ldots, y_m), \tag{7.27}$$

where

$$K_m(y_1, \ldots, y_m) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |\eta_1 \cdots \eta_m| D_m(0, \eta; y_1, \ldots, y_m)\, d\eta_1 \cdots d\eta_m. \tag{7.28}$$

□

Remarks 7.3. (1) We will prove, in fact, that for any $\delta > 0$ and any $\varepsilon > 0$, there exists $C > 0$ such that for any θ^0 and $y = (y_1, \ldots, y_m)$ such that $\delta < |\theta^0| < \pi/2 - \delta$ and $\delta < |y_i - y_j|$, $i \neq j$,

$$\left| K_{nm}(y_1, \ldots, y_m) - K_m(y_1, \ldots, y_m) \right| \le C n^{-1/4+\varepsilon}. \tag{7.29}$$

This estimates the rate of convergence of $K_{nm}(y_1, \ldots, y_m)$ to $K_m(y_1, \ldots, y_m)$.

(2) Conditions (7.25), (7.26) are fulfilled for any density $r(t)$ of the class $\mathcal{D}_\infty$.

We will prove Theorem 7.2 by showing that $D_{nm}(\xi, \eta; y)$ converges to $D_m(\xi, \eta; y)$, where $y = (y_1, \ldots, y_m)$, in an appropriate sense, so that we will be able to prove that the integral in (7.18) converges to the one in (7.28).

8 Proof of Theorem 7.2

Let

$$\Phi_n(\gamma) = \Phi_n(\gamma; y) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} D_{nm}(\xi, \eta; y) e^{i(\alpha,\xi)+i(\beta,\eta)} d\xi\, d\eta, \quad y = (y_1, \dots, y_m), \tag{8.1}$$

be the characteristic function of the random vector (ξ, η), where

$$\alpha = (\alpha_1, \dots, \alpha_m), \qquad \beta = (\beta_1, \dots, \beta_m), \qquad \gamma = (\alpha, \beta) = (\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m). \tag{8.2}$$

From (7.20), we have that

$$\Phi_n(\gamma) = \prod_{k=0}^{n} \varphi(\omega_k), \tag{8.3}$$

where φ is the characteristic function of c_k and

$$\omega_k = \sum_{i=1}^{m} \left[\mu_k(y_i)\alpha_i + \lambda_k(y_i)\beta_i\right]. \tag{8.4}$$

We will prove the following basic lemma.

Lemma 8.1. If $\varphi(s)$ satisfies (7.25), then for any $L > 0$, there exist $a_0 > 0$ and $N_0 > 0$ such that for all $n \geq N_0$,

$$\left|\Phi_n(\gamma)\right| \leq \frac{1}{\left(1 + a_0|\gamma|^2\right)^L}. \tag{8.5}$$

□

A proof of Lemma 8.1 is given in the appendix. We will prove the following addition to the basic lemma.

Lemma 8.2. If $\varphi(s)$ satisfies (7.25) and (7.26), then for any $L > 0$, there exist $a_0 > 0$ and $N_0 > 0$ such that for any multi-index $k = (k_1, \dots, k_{2m})$, there exists $C_k > 0$ such that for all $n \geq N_0$,

$$\left|D^k \Phi_n(\gamma)\right| \leq \frac{C_k}{\left(1 + a_0|\gamma|^2\right)^L}, \qquad D^k \equiv \frac{\partial^{k_1+\cdots+k_{2m}}}{\partial\gamma_1^{k_1} \cdots \partial\gamma_{2m}^{k_{2m}}}. \tag{8.6}$$

□

A proof of Lemma 8.2 is given in the appendix. Lemma 8.2 implies the following useful estimate.

Lemma 8.3. If $\varphi(s)$ satisfies (7.25) and (7.26), then for any $K > 0$, there exist $C > 0$ and $N_0 > 0$ such that for all $n \geq N_0$,

$$\left|D_{nm}(\xi, \eta; y)\right| \leq \frac{C}{\left(1 + |\xi|^2 + |\eta|^2\right)^K}. \tag{8.7}$$

□

Proof. Observe that by (8.1),

$$\left(1 + |\xi|^2 + |\eta|^2\right)^K D_{nm}(\xi, \eta; y) = \frac{1}{(2\pi)^{2m}} \int_{\mathbb{R}^{2m}} e^{-i(\alpha,\xi)-i(\beta,\eta)}(1-\Delta)^K \Phi_n(\gamma)\,d\gamma, \tag{8.8}$$

where Δ is the Laplacian; hence

$$\left(1 + |\xi|^2 + |\eta|^2\right)^K \left|D_{nm}(\xi, \eta; y)\right| \leq \int_{\mathbb{R}^{2m}} \left|(1-\Delta)^K \Phi_n(\gamma)\right| d\gamma. \tag{8.9}$$

From (8.6), we obtain that there exists a constant C_0 such that

$$\left|(1-\Delta)^K \Phi_n(\gamma)\right| \leq \frac{C}{\left(1 + a_0|\gamma|^2\right)^L}; \tag{8.10}$$

hence

$$\left(1 + |\xi|^2 + |\eta|^2\right)^K \left|D_{nm}(\xi, \eta; y)\right| \leq \int_{\mathbb{R}^{2m}} \frac{C}{\left(1 + a_0|\gamma|^2\right)^L} d\gamma \leq C, \tag{8.11}$$

if L is taken greater than m. This implies (8.7). Lemma 8.3 is proved. ■

Remark 8.4. The constants a_0 in (8.5) and (8.6), C_k in (8.6), and C in (8.7) are uniform with respect to θ^0 and y, assuming that $\delta < |\theta^0| < \pi/2 - \delta$ and $|y_i - y_j| > \delta$, $i \neq j$, for some $\delta > 0$.

Proof of Theorem 7.2. Let $\kappa > 0$ be a small fixed number:

$$\kappa < \frac{1}{8(2m+3)}. \tag{8.12}$$

Set

$$\Lambda_n = \left\{\gamma \mid |\gamma| \leq n^\kappa\right\}. \tag{8.13}$$

By (2.36), there exists a constant $C_0 > 0$ such that

$$\max_k \left|\mu_k(y_i)\right| \leq C_0 n^{-1/4}, \quad \max_k \left|\lambda_k(y_i)\right| \leq C_0 n^{-1/4}, \quad i = 1, \ldots, m. \tag{8.14}$$

Therefore, for each $\gamma \in \Lambda_n$, we have that

$$|\omega_k| \le \sum_{i=1}^{m} \left(|\mu_k(y_i)\alpha_i| + |\lambda_k(y_i)\beta_i|\right) \le C_1 n^{-1/4+\kappa}, \quad C_1 = 2mC_0; \tag{8.15}$$

hence

$$\sum_{k=0}^{n} |\omega_k^3| \le C_1 n^{-1/4+\kappa} \sum_{k=0}^{n} \omega_k^2. \tag{8.16}$$

From (8.4), we have that

$$\sum_{k=0}^{n} \omega_k^2 = \gamma \Delta_n \gamma^T, \tag{8.17}$$

where γ^T is the transpose vector of γ, and

$$\Delta_n = \begin{pmatrix} A_n & B_n \\ B_n^T & C_n \end{pmatrix}, \tag{8.18}$$

with

$$A_n = \left(a_n(y_i, y_j)\right)_{i,j=1}^{m}, \qquad B_n = \left(b_n(y_i, y_j)\right)_{i,j=1}^{m}, \qquad C_n = \left(c_n(y_i, y_j)\right)_{i,j=1}^{m}. \tag{8.19}$$

By (7.16), as $n \to \infty$,

$$\Delta_n = \Delta + O(n^{-1}), \tag{8.20}$$

where

$$\Delta = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}, \tag{8.21}$$

with

$$A = \left(a(y_i, y_j)\right)_{i,j=1}^{m}, \qquad B = \left(b(y_i, y_j)\right)_{i,j=1}^{m}, \qquad C = \left(c(y_i, y_j)\right)_{i,j=1}^{m}. \tag{8.22}$$

The matrix Δ is invertible [5] (cf. (7.23)). From (8.17) and (8.20), we have that if $\gamma \in \Lambda_n$, then

$$\sum_{k=0}^{n} \omega_k^2 = \gamma \Delta \gamma^T + O(n^{-1+2\kappa}). \tag{8.23}$$

We will prove the following lemma.

Lemma 8.5. There exist $C > 0$ and $N_0 > 0$ such that for all $n > N_0$,

$$\sup_{\gamma\in\Lambda_n} \left|\Phi_n(\gamma) - e^{-(1/2)\gamma\Delta\gamma^T}\right| \le Cn^{-1/4+\kappa_0}, \quad \kappa_0 = 3\kappa. \tag{8.24}$$

□

Proof. We will prove first that

$$\sup_{\gamma\in\Lambda_n} \left|\Phi_n(\gamma) - e^{-(1/2)\sum_{k=0}^n \omega_k^2}\right| \le Cn^{-1/4+\kappa_0}. \tag{8.25}$$

Then we will use (8.23). To prove (8.25), we write that

$$\begin{aligned}\Phi_n(\gamma) - e^{-(1/2)\sum_{k=0}^n \omega_k^2} &= \prod_{k=0}^n \varphi(\omega_k) - \prod_{k=0}^n e^{-(1/2)\omega_k^2} \\ &= \sum_{j=0}^n \left(\prod_{k=0}^{j-1} \varphi(\omega_k)\right)\left(\varphi(\omega_j) - e^{-(1/2)\omega_j^2}\right) \prod_{k=j+1}^n e^{-(1/2)\omega_k^2}.\end{aligned} \tag{8.26}$$

Due to (1.2), we have that as $s \to 0$, $\varphi(s) = 1 - (1/2)s^2 + O(|s|^3)$; hence there exists some constant $C_0 > 0$ such that

$$\left|\varphi(\omega_j) - e^{-(1/2)\omega_j^2}\right| \le C_0|\omega_j|^3, \quad \gamma \in \Lambda_n. \tag{8.27}$$

In addition, $|\varphi(s)| \le 1$; hence from (8.26), we obtain that

$$\left|\Phi_n(\gamma) - e^{-(1/2)\sum_{k=0}^n \omega_k^2}\right| \le C_0 \sum_{k=0}^n |\omega_k|^3. \tag{8.28}$$

From (8.16) and (8.17), we obtain now that there exists some constant C_1 such that

$$\left|\Phi_n(\gamma) - e^{-(1/2)\sum_{k=0}^n \omega_k^2}\right| \le C_1 n^{-1/4+\kappa_0}. \tag{8.29}$$

Finally, from (8.23), we have that

$$\left|e^{-(1/2)\sum_{k=0}^n \omega_k^2} - e^{-(1/2)\gamma\Delta\gamma^T}\right| \le C_2 n^{-1+2\kappa}. \tag{8.30}$$

Combining (8.29) with (8.30), we obtain (8.24). Lemma 8.5 is proved. ■

From Lemmas 8.1 and 8.5, we will derive the following lemma.

Lemma 8.6. There exist $C > 0$ and $N_0 > 0$ such that for all $n > N_0$,

$$\sup_{\xi,\eta} \left|D_{nm}(\xi,\eta;y) - D_m(\xi,\eta;y)\right| \le Cn^{-1/4+\kappa_1}, \quad \kappa_1 = (2m+3)\kappa. \tag{8.31}$$

□

Proof. From (8.1) and the definition of D_m, we have that

$$D_{nm}(\xi,\eta;y) - D_m(\xi,\eta;y) = \frac{1}{(2\pi)^{2m}} \int_{\mathbb{R}^{2m}} e^{-i(\xi,\alpha)-i(\eta,\beta)} \left(\Phi_n(\gamma) - e^{-(1/2)\gamma\Delta\gamma^T}\right) d\gamma. \tag{8.32}$$

From Lemma 8.5, we obtain that there exists $C_0 > 0$ such that

$$\left| \int_{\Lambda_n} e^{-i(\xi,\alpha)-i(\eta,\beta)} \left(\Phi_n(\gamma) - e^{-(1/2)\gamma\Delta\gamma^T}\right) d\gamma \right| \le C_0 n^{-1/4+\kappa_0} \operatorname{Vol}\Lambda_n = C_1 n^{-1/4+\kappa_1}. \tag{8.33}$$

By Lemma 8.1, there exists $C_0 > 0$ such that

$$\left| \int_{\mathbb{R}^{2m}\setminus\Lambda_n} e^{-i(\xi,\alpha)-i(\eta,\beta)} \Phi_n(\gamma) d\gamma \right| \le C_0 \int_{\mathbb{R}^{2m}\setminus\Lambda_n} \frac{1}{\left(1+a_0|\gamma|^2\right)^L} d\gamma \le C_1 n^{-1}, \tag{8.34}$$

if we choose $L = L(\kappa)$ sufficiently large. Since $e^{-(1/2)\gamma\Delta\gamma^T}$ is a nondegenerate Gaussian kernel, there exists, obviously, $C_0 > 0$ such that

$$\left| \int_{\mathbb{R}^{2m}\setminus\Lambda_n} e^{-i(\xi,\alpha)-i(\eta,\beta)} e^{-(1/2)\gamma\Delta\gamma^T} d\gamma \right| \le \int_{\mathbb{R}^{2m}\setminus\Lambda_n} e^{-(1/2)\gamma\Delta\gamma^T} d\gamma \le C_0 n^{-1}. \tag{8.35}$$

Combining estimates (8.33), (8.34), and (8.35) with equation (8.32), we obtain (8.31). Lemma 8.6 is proved. ∎

We will prove Theorem 7.2 from Lemmas 8.3 and 8.6. Let $\tau > 0$ be a fixed small number. By (7.18) and (7.28),

$$K_{nm}(y) - K_m(y) = \int_{\mathbb{R}^m} |\eta_1 \cdots \eta_m| \left[D_{nm}(0,\eta;y) - D_m(0,\eta;y)\right] d\eta, \quad y = (y_1,\ldots,y_m). \tag{8.36}$$

By Lemma 8.6,

$$\begin{aligned} &\int_{\{\eta:|\eta|\le n^\tau\}} |\eta_1\cdots\eta_m| \left|D_{nm}(0,\eta;y) - D_m(0,\eta;y)\right| d\eta \\ &\quad \le C n^{-1/4+\kappa_1} \int_{\{\eta:|\eta|\le n^\tau\}} |\eta_1\cdots\eta_m| d\eta \le C_1 n^{-1/4+\tau_1}, \quad \tau_1 = \kappa_1 + 2m\tau. \end{aligned} \tag{8.37}$$

By Lemma 8.3,

$$\begin{aligned} &\int_{\{\eta:|\eta|\ge n^\tau\}} |\eta_1\cdots\eta_m D_{nm}(0,\eta;y)| d\eta \\ &\quad \le C \int_{\{\eta:|\eta|\ge n^\tau\}} |\eta_1\cdots\eta_m| \frac{1}{\left(1+|\eta|^2\right)^K} d\eta \le C_1 n^{-1}, \end{aligned} \tag{8.38}$$

if we take K sufficiently large. Since D_m is a Gaussian kernel, there exists, obviously, $C_0 > 0$ such that

$$\int_{\{\eta:|\eta|\geq n^{\tau}\}} \left|\eta_1 \cdots \eta_m D_m(0,\eta;y)\right| d\eta \leq C_0 n^{-1}. \tag{8.39}$$

From (8.36), (8.37), (8.38), and (8.39), we obtain that there exists $C > 0$ such that

$$\left|K_{nm}(y) - K_m(y)\right| \leq C n^{-1/4+\tau_1}. \tag{8.40}$$

Since $\tau_1 = (2m+3)\kappa + 2m\tau$ can be made as small as we want, Theorem 7.2 is proved. ■

9 Conclusion

In the present work, we have proved that the correlation functions of real zeros of a random polynomial of form (1.1) have a universal scaling limit if we stay away from the origin. The method of the proof is based on the Kac-Rice type formula for the correlation functions and on the convergence of the Kac-Rice kernel to a universal Gaussian limit. The convergence to the Gaussian kernel is established as a local central limit theorem for not identically distributed multivariate random variables, with some appropriate additional estimates. The method of the proof is rather general and it can be extended to ensembles of multivariate random polynomials. We consider briefly these extensions.

Real zeros of non-Gaussian multivariate random polynomials. Let $x = (x_0, \ldots, x_d) \in \mathbb{R}^{d+1}$. Consider a random homogeneous multivariate polynomial in x,

$$f_n(x) = \sum_{|k|=n} \sqrt{\binom{n}{k}} c_k x^k, \tag{9.1}$$

where

$$\begin{aligned} &k = (k_0, \ldots, k_d) \in \mathbb{Z}_+^{d+1}, \qquad |k| = k_0 + \cdots + k_d, \\ &x^k = x_0^{k_0} \cdots x_d^{k_d}, \qquad \binom{n}{k} = \frac{n!}{k_0! \cdots k_d!}, \end{aligned} \tag{9.2}$$

$\mathbb{Z}_+ = \{n \in \mathbb{Z},\ n \geq 0\}$, and c_k are identically distributed real random variables. Assume that

$$\mathbf{E}c_k = 0, \qquad \mathbf{E}c_k^2 = 1. \tag{9.3}$$

In the case when $\{c_k\}$ are independent standard Gaussian random variables, we obtain the Gaussian SO(d + 1) ensemble. It is worth mentioning that the Gaussian SO(d + 1) ensemble was introduced and studied by Kostlan [18].

Consider p independent copies $f = (f_{n1}(x), \ldots, f_{np}(x))$ of polynomial (9.1), $p \le d$, and the set of common zeros of these polynomials

$$Z_f = \{x : f_{n1}(x) = \cdots = f_{np}(x) = 0\}. \tag{9.4}$$

Since $f_{nj}(x)$ are homogeneous polynomials, we can view Z_f as a real algebraic variety in the projective space $\mathbb{RP}^d$. Assume that the distribution of c_k is Lebesgue absolutely continuous with a smooth density. Let $K_{nm}(x^{(1)}, \ldots, x^{(m)})$ be the m-point correlation function of the common zeros (see [7, 8, 9, 10]). In the case when $\{c_k\}$ are independent standard Gaussian random variables, the correlation functions are $SO(d+1)$-invariant. In the Gaussian case with $p = d$, as was shown by Kostlan [18] and Shub and Smale [28] (see also [13, 14]), the mathematical expectation of the number of the common real zeros is exactly equal to the square root of the total number of common complex zeros.

Our approach enables us to prove the following multivariate extension of Theorem 7.2. Consider the random Gaussian multivariate analytic function of d variables

$$g(y) = e^{-|y|^2/2} \sum_{k \in \mathbb{Z}_+^d} \frac{1}{\sqrt{k!}} c_k y^k, \quad y = (y_1, \ldots, y_d), \tag{9.5}$$

where c_k are independent standard Gaussian random variables. Observe that the random series in (9.5) converges almost surely and it defines $g(y)$ as an entire function. Let $g_1(y), \ldots, g_p(y)$ be p independent copies of random function (9.5). Set

$$\vec{g}(y) = \begin{pmatrix} g_1(y) \\ \vdots \\ g_p(y) \end{pmatrix}, \qquad h(y) = \begin{pmatrix} \dfrac{\partial g_1(y)}{\partial y_1} & \cdots & \dfrac{\partial g_1(y)}{\partial y_d} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_p(y)}{\partial y_1} & \cdots & \dfrac{\partial g_p(y)}{\partial y_d} \end{pmatrix}, \tag{9.6}$$

$$\|h(y)\| = \left[\sum_{k,j} \left| \frac{\partial g_k(y)}{\partial y_j} \right|^2 \right]^{1/2}. \tag{9.7}$$

Let $y^{(1)}, \ldots, y^{(m)} \in \mathbb{R}^d$, and let $D_m(\xi, \eta; y^{(1)}, \ldots, y^{(m)})$ be the joint distribution density of the Gaussian random tensors

$$\xi = \left(\vec{g}(y^{(1)}), \ldots, \vec{g}(y^{(m)})\right), \qquad \eta = \left(h(y^{(1)}), \ldots, h(y^{(m)})\right). \tag{9.8}$$

Consider any point $\theta^0 \in \mathbb{RP}^d$ different from $P = (1, 0, \ldots, 0)$. Consider any coordinate system in a neighborhood of θ^0, with the origin at θ^0, such that the metric tensor on $\mathbb{RP}^d$ at θ^0 is an identity tensor. By $\theta^0 + y/\sqrt{n}$, where $y \in \mathbb{R}^d$, we understand (for large n) a point in $\mathbb{RP}^d$ with coordinates $(y_1/\sqrt{n}, \ldots, y_d/\sqrt{n})$ in the chosen coordinate system.

Theorem 9.1. Assume that $\varphi(s)$, the characteristic function of c_k, satisfies estimates (7.25), (7.26). Assume that the reference point $\theta^0 \neq P \equiv (1, 0, \dots, 0) \in \mathbb{RP}^d$ and $y^{(i)} \neq y^{(j)}$ for $i \neq j$. Then for every $m \geq 1$, there exists the limit

$$\lim_{n\to\infty} \frac{1}{n^{mp/2}} K_{nm}\left(\theta^0 + \frac{y^{(1)}}{\sqrt{n}}, \dots, \theta^0 + \frac{y^{(m)}}{\sqrt{n}}\right) = K_m\left(y^{(1)}, \dots, y^{(m)}\right), \tag{9.9}$$

where

$$K_m\left(y^{(1)}, \dots, y^{(m)}\right) = \int_{\mathbb{R}^{pd}} \cdots \int_{\mathbb{R}^{pd}} \|\eta^{(1)}\| \cdots \|\eta^{(m)}\| D_m\left(0, \eta; y^{(1)}, \dots, y^{(m)}\right) d\eta^{(1)} \cdots d\eta^{(m)}. \tag{9.10}$$

□

Complex zeros of complex non-Gaussian multivariate random polynomials. There is a complex counterpart of Theorem 9.1. Consider multivariate polynomial (9.1) with complex coefficients c_k, where $\{c_k\}$ are independent identically distributed complex random variables such that

$$\mathbf{E}c_k = 0, \qquad \mathbf{E}c_k^2 = 0, \qquad \mathbf{E}|c_k|^2 = 1. \tag{9.11}$$

Let $K_{nm}(z^{(1)}, \dots, z^{(m)})$ be the m-point correlation function of complex zeros of $f_n(z)$ in $\mathbb{CP}^d$ (see [7, 8, 9, 10]) (for $d = 1$ case, see also earlier works [11, 12, 16]). Let $D_m(\xi, \eta; y^{(1)}, \dots, y^{(m)})$ be the joint distribution density of the complex Gaussian random tensors ξ and η defined in (9.6) and (9.8). By using our approach, we are able to prove the following theorem. We assume that the probability distribution of c_k is absolutely continuous with respect to the Lebesgue measure on the plane, and its characteristic function $\varphi(s)$ is infinitely differentiable. As above, consider any point $\theta^0 \in \mathbb{CP}^d$, different from $P = (1, 0, \dots, 0)$, as a reference point. Consider also any coordinate system in a neighborhood of θ^0, with the origin at θ^0, such that the Fubini-Study metric tensor on $\mathbb{CP}^d$ at θ^0 is an identity tensor. By $\theta^0 + y/\sqrt{n}$, where $y \in \mathbb{C}^d$, we denote (for large n) a point in $\mathbb{CP}^d$ with coordinates $(y_1/\sqrt{n}, \dots, y_d/\sqrt{n})$ in the chosen coordinate system.

Theorem 9.2. Assume that $\varphi(s)$, the characteristic function of c_k, satisfies the estimate

$$|\varphi(s)| \leq \frac{1}{(1 + as^2)^q}, \tag{9.12}$$

for some $a, q > 0$; and for any $j = (j_1, j_2)$ with $j_1 + j_2 \geq 2$, there exists $C_j > 0$ such that

$$|D^j \varphi(s)| < C_j, \quad D^j = \frac{\partial^{j_1+j_2}}{\partial s_1^{j_1} \partial s_2^{j_2}}. \tag{9.13}$$

Assume that the reference point $\theta^0 \neq P \equiv (1, 0, \dots, 0) \in \mathbb{CP}^d$ and $y^{(i)} \neq y^{(j)}$ for $i \neq j$. Then for every $m \geq 1$, there exists the limit

$$\lim_{n\to\infty} \frac{1}{n^{mp}} K_{nm}\left(\theta^0 + \frac{y^{(1)}}{\sqrt{n}}, \dots, \theta^0 + \frac{y^{(m)}}{\sqrt{n}}\right) = K_m\left(y^{(1)}, \dots, y^{(m)}\right), \tag{9.14}$$

where

$$\begin{aligned} &K_m\left(y^{(1)}, \dots, y^{(m)}\right) \\ &\quad = \int_{\mathbb{R}^{pd}} \cdots \int_{\mathbb{R}^{pd}} \left\|\eta^{(1)}\right\|^2 \cdots \left\|\eta^{(m)}\right\|^2 D_m\left(0, \eta; y^{(1)}, \dots, y^{(m)}\right) d\eta^{(1)} \cdots d\eta^{(m)}. \end{aligned} \tag{9.15}$$

□

Theorems 9.1 and 9.2 can be further extended to non-Gaussian random sections of powers of line bundles over compact manifolds (cf. [7, 8, 9, 10]), but we will not consider these extensions here.

Appendix

Proof of Lemmas 8.1 and 8.2

Proof of Lemma 8.1. From (8.3) and (7.25), we have that

$$\left|\Phi_n(\gamma)\right| \leq \prod_{k=0}^{n} \frac{1}{\left(1 + a\omega_k^2\right)^q}. \tag{A.1}$$

This implies that

$$\left|\Phi_n(\gamma)\right| \leq \frac{1}{\left(1 + a\sum_{k=0}^{n} \omega_k^2\right)^q}. \tag{A.2}$$

To prove (8.5), we show that

$$\sum_{k=0}^{n} \omega_k^2 \geq C|\gamma|^2. \tag{A.3}$$

To do it, observe that

$$\sum_{k=0}^{n} \omega_k^2 = \sum_{i,j=1}^{2m} d_n(i,j)\gamma_i\gamma_j, \tag{A.4}$$

where the matrix $D_n = (d_n(i,j))_{i,j=1}^{2m}$ is defined as

$$D_n = \begin{pmatrix} A_n & B_n \\ B_n^T & C_n \end{pmatrix}, \tag{A.5}$$
$$A_n = \left(a_n(s_i, s_j)\right)_{i,j=1}^m, \qquad B_n = \left(b_n(s_i, s_j)\right)_{i,j=1}^m, \qquad C_n = \left(c_n(s_i, s_j)\right)_{i,j=1}^m.$$

By (7.15),

$$\lim_{n\to\infty} D_n = D = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}, \tag{A.6}$$
$$A = \left(a(s_i, s_j)\right)_{i,j=1}^m, \qquad B = \left(b(s_i, s_j)\right)_{i,j=1}^m, \qquad C = \left(c(s_i, s_j)\right)_{i,j=1}^m.$$

The matrix D is positive definite (see [5]); hence

$$\sum_{i,j=1}^{2m} d(i,j)\gamma_i\gamma_j \geq C|\gamma|^2. \tag{A.7}$$

Due to (A.6), D_n is also positive definite for large n, uniformly in n. This proves (A.3).

Partition now all k's into T groups M_j so that for each group,

$$\sum_{k\in M_j} \omega_k^2 \geq \frac{C}{2T}|\gamma|^2. \tag{A.8}$$

Then

$$\left|\Phi_n(\gamma)\right| \leq \prod_{j=1}^{T} \prod_{k\in M_j} \frac{1}{\left(1 + a\omega_k^2\right)^q} \leq \frac{1}{\left(1 + a_0|\gamma|^2\right)^{Tq}}, \quad a_0 = \frac{C}{2T}a. \tag{A.9}$$

Lemma 8.1 is proved. ■

Proof of Lemma 8.2. From (7.26), we obtain that

$$\left|\frac{d\varphi(s)}{ds}\right| < c_1|s|, \tag{A.10}$$

where $c_1 = c_2$. Consider first $k = (1, 0, \ldots, 0)$. From (8.3), we have that

$$\left|\frac{\partial\Phi_n(\gamma)}{\partial\alpha_1}\right| = \left|\sum_{k=0}^{n} \mu_k(s_1)\varphi'(\omega_k) \prod_{l\neq k} \varphi(\omega_l)\right|. \tag{A.11}$$

Observe that

$$\left|\prod_{l\neq k}\varphi(\omega_l)\right| \leq \prod_{l\neq k}\frac{1}{(1+a\omega_l^2)^q} \leq (1+|\gamma|^2)\prod_{l=0}^{n}\frac{1}{(1+a\omega_l^2)^q}; \tag{A.12}$$

hence by (A.10) and Cauchy,

$$\left|\frac{\partial\Phi_n(\gamma)}{\partial\alpha_1}\right| \leq c_1\left(\sum_{k=0}^{n}\mu_k^2(s_1)\right)^{1/2}\left(\sum_{k=0}^{n}\omega_k^2\right)^{1/2}(1+|\gamma|^2)\prod_{l=0}^{n}\frac{1}{(1+a\omega_l^2)^q}. \tag{A.13}$$

From (A.4),

$$\left(\sum_{k=0}^{n}\omega_k^2\right)^{1/2} \leq C|\gamma|; \tag{A.14}$$

hence

$$\left|\frac{\partial\Phi_n(\gamma)}{\partial\alpha_1}\right| \leq C_0|\gamma|(1+|\gamma|^2)\prod_{l=0}^{n}\frac{1}{(1+a\omega_l^2)^q}. \tag{A.15}$$

From (A.9), we obtain now that

$$\left|\frac{\partial\Phi_n(\gamma)}{\partial\alpha_1}\right| \leq C_0|\gamma|(1+|\gamma|^2)\frac{1}{(1+a_0|\gamma|^2)^{Tq}}, \tag{A.16}$$

which implies that

$$\left|\frac{\partial\Phi_n(\gamma)}{\partial\alpha_1}\right| \leq \frac{C_1}{(1+a_0|\gamma|^2)^L} \tag{A.17}$$

if we take T sufficiently large. This proves estimate (8.6) for $k=(1,0,\ldots,0)$. A similar argument proves it for any k with $|k|\equiv k_1+\cdots+k_{2m}=1$. If $|k|\geq 2$, then formula (A.11) involves higher-order derivatives of φ and different s_j's. If the derivative of φ is of the second order or higher, we get $\mu_k(s_j)$ or $\lambda_k(s_j)$ at least second power. If the derivative of φ is of the first order, we get $\mu_k(s_j)\varphi'(\omega_k)$ or $\lambda_k(s_j)\varphi'(\omega_k)$. In both cases, we use (A.10), (7.26), and (A.14) to prove that

$$\left|\frac{\partial^{|k|}\Phi_n(\gamma)}{\partial\gamma^k}\right| \leq C_0|\gamma|^{|k|}(1+|\gamma|^2)^{|k|}\frac{1}{(1+a_0|\gamma|^2)^{Tq}}, \tag{A.18}$$

which implies that

$$\left|\frac{\partial^{|k|}\Phi_n(\gamma)}{\partial\gamma^k}\right| \le \frac{C_k}{\left(1+a_0|\gamma|^2\right)^L} \tag{A.19}$$

if we take T sufficiently large. This proves estimate (8.6) for any multi-index k. Lemma 8.2 is proved. ■

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