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Random Matrices with External Source and Multiple Orthogonal Polynomials

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1 Random matrices with external source

Following Brézin and Hikami [5, 6, 7, 8] and Zinn-Justin [20, 21], we consider a random matrix ensemble with an external source

$$\frac{1}{Z_n} e^{-\operatorname{Tr}(V(M)-AM)} dM \tag{1.1}$$

defined on $n \times n$ Hermitian matrices M. The ensemble (1.1) consists of a general unitary invariant part $V(M)$ and an extra term AM, where A is a fixed $n \times n$ Hermitian matrix, the external source or the external field. Due to the external source, the ensemble (1.1) is not unitary invariant. For the special Gaussian case $V(x) = (1/2)x^2$, we can write M in (1.1) as $M = H + A$, where H is a random matrix from the GUE ensemble and A is deterministic, hence in this case it reduces to the class of deterministic plus random matrices studied in [5, 6, 7, 8, 9, 10, 18].

Zinn-Justin [20] showed that the eigenvalue correlations of ensemble (1.1) can be expressed in the determinantal form

$$R_m(\lambda_1, \ldots, \lambda_m) = \det \left(K_n(\lambda_i, \lambda_j)\right)_{i,j=1,\ldots,m} \tag{1.2}$$

for some kernel K_n. In this paper, we give different expressions for K_n. We believe that our formulation is useful for asymptotic analysis. Indeed, for the Gaussian case $V(x) = (1/2)x^2$ and for the case where A has only two distinct eigenvalues, we have been

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able to carry out the asymptotic analysis almost completely. This will be reported elsewhere.

Our approach is based on the observation that the average characteristic polynomial

$$P_n(z) = \mathbb{E}\big[\det(z - M)\big] \tag{1.3}$$

of ensemble (1.1) can be characterized by the property that

$$\int_{-\infty}^{\infty} P_n(x) x^k e^{-(V(x)-a_j x)} dx = 0 \tag{1.4}$$

for every eigenvalue a_j of A and for $k = 0, \ldots, n_j - 1$, where n_j is the multiplicity of a_j, see Section 2. We can embed the polynomial P_n in a sequence of polynomials $\{P_k\}_0^n$, where P_k has degree k. Then our kernel has the form

$$K_n(x, y) = e^{-(1/2)(V(x)+V(y))} \sum_{k=0}^{n-1} P_k(x) Q_k(y), \tag{1.5}$$

where the Q_k are certain dual functions (not polynomials in general), see Section 3.

When $A = 0$ (no external source), the polynomials P_k are usual monic orthogonal polynomials with respect to the weight $e^{-V(x)}$ on $\mathbb{R}$. In that case, the function Q_k is a multiple of P_k and the kernel (1.5) reduces to the orthogonal polynomial kernel which is familiar in the theory of random matrices. By the Christoffel-Darboux formula, we then have

$$K_n(x, y) = e^{-(1/2)(V(x)+V(y))} \gamma_{n-1}^2 \frac{P_n(x)P_{n-1}(y) - P_{n-1}(x)P_n(y)}{x - y}, \tag{1.6}$$

where γ_k is the leading coefficient of the orthonormal polynomial of degree k.

In Section 4, we present an analog of the Christoffel-Darboux formula for the kernel (1.5) in the case where A has only two eigenvalues. We also relate it to a Riemann-Hilbert problem in Section 5.

2 The average characteristic polynomial

We define the monic polynomial

$$P_n(z) = \mathbb{E}\big[\det(z - M)\big], \tag{2.1}$$

where the expectation is with respect to ensemble (1.1).

Proposition 2.1. Suppose A has eigenvalues a_j, $j = 1, \dots, n$, with $a_i \neq a_j$ if $i \neq j$. Then the following hold.

(a) There is a constant $\tilde{Z}_n$ such that

$$P_n(z) = \frac{1}{\tilde{Z}_n} \int \prod_{j=1}^{n} (z - \lambda_j) \prod_{j=1}^{n} e^{-(V(\lambda_j) - a_j \lambda_j)} \Delta(\lambda) d\lambda, \tag{2.2}$$

where

$$\Delta(\lambda) = \prod_{i>j} (\lambda_i - \lambda_j) \tag{2.3}$$

and $d\lambda = d\lambda_1 d\lambda_2 \cdots d\lambda_n$.

(b) Let

$$m_{jk} = \int_{-\infty}^{\infty} x^k e^{-(V(x) - a_j x)} dx. \tag{2.4}$$

Then the following determinantal formula holds:

$$P_n(z) = \frac{1}{\tilde{Z}_n} \begin{vmatrix} m_{10} & m_{11} & \cdots & m_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n0} & m_{n1} & \cdots & m_{nn} \\ 1 & z & \cdots & z^n \end{vmatrix}. \tag{2.5}$$

(c) For $j = 1, \dots, n$,

$$\int_{-\infty}^{\infty} P_n(x) e^{-(V(x) - a_j x)} dx = 0, \tag{2.6}$$

and these equations uniquely determine the monic polynomial P_n. □

Proof. Write $M = U \Lambda U^*$, where U is unitary and Λ is diagonal, $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Then by the Weyl integration formula (see, e.g., [11, 17]) we have for every integrable function f on the space of Hermitian $n \times n$ matrices,

$$\int f(M) dM = \pi^{-n(n-1)/2} \left(\prod_{j=0}^{n} j! \right) \iint f(U \Lambda U^*) \Delta(\lambda)^2 d\lambda \, dU, \tag{2.7}$$

where dU denotes the normalized Haar measure on the unitary group $U(n)$. Thus,

$$P_n(z) = \frac{\left(\prod_{j=0}^{n} j! \right)}{Z_n \pi^{n(n-1)/2}} \int \prod_{j=1}^{n} (z - \lambda_j) \prod_{j=1}^{n} e^{-V(\lambda_j)} \left(\int e^{A U \Lambda U^*} dU \right) \Delta(\lambda)^2 d\lambda. \tag{2.8}$$

Because of the Harish-Chandra/Itzykson-Zuber integral [15, 16]

$$\int e^{AU\Lambda U^*}\, dU = \left(\prod_{j=0}^{n-1} j!\right) \frac{\det\left(e^{a_j\lambda_k}\right)}{\Delta(a)\Delta(\lambda)}, \tag{2.9}$$

we obtain that

$$P_n(z) = \frac{n!\left(\prod_{j=0}^{n-1} j!\right)^2}{Z_n \pi^{n(n-1)/2}} \int \prod_{j=1}^{n} (z-\lambda_j) \prod_{j=1}^{n} e^{-V(\lambda_j)} \frac{\det\left(e^{a_j\lambda_k}\right)}{\Delta(a)} \Delta(\lambda)\, d\lambda. \tag{2.10}$$

We expand the determinant

$$\det\left(e^{a_j\lambda_k}\right) = \sum_{\sigma\in S_n} (-1)^\sigma \prod_{j=1}^{n} e^{a_j \lambda_{\sigma(j)}}, \tag{2.11}$$

where S_n is the symmetric group. Hence,

$$P_n(z) = \frac{n!\left(\prod_{j=0}^{n-1} j!\right)^2}{Z_n \pi^{n(n-1)/2}\Delta(a)} \sum_{\sigma\in S_n} (-1)^\sigma \int \prod_{j=1}^{n} (z-\lambda_j) \prod_{j=1}^{n} e^{-V(\lambda_j)} \prod_{j=1}^{n} e^{a_j\lambda_{\sigma(j)}} \Delta(\lambda)\, d\lambda. \tag{2.12}$$

We make a change of variables $\lambda_j' = \lambda_{\sigma(j)}$. Then $(-1)^\sigma \Delta(\lambda) = \Delta(\lambda')$, hence, in the sum over S_n, we have $n!$ equal terms and, by dropping the prime, we obtain (2.2) with constant

$$\tilde{Z}_n = Z_n \frac{\pi^{n(n-1)/2}\Delta(a)}{\left(\prod_{j=0}^{n} j!\right)^2}. \tag{2.13}$$

This proves part (a).

Observe that

$$\prod_{j=1}^{n} (z-\lambda_j)\Delta(\lambda) = \Delta(\lambda, z) = \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^n \\ 1 & z & \cdots & z^n \end{vmatrix} \tag{2.14}$$

so that

$$\prod_{j=1}^{n}(z-\lambda_j)\prod_{j=1}^{n}e^{-(V(\lambda_j)-a_j\lambda_j)}\Delta(\lambda) = \begin{vmatrix} e^{-(V(\lambda_1)-a_1\lambda_1)} & \lambda_1 e^{-(V(\lambda_1)-a_1\lambda_1)} & \cdots & \lambda_1^n e^{-(V(\lambda_1)-a_1\lambda_1)} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-(V(\lambda_n)-a_n\lambda_n)} & \lambda_n e^{-(V(\lambda_n)-a_n\lambda_n)} & \cdots & \lambda_n^n e^{-(V(\lambda_n)-a_n\lambda_n)} \\ 1 & z & \cdots & z^n \end{vmatrix}. \tag{2.15}$$

Then (2.5) follows immediately from this and (2.2). This proves part (b).

From (2.5), it follows that

$$\int_{-\infty}^{\infty} P_n(x)e^{-(V(x)-a_jx)}dx = \frac{1}{\tilde{Z}_n}\begin{vmatrix} m_{10} & m_{11} & \cdots & m_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n0} & m_{n1} & \cdots & m_{nn} \\ m_{j0} & m_{j1} & \cdots & m_{jn} \end{vmatrix} = 0, \tag{2.16}$$

for every $j = 1,\dots,n$. This proves (2.6). To prove the uniqueness of P_n satisfying (2.6), observe that by equating the coefficients of x^n in (2.5), we obtain that

$$\begin{vmatrix} m_{10} & m_{11} & \cdots & m_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n0} & m_{n1} & \cdots & m_{n,n-1} \end{vmatrix} = \tilde{Z}_n \neq 0. \tag{2.17}$$

Let $P_n(x) = x^n + p_{n-1}x^{n-1} + \cdots + p_0$ and set $p = (p_0 \cdots p_{n-1})^T$. Then equations (2.6) are written in terms of the vector p as

$$Mp = -m, \quad M = (m_{jk})_{j=1,\dots,n;\,k=0,\dots,n-1}, \; m = (m_{jn})_{j=1,\dots,n}. \tag{2.18}$$

By (2.17), $\det M \neq 0$, hence p, and therefore P_n, is unique. ■

Proposition 2.1 can be extended to the case of multiple a_j's as follows.

Proposition 2.2. Suppose A has distinct eigenvalues a_i, $i = 1,\dots,p$, with respective multiplicities n_i so that $n_1 + \cdots + n_p = n$. Let $n^{(i)} = n_1 + \cdots + n_i$ and $n^{(0)} = 0$. Define

$$w_j(x) = x^{d_j-1}e^{-(V(x)-a_ix)}, \quad j = 1,\dots,n, \tag{2.19}$$

where $i = i_j$ is such that $n^{(i-1)} < j \leq n^{(i)}$ and $d_j = j - n^{(i-1)}$. Then the following hold.

(a) There is a constant $\tilde{Z}_n > 0$ such that

$$P_n(z) = \frac{1}{\tilde{Z}_n} \int \prod_{j=1}^{n} (z - \lambda_j) \prod_{j=1}^{n} w_j(\lambda_j) \Delta(\lambda) d\lambda. \tag{2.20}$$

(b) Let

$$m_{jk} = \int_{-\infty}^{\infty} x^k w_j(x) dx. \tag{2.21}$$

Then we have the determinantal formula

$$P_n(z) = \frac{1}{\tilde{Z}_n} \begin{vmatrix} m_{10} & m_{11} & \cdots & m_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n0} & m_{n1} & \cdots & m_{nn} \\ 1 & z & \cdots & z^n \end{vmatrix}. \tag{2.22}$$

(c) For $i = 1, \ldots, p$,

$$\int_{-\infty}^{\infty} P_n(x) x^j e^{-(V(x) - a_i x)} dx = 0, \quad j = 0, \ldots, n_i - 1, \tag{2.23}$$

and these equations uniquely determine the monic polynomial P_n. □

Proof. We write

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) = (\underbrace{a_1, \ldots, a_1}_{n_1 \text{ times}}, a_2, \ldots, a_{p-1}, \underbrace{a_p, \ldots, a_p}_{n_p \text{ times}}). \tag{2.24}$$

Apply formula (2.10) in the case when all $a_j = \tilde{a}_j$ are different and take a limit to the multiple a_j's. In this limit, we have that

$$\lim \frac{\det\left(e^{\tilde{a}_j \lambda_k}\right)}{\Delta(\tilde{a})} = \frac{\det\left(\lambda_k^{d_j - 1} e^{\alpha_j \lambda_k}\right)}{\Delta_0(a) \prod_{i=1}^{p} \prod_{k=1}^{n_i - 1} k!}, \tag{2.25}$$

where d_j is as in the statement of the proposition, and

$$\Delta_0(a) = \prod_{i>j} (a_i - a_j)^{n_i n_j}. \tag{2.26}$$

Thus, formula (2.10) becomes

$$P_n(z) = \frac{n!\left(\prod_{j=0}^{n-1} j!\right)^2}{Z_n \pi^{n(n-1)/2}} \int \prod_{j=1}^{n} (z-\lambda_j) \prod_{j=1}^{n} e^{-V(\lambda_j)} \frac{\det\left(\lambda_k^{d_j-1} e^{\alpha_j \lambda_k}\right)}{\Delta_0(a) \prod_{i=1}^{p} \prod_{k=1}^{n_i-1} k!} \Delta(\lambda)\, d\lambda. \tag{2.27}$$

Then we continue as in the proof of Proposition 2.1, that is, we write

$$\det\left(\lambda_k^{d_j-1} e^{\alpha_j \lambda_k}\right) = \sum_{\sigma \in S_n} (-1)^\sigma \prod_{j=1}^{n} \lambda_{\sigma(j)}^{d_j-1} e^{\alpha_j \lambda_{\sigma(j)}}, \tag{2.28}$$

and insert this into (2.27) to obtain a sum of $n!$ equal terms, which leads to (2.20) with

$$\tilde{Z}_n = Z_n \frac{\pi^{n(n-1)/2} \Delta_0(a) \prod_{i=1}^{p} \prod_{k=1}^{n_i-1} k!}{\left(\prod_{j=0}^{n} j!\right)^2}. \tag{2.29}$$

This proves part (a).

Parts (b) and (c) follow from (2.20) in the same way as Proposition 2.1(b) and (c) followed from (2.2). Note that in particular we have $\tilde{Z}_n$ as in (2.17),

$$\begin{vmatrix} m_{10} & m_{11} & \cdots & m_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n0} & m_{n1} & \cdots & m_{n,n-1} \end{vmatrix} = \tilde{Z}_n \neq 0. \tag{2.30}$$

■

Remark 2.3. Formula (2.20) can also be written in the following form:

$$P_n(z) = \frac{1}{\widehat{Z}_n} \int \prod_{j=1}^{n} (z-\lambda_j) \prod_{j=1}^{n} e^{-(V(\lambda_j) - a_{i_j} \lambda_j)} \prod_{i=1}^{p} \Delta(\lambda^{(i)}) \Delta(\lambda)\, d\lambda, \tag{2.31}$$

where $\lambda^{(i)} = (\lambda_{n^{(i-1)}+1}, \ldots, \lambda_{n^{(i)}})$ and

$$\widehat{Z}_n = \tilde{Z}_n n_1! \cdots n_p!. \tag{2.32}$$

When $A = 0$, (2.31) reduces to the usual formula for $P_n(z)$ with respect to the random matrix ensemble without external source.

Corollary 2.4. Under the same assumptions as in Proposition 2.2,

$$\int_{-\infty}^{\infty} P_n(x) x^{n_i} e^{-(V(x)-a_i x)} dx \neq 0 \tag{2.33}$$

for $i = 1, \ldots, p$. □

Proof. Let P_{n+1} be the average characteristic polynomial of an ensemble of $(n+1) \times (n+1)$ Hermitian random matrices whose external source has the same eigenvalues as A plus an additional eigenvalue a_i. Then by Proposition 2.2(c), we have that P_{n+1} is the unique monic polynomial that satisfies the relations (2.6) with n_i replaced by $n_i + 1$. If $\int_{-\infty}^{\infty} P_n(x) x^{n_i} e^{-(V(x)-a_i x)} dx$ would vanish, then $P_{n+1} + P_n$ would satisfy these relations as well, which would contradict the uniqueness of P_{n+1}. ■

Remark 2.5. The relations (2.6) can be viewed as multiple orthogonality conditions for the polynomial P_n. There are p weights $e^{-(V(x)-a_j x)}$, $j = 1, \ldots, p$, and for each weight there are a number of orthogonality conditions, so that the total number of them is n. This point of view is especially useful in case A has only a small number of distinct eigenvalues. We will come back to this in Section 5 when we are considering the case of two distinct eigenvalues in detail.

There is a considerable literature on multiple orthogonal polynomials (also called Hermite-Padé polynomials) (see, e.g., [1, 2, 19] and the references therein).

3 Determinantal form of joint probability density function

As in Proposition 2.2, we assume that A is a fixed Hermitian matrix whose eigenvalues $a_1, \ldots, a_p$ have respective multiplicities $n_1, \ldots, n_p$, so that $\sum_{i=1}^{p} n_i = n$. We let Σ_n be the collection of functions

$$\Sigma_n := \left\{ x^j e^{a_i x} \mid i = 1, \ldots, p,\ j = 0, \ldots, n_i - 1 \right\}. \tag{3.1}$$

We start with the following lemma.

Lemma 3.1. There exists a unique function Q_{n-1} in the linear span of Σ_n such that

$$\int_{-\infty}^{\infty} x^j Q_{n-1}(x) e^{-V(x)} dx = 0 \quad \text{for } j = 0, \ldots, n-2, \tag{3.2}$$

$$\int_{-\infty}^{\infty} x^{n-1} Q_{n-1}(x) e^{-V(x)} dx = 1. \tag{3.3}$$

□

Proof. Conditions (3.2) and (3.3) give us n linear equations for the n coefficients of Q_{n-1} with respect to the basis Σ_n with coefficient matrix

$$\begin{pmatrix} m_{10} & m_{20} & \cdots & m_{n0} \\ m_{11} & m_{21} & \cdots & m_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{1,n-1} & m_{2,n-1} & \cdots & m_{n,n-1} \end{pmatrix}, \tag{3.4}$$

where m_{jk} is as in Proposition 2.2(b). This matrix is nonsingular by (2.30), so that the linear equations have a unique solution, and therefore, Q_{n-1} exists and is unique. ■

For the rest of this section, we choose some ordering of the eigenvalues of A taking into account the multiplicities, say

$$\alpha_1, \alpha_2, \ldots, \alpha_n, \tag{3.5}$$

so that each a_i appears exactly n_i times among the α's. For each $k = 0, 1, \ldots, n$, we can construct P_k as in Section 2, but based on $\alpha_1, \ldots, \alpha_k$. Thus, P_k is a monic polynomial of degree k such that

$$\int_{-\infty}^{\infty} P_k(x) x^j e^{-(V(x)-a_i x)} dx = 0, \quad i = 1, \ldots, p,\ j = 0, \ldots, k_i - 1, \tag{3.6}$$

where k_i is the number of times that a_i appears among $\alpha_1, \ldots, \alpha_k$. We also have that P_k is the average characteristic polynomial of the ensemble of $k \times k$ Hermitian matrices with external source having eigenvalues a_i with multiplicity k_i.

For each $k = 1, \ldots, n$, we also have by Lemma 3.1 a function Q_{k-1} from the linear span of the functions

$$\Sigma_k := \left\{ x^j e^{a_i x} \mid i = 1, \ldots, p,\ j = 0, \ldots, k_i - 1 \right\} \tag{3.7}$$

such that

$$\int_{-\infty}^{\infty} x^i Q_{k-1}(x) e^{-V(x)} dx = 0 \quad \text{for } i = 0, \ldots, k-2, \tag{3.8}$$

$$\int_{-\infty}^{\infty} x^{k-1} Q_{k-1}(x) e^{-V(x)} dx = 1. \tag{3.9}$$

It follows from (3.6), (3.8), and (3.9) that the P's and Q's are a biorthogonal system in the sense that

$$\int_{-\infty}^{\infty} P_j(x) Q_k(x) e^{-V(x)} dx = \delta_{jk} \quad \text{for } j, k = 0, \ldots, n-1. \tag{3.10}$$

This property explains why we used Q_{k-1} for the function that satisfies (3.8) and (3.9) (and not Q_k).

We now introduce the kernel K_n.

Definition 3.2. With the polynomials P_k and the functions Q_k introduced above, define

$$K_n(x,y) = e^{-(1/2)(V(x)+V(y))} \sum_{k=0}^{n-1} P_k(x) Q_k(y). \tag{3.11}$$

Note that the P's and the Q's depend on the specific ordering (3.5) that we choose for the eigenvalues of A. However, it will turn out that K_n does not depend on this ordering.

Because of the biorthogonality property (3.10) it is easy to see that from definition (3.11) we have

$$\int_{-\infty}^{\infty} K_n(x,x)dx = n \tag{3.12}$$

and the reproducing kernel property

$$\int_{-\infty}^{\infty} K_n(x,y)K_n(y,z)dy = K_n(x,z). \tag{3.13}$$

The following theorem is the main result of this paper.

Theorem 3.3. The joint probability density function on eigenvalues has the determinantal form

$$\frac{1}{n!} \det\left(K_n(\lambda_j, \lambda_k)\right)_{1\le j,k\le n}. \tag{3.14}$$

The m-point correlation function has the form

$$R_m(\lambda_1, \ldots, \lambda_m) = \det\left(K_n(\lambda_j, \lambda_k)\right)_{1\le j,k\le m}. \tag{3.15}$$

□

Proof. Any joint probability density function of the form (3.14) with a kernel K_n, satisfying (3.12) and (3.13), leads to m-point correlation functions of the form (3.15). So it suffices to prove that (3.14) is the joint probability density function of the eigenvalues.

For each j, we define

$$w_j(x) = x^{d_j-1} e^{a_i x} \tag{3.16}$$

if $a_i = \alpha_j$ and a_i appears d_j times in the sequence $\alpha_1, \ldots, \alpha_j$. Note that the functions (3.16) differ from the functions w_j used in Proposition 2.2 in two respects. First, there is an extra factor $e^{-V(x)}$ in Proposition 2.2, and second, we used a specific ordering of the eigenvalues of A in Proposition 2.2 (which only amounts to a renumbering).

A similar calculation leading to (2.10) in the proof of Proposition 2.1 shows that the joint probability density of eigenvalues is proportional to

$$\prod_{j=1}^{n} e^{-V(\lambda_j)} \det\left(w_i(\lambda_j)\right)_{1\leq i,j\leq n} \Delta(\lambda). \tag{3.17}$$

Since Q_{i-1} is a linear combination of $w_1, \ldots, w_i$, we can take appropriate row combinations to find that

$$\det\left(w_i(\lambda_j)\right)_{1\leq i,j\leq n} \propto \det\left(Q_{i-1}(\lambda_j)\right)_{1\leq i,j\leq n}. \tag{3.18}$$

We write $\Delta(\lambda)$ as a Vandermonde determinant which we similarly rewrite as

$$\Delta(\lambda) = \det\left(P_{i-1}(\lambda_k)\right)_{1\leq i,k\leq n}. \tag{3.19}$$

Thus, the joint probability density of eigenvalues is proportional to

$$\det\left(e^{-(1/2)V(\lambda_j)} Q_{i-1}(\lambda_j)\right)_{1\leq i,j\leq n} \det\left(e^{-(1/2)V(\lambda_k)} P_{i-1}(\lambda_k)\right)_{1\leq i,k\leq n}. \tag{3.20}$$

Taking the transpose of the matrix in the first determinant, and then using the multiplicative property of determinants, we find that the joint probability density is equal to

$$c \det\left(K_n(\lambda_j, \lambda_k)\right)_{1\leq j,k\leq n} \tag{3.21}$$

for some constant c, which should be such that the integral with respect to $d\lambda_1 \cdots d\lambda_n$ is 1. Because of the properties (3.12) and (3.13) this is so for $c = 1/n!$ and the theorem is proved. ∎

Remark 3.4. Renumbering the eigenvalues $a_1, a_2, \ldots, a_n$ leads to the same kernel K_n but to different P_k and Q_k.

4 Special form of the kernel in case of two eigenvalues

In this section, we assume that we have only two distinct eigenvalues a_1 and a_2 with multiplicities n_1 and n_2, respectively. We order the eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_n$ in some arbitrary way (a_j appear n_j times in the sequence), but for convenience, we assume that

$$\alpha_{n-1} = a_1, \qquad \alpha_n = a_2. \tag{4.1}$$

We also put

$$\alpha_{n+1} = a_1, \qquad \alpha_{n+2} = a_2. \tag{4.2}$$

As in Section 3, we have polynomials P_k and functions Q_k for every $k = 0, \ldots, n-1$ such that

$$K_n(x,y) = e^{-(1/2)(V(x)+V(y))} \sum_{k=0}^{n-1} P_k(x) Q_k(y). \tag{4.3}$$

Our aim in this section is to simplify this expression. The formula we will find is an analogue of the well-known Christoffel-Darboux formula for orthogonal polynomials.

To present the formula we are going to use multi-index notation. For nonnegative integers k_1 and k_2, we use P_{k_1,k_2} to denote the monic polynomial of degree $k_1 + k_2$ having k_j orthogonality relations with respect to the weight

$$w_j(x) = e^{-(V(x)-a_j x)}, \quad j = 1, 2. \tag{4.4}$$

Thus,

$$\int_{-\infty}^{\infty} P_{k_1,k_2}(x) x^i w_j(x)\, dx = 0, \quad i = 0, \ldots, k_j - 1,\ j = 1, 2. \tag{4.5}$$

The polynomial P_{k_1,k_2} is called a multiple orthogonal polynomial of type II (see, e.g., [1, 2]). We also define

$$Q_{k_1,k_2}(x) = A_{k_1,k_2}(x) e^{a_1 x} + B_{k_1,k_2}(x) e^{a_2 x}, \tag{4.6}$$

where the degree of A_{k_1,k_2} is $k_1 - 1$, the degree of B_{k_1,k_2} is $k_2 - 1$, and

$$\int_{-\infty}^{\infty} x^j Q_{k_1,k_2}(x) e^{-V(x)}\, dx = \begin{cases} 0, & j = 0, \ldots, k_1 + k_2 - 2, \\ 1 & j = k_1 + k_2 - 1. \end{cases} \tag{4.7}$$

The polynomials A_{k_1,k_2} and B_{k_1,k_2} are called multiple orthogonal polynomials of type I (see [1, 2]). For each pair (k_1, k_2) of nonnegative integers, the polynomials P_{k_1,k_2}, A_{k_1,k_2}, and B_{k_1,k_2} exist and are uniquely defined by their degree requirements and the relations (4.5), (4.6), and (4.7).

We can express P_k and Q_k in this new notation as

$$P_k = P_{k_1,k_2}, \qquad Q_{k-1} = Q_{k_1,k_2}, \tag{4.8}$$

provided a_j appears k_j times among the numbers $\alpha_1, \ldots, \alpha_k$ (for $j = 1, 2$). In particular, because of our assumptions (4.1) and (4.2) we have

$$P_n = P_{n_1,n_2}, \qquad P_{n-1} = P_{n_1,n_2-1}, \qquad P_{n-2} = P_{n_1-1,n_2-1}, \tag{4.9}$$

$$Q_{n-1} = Q_{n_1,n_2}, \qquad Q_n = Q_{n_1+1,n_2}, \qquad Q_{n+1} = Q_{n_1+1,n_2+1}. \tag{4.10}$$

We also need the numbers

$$h^{(j)}_{k_1,k_2} = \int_{-\infty}^{\infty} P_{k_1,k_2}(x) x^{k_j} w_j(x)\,dx, \quad j = 1, 2, \tag{4.11}$$

which are nonzero, see (2.33). For later use, we note that

$$\begin{aligned} 1 &= \int P_{k_1,k_2}(x) Q_{k_1+1,k_2}(x) e^{-V(x)}\,dx \\ &= \int P_{k_1,k_2}(x) \big(A_{k_1+1,k_2}(x) w_1(x) + B_{k_1+1,k_2}(x) w_2(x)\big)\,dx \\ &= \int P_{k_1,k_2}(x) A_{k_1+1,k_2}(x) w_1(x)\,dx \\ &= \big(\text{leading coefficient of } A_{k_1+1,k_2}\big) \times h^{(1)}_{k_1,k_2}, \end{aligned} \tag{4.12}$$

so that

$$\text{leading coefficient of } A_{k_1+1,k_2} = \frac{1}{h^{(1)}_{k_1,k_2}}. \tag{4.13}$$

Similarly,

$$\text{leading coefficient of } B_{k_1,k_2+1} = \frac{1}{h^{(2)}_{k_1,k_2}}. \tag{4.14}$$

It also follows from (4.13) and (4.14) that $h^{(j)}_{k_1,k_2} \neq 0$ for $j = 1, 2$.

Then we can state the following theorem.

Theorem 4.1. With the notation introduced above,

$$\begin{aligned} (x-y) e^{(1/2)(V(x)+V(y))} K_n(x,y) = {} & P_{n_1,n_2}(x) Q_{n_1,n_2}(y) \\ & - \frac{h^{(1)}_{n_1,n_2}}{h^{(1)}_{n_1-1,n_2}} P_{n_1-1,n_2}(x) Q_{n_1+1,n_2}(y) \\ & - \frac{h^{(2)}_{n_1,n_2}}{h^{(2)}_{n_1,n_2-1}} P_{n_1,n_2-1}(x) Q_{n_1,n_2+1}(y). \end{aligned} \tag{4.15}$$

□

The proof of the theorem needs some preparation. We start working again with the P_k's and Q_j's (single index) as before. For each j and k, we put

$$c_{jk} = \int x P_k(x) Q_j(x) e^{-V(x)} dx. \tag{4.16}$$

The coefficients c_{jk} appear in the expansion

$$x P_k(x) = \sum_{j=0}^{k+1} c_{jk} P_j(x), \tag{4.17}$$

since by the biorthogonality relation, we have indeed

$$c_{jk} = \int x P_k(x) Q_j(x) e^{-V(x)} dx. \tag{4.18}$$

Similarly, we have for $j = 0, \dots, n-1$,

$$x Q_j(x) = \sum_{k=0}^{n+1} c_{jk} Q_k(x). \tag{4.19}$$

Note that by adding the two values α_{n+1} and α_{n+2} as we did in (4.2), we have this expansion for every $j \leq n-1$.

Lemma 4.2. (a) If $j \geq k+2$, then $c_{jk} = 0$.

(b) If $k \geq j+3$ and if both a_1 and a_2 appear at least once among $\alpha_{j+2}, \alpha_{j+3}, \dots, \alpha_k$, then $c_{jk} = 0$. □

Proof. (a) We have that

$$\int P(x) Q_j(x) e^{-V(x)} dx = 0 \tag{4.20}$$

for every polynomial P of degree less than or equal to $j-1$. Since xP_k is a polynomial of degree $k+1$, it follows that $c_{jk} = 0$ if $k+1 \leq j-1$. This proves part (a).

(b) Let k and j be such that the conditions of part (b) are satisfied. Suppose that a_1 appears k_1 times among $\alpha_1, \dots, \alpha_k$ and j_1 times among $\alpha_1, \dots, \alpha_{j+1}$. We put $k_2 = k - k_1$ and $j_2 = j + 1 - j_1$. It follows from the assumptions that $j_1 < k_1$ and $j_2 < k_2$. Then $Q_j(x) = Q_{j_1,j_2}(x) = A_{j_1,j_2}(x) e^{a_1 x} + B_{j_1,j_2}(x) e^{a_2 x}$, where A_{j_1,j_2} has degree $j_1 - 1$ and B_{j_1,j_2} has degree

$j_2 - 1$. It follows that

$$xQ_j(x) = xA_{j_1,j_2}(x)e^{a_1 x} + xB_{j_1,j_2}(x)e^{a_2 x} \tag{4.21}$$

and $xA_{j_1,j_2}(x)$ has degree $j_1 \le k_1 - 1$ and $xB_{j_1,j_2}(x)$ has degree $j_2 \le k_2 - 1$. Thus, by the multiple orthogonality property of $P_k = P_{k_1,k_2}$, we have

$$\int P_k(x)xQ_j(x)e^{-V(x)}dx = 0. \tag{4.22}$$

This proves part (b). ■

We also need the following relations between nearby P's and Q's.

Lemma 4.3. The following relations hold:

$$\begin{aligned} P_{n_1-1,n_2-1} &= \frac{h^{(1)}_{n_1-1,n_2-1}}{h^{(1)}_{n_1-1,n_2}}\left(P_{n_1-1,n_2} - P_{n_1,n_2-1}\right) \\ &= -\frac{h^{(2)}_{n_1-1,n_2-1}}{h^{(2)}_{n_1,n_2-1}}\left(P_{n_1-1,n_2} - P_{n_1,n_2-1}\right), \end{aligned} \tag{4.23}$$

$$\begin{aligned} Q_{n_1+1,n_2+1} &= -\frac{h^{(1)}_{n_1,n_2}}{h^{(1)}_{n_1,n_2+1}}\left(Q_{n_1,n_2+1} - Q_{n_1+1,n_2}\right) \\ &= \frac{h^{(2)}_{n_1,n_2}}{h^{(2)}_{n_1+1,n_2}}\left(Q_{n_1,n_2+1} - Q_{n_1+1,n_2}\right). \end{aligned} \tag{4.24}$$

□

Proof. Since P_{n_1-1,n_2} and P_{n_1,n_2-1} are both monic polynomials of degree n, their difference is a polynomial of degree less than or equal to $n-1$. Since this difference has $n_j - 1$ orthogonality conditions with respect to w_j for $j = 1, 2$, it must be a multiple of P_{n_1-1,n_2-1}. Thus,

$$P_{n_1-1,n_2} - P_{n_1,n_2-1} = \gamma P_{n_1-1,n_2-1} \tag{4.25}$$

for some γ. Integrating this equation with respect to $x^{n_1-1}w_1(x)$ and $x^{n_2-1}w_2(x)$, we get $h^{(1)}_{n_1-1,n_2} = \gamma h^{(1)}_{n_1-1,n_2-1}$, and $-h^{(2)}_{n_1,n_2-1} = \gamma h^{(2)}_{n_1-1,n_2-1}$, respectively. This gives (4.23).

Next, we note that we have

$$\begin{aligned} &\int x^j\left(Q_{n_1,n_2+1} - Q_{n_1+1,n_2}\right)e^{-V(x)}dx = 0 - 0 = 0, \quad j = 0, \dots, n_1 + n_2 - 1, \\ &\int x^{n_1+n_2}\left(Q_{n_1,n_2+1} - Q_{n_1+1,n_2}\right)e^{-V(x)}dx = 1 - 1 = 0. \end{aligned} \tag{4.26}$$

Since $Q_{n_1,n_2+1}(x) - Q_{n_1+1,n_2}(x) = A(x)e^{a_1 x} + B(x)e^{a_2 x}$, where A has degree n_1 and B has degree n_2, it follows that $Q_{n_1,n_2+1} - Q_{n_1+1,n_2}$ is a multiple of Q_{n_1+1,n_2+1}, say

$$Q_{n_1,n_2+1} - Q_{n_1+1,n_2} = \beta Q_{n_1+1,n_2+1}. \tag{4.27}$$

This means for the A-polynomials that

$$A_{n_1,n_2+1} - A_{n_1+1,n_2} = \beta A_{n_1+1,n_2+1}, \tag{4.28}$$

and looking at the leading coefficient (= coefficient of x^{n_1}), we get

$$\beta = -\frac{\text{leading coefficient of } A_{n_1+1,n_2}}{\text{leading coefficient of } A_{n_1+1,n_2+1}} = -\frac{h^{(1)}_{n_1,n_2+1}}{h^{(1)}_{n_1,n_2}}, \tag{4.29}$$

where we used (4.13). We also get, by considering the B-polynomials, that

$$\beta = \frac{\text{leading coefficient of } B_{n_1,n_2+1}}{\text{leading coefficient of } B_{n_1+1,n_2+1}} = \frac{h^{(2)}_{n_1+1,n_2}}{h^{(2)}_{n_1,n_2}} \tag{4.30}$$

because of (4.14). This proves (4.24). ■

Now, we are ready for the proof of Theorem 4.1.

Proof. We note that $(x-y)e^{(1/2)(V(x)+V(y))}K_n(x,y)$ has a telescoping character. Indeed, we have, by (4.17) and (4.19),

$$\begin{aligned}
(x-y)&\sum_{k=0}^{n-1} P_k(x)Q_k(y) \\
&= \sum_{k=0}^{n-1} xP_k(x)Q_k(y) - \sum_{j=0}^{n-1} yP_j(x)Q_j(y) \\
&= \sum_{k=0}^{n-1}\sum_{j=0}^{k+1} c_{jk}P_j(x)Q_k(y) - \sum_{j=0}^{n-1}\sum_{k=0}^{n+1} c_{jk}P_j(x)Q_k(y) \\
&= c_{n,n-1}P_n(x)Q_{n-1}(y) - \sum_{j=0}^{n-1} c_{jn}P_j(x)Q_n(y) - \sum_{j=0}^{n-1} c_{j,n+1}P_j(x)Q_{n+1}(y).
\end{aligned} \tag{4.31}$$

Now, observe that $c_{n,n-1} = 1$, $c_{jn} = 0$ for $j = 0, \dots, n-3$, and $c_{j,n+1} = 0$ for $j = 0, \dots, n-2$, which follows from Lemma 4.2. Thus,

$$(x-y)e^{(1/2)(V(x)+V(y))}K_n(x,y) = P_n(x)Q_{n-1}(y) - c_{n-2,n}P_{n-2}(x)Q_n(y) - c_{n-1,n}P_{n-1}(x)Q_n(y) - c_{n-1,n+1}P_{n-1}(x)Q_{n+1}(y). \tag{4.32}$$

In formula (4.32), we have reduced the n-term expression to four terms, which is already quite nice. However, we want to reduce it to three terms only. Changing back to multi-index notation and using (4.9) and (4.10), we see that (4.32) leads to

$$(x-y)e^{(1/2)(V(x)+V(y))}K_n(x,y) = P_{n_1,n_2}(x)Q_{n_1,n_2}(y) - c_{n-2,n}P_{n_1-1,n_2-1}(x)Q_{n_1+1,n_2}(y) - c_{n-1,n}P_{n_1,n_2-1}(x)Q_{n_1+1,n_2}(y) - c_{n-1,n+1}P_{n_1,n_2-1}(x)Q_{n_1+1,n_2+1}(y). \tag{4.33}$$

Comparing (4.33) and (4.15), we see that we need to get rid of P_{n_1-1,n_2-1} and Q_{n_1+1,n_2+1} which can be done using the relations (4.23) and (4.24).

Our next task now is to express the recurrence coefficients $c_{n-2,n}$, $c_{n-1,n}$, and $c_{n-1,n+1}$ that appear in (4.33) in terms of the h-numbers. This is rather straightforward from definition (4.18). Indeed, we have

$$\begin{aligned} c_{n-2,n} &= \int xP_{n_1,n_2}(x)Q_{n_1,n_2-1}(x)e^{-V(x)}\,dx \\ &= \int P_{n_1,n_2}(x)\big(xA_{n_1,n_2-1}(x)w_1(x) + xB_{n_1,n_2-1}(x)w_2(x)\big)\,dx \\ &= \int P_{n_1,n_2}(x)xA_{n_1,n_2-1}(x)w_1(x)\,dx \\ &= \big(\text{leading coefficient of } A_{n_1,n_2-1}\big) \times h^{(1)}_{n_1,n_2} \\ &= \frac{h^{(1)}_{n_1,n_2}}{h^{(1)}_{n_1-1,n_2-1}}, \end{aligned} \tag{4.34}$$

where we used (4.13), and similarly,

$$c_{n-1,n} = \frac{h^{(1)}_{n_1,n_2}}{h^{(1)}_{n_1-1,n_2}} + \frac{h^{(2)}_{n_1,n_2}}{h^{(2)}_{n_1,n_2-1}}, \tag{4.35}$$

$$c_{n-1,n+1} = \frac{h^{(2)}_{n_1+1,n_2}}{h^{(2)}_{n_1,n_2-1}}. \tag{4.36}$$

Now, we substitute all formulas (4.23), (4.24), (4.34), (4.35), and (4.36) into (4.33). Then straightforward calculations lead to (4.15). ■

5 Riemann-Hilbert problem

We use the notation of Section 4.

The Christoffel-Darboux formula (4.15) can be expressed in terms of the solution of a Riemann-Hilbert problem that was given by Van Assche, Geronimo, and Kuijlaars [19] to characterize the multiple orthogonal polynomials, and which generalizes the Riemann-Hilbert problem for orthogonal polynomials due to Fokas, Its, and Kitaev [14]. The Rieman-Hilbert problem is to find $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{3\times 3}$ such that

(i) Y is analytic on $\mathbb{C} \setminus \mathbb{R}$,

(ii) for $x \in \mathbb{R}$, we have

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_1(x) & w_2(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.1}$$

where $Y_+(x)$ ($Y_-(x)$) denotes the limit of $Y(z)$ as $z \to x$ from the upper (resp., lower) half-plane,

(iii) as $z \to \infty$, we have

$$Y(z) = \left(I + O\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 & 0 \\ 0 & z^{-n_1} & 0 \\ 0 & 0 & z^{-n_2} \end{pmatrix}, \tag{5.2}$$

where I denotes the 3×3 identity matrix.

In [19], it was shown that there is a unique solution

$$Y = \begin{pmatrix} P_{n_1,n_2} & C(P_{n_1,n_2} w_1) & C(P_{n_1,n_2} w_2) \\ c_1 P_{n_1-1,n_2} & c_1 C(P_{n_1-1,n_2} w_1) & c_1 C(P_{n_1-1,n_2} w_2) \\ c_2 P_{n_1,n_2-1} & c_2 C(P_{n_1,n_2-1} w_1) & c_2 C(P_{n_1,n_2-1} w_2) \end{pmatrix} \tag{5.3}$$

with constants

$$c_1 = -2\pi i \left(h^{(1)}_{n_1-1,n_2}\right)^{-1}, \qquad c_2 = -2\pi i \left(h^{(2)}_{n_1,n_2-1}\right)^{-1}, \tag{5.4}$$

and where Cf denotes the Cauchy transform of f, that is,

$$Cf(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(s)}{s - z} ds. \tag{5.5}$$

The multiple orthogonal polynomials of type I, A_{n_1,n_2} and B_{n_1,n_2}, have a Riemann-Hilbert characterization as well. We seek $X : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{3\times 3}$ such that

(i) X is analytic on $\mathbb{C} \setminus \mathbb{R}$,

(ii) for $x \in \mathbb{R}$, we have

$$X_+(x) = X_-(x) \begin{pmatrix} 1 & 0 & 0 \\ -w_1(x) & 1 & 0 \\ -w_2(x) & 0 & 1 \end{pmatrix}, \tag{5.6}$$

(iii) as $z \to \infty$, we have

$$X(z) = \left(I + O\left(\frac{1}{z}\right) \right) \begin{pmatrix} z^{-n} & 0 & 0 \\ 0 & z^{n_1} & 0 \\ 0 & 0 & z^{n_2} \end{pmatrix}. \tag{5.7}$$

The solution to this Riemann-Hilbert problem [19] is

$$X = \begin{pmatrix} -2\pi i C\big(A_{n_1,n_2} w_1 + B_{n_1,n_2} w_2\big) & 2\pi i A_{n_1,n_2} & 2\pi i B_{n_1,n_2} \\ -k_1 C\big(A_{n_1+1,n_2} w_1 + B_{n_1+1,n_2} w_2\big) & k_1 A_{n_1+1,n_2} & k_1 B_{n_1+1,n_2} \\ -k_2 C\big(A_{n_1,n_2+1} w_2 + B_{n_1,n_2+1} w_2\big) & k_2 A_{n_1,n_2+1} & k_2 B_{n_1,n_2+1} \end{pmatrix}, \tag{5.8}$$

where

$$k_1 = \frac{1}{\text{leading coefficient of } A_{n_1+1,n_2}} = h^{(1)}_{n_1,n_2}, \tag{5.9}$$

$$k_2 = \frac{1}{\text{leading coefficient of } B_{n_1,n_2+1}} = h^{(2)}_{n_1,n_2}. \tag{5.10}$$

It is easy to see that

$$X = Y^{-t} \quad \text{(inverse transpose)} \tag{5.11}$$

(see also [19]).

Now, we form the product $Y^{-1}(y)Y(x) = X^t(y)Y(x)$ and we compute the 21-entry using (5.3), (5.4), (5.8), (5.9), and (5.10),

$$\begin{aligned} &\big[Y^{-1}(y)Y(x)\big]_{21} \\ &\quad = \Big(2\pi i A_{n_1,n_2}(y) \quad k_1 A_{n_1+1,n_2}(y) \quad k_2 A_{n_1,n_2+1}(y)\Big) \begin{pmatrix} P_{n_1,n_2}(x) \\ c_1 P_{n_1-1,n_2}(x) \\ c_2 P_{n_1,n_2-1}(x) \end{pmatrix} \\ &\quad = 2\pi i \Bigg(P_{n_1,n_2}(x) A_{n_1,n_2}(y) - \frac{h^{(1)}_{n_1,n_2}}{h^{(1)}_{n_1-1,n_2}} P_{n_1-1,n_2}(x) A_{n_1+1,n_2}(y) \\ &\qquad\qquad - \frac{h^{(2)}_{n_1,n_2}}{h^{(2)}_{n_1,n_2-1}} P_{n_1,n_2-1}(x) A_{n_1,n_2+1}(y) \Bigg). \end{aligned} \tag{5.12}$$

We get a similar expression for the 31-entry $[Y^{-1}(y)Y(x)]_{31}$, but with the B-polynomials instead of the A-polynomials. Then it follows that we can rewrite the Christoffel-Darboux formula (4.15) as

$$K_n(x,y) = e^{-(1/2)(V(x)+V(y))} \frac{e^{a_1 y}\left[Y^{-1}(y)Y(x)\right]_{21} + e^{a_2 y}\left[Y^{-1}(y)Y(x)\right]_{31}}{2\pi i(x-y)} \tag{5.13}$$

which is a compact form for the kernel in terms of the solution of the Riemann-Hilbert problem. We expect that the Riemann-Hilbert problem for Y is tractable to asymptotic analysis using the methods of [3, 4, 12, 13, 11].

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References

[1] A. I. Aptekarev, *Multiple orthogonal polynomials*, J. Comput. Appl. Math. **99** (1998), no. 1-2, 423–447.

[2] A. I. Aptekarev, A. Branquinho, and W. Van Assche, *Multiple orthogonal polynomials for classical weights*, Trans. Amer. Math. Soc. **355** (2003), no. 10, 3887–3914.

[3] P. Bleher and A. Its, *Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problem, and universality in the matrix model*, Ann. of Math. (2) **150** (1999), no. 1, 185–266.

[4] ———, *Double scaling limit in the random matrix model: the Riemann-Hilbert approach*, Comm. Pure Appl. Math. **56** (2003), no. 4, 433–516.

[5] E. Brézin and S. Hikami, *Correlations of nearby levels induced by a random potential*, Nuclear Phys. B **479** (1996), no. 3, 697–706.

[6] ———, *Extension of level-spacing universality*, Phys. Rev. E **56** (1997), no. 1, 264–269.

[7] ———, *Spectral form factor in a random matrix theory*, Phys. Rev. E (3) **55** (1997), no. 4, 4067–4083.

[8] ———, *Level spacing of random matrices in an external source*, Phys. Rev. E (3) **58** (1998), no. 6, 7176–7185.

[9] ———, *Universal singularity at the closure of a gap in a random matrix theory*, Phys. Rev. E (3) **57** (1998), no. 4, 4140–4149.

[10] E. Brézin, S. Hikami, and A. Zee, *Universal correlations for deterministic plus random Hamiltonians*, Phys. Rev. E **51** (1995), no. 6, 5442–5452.

[11] P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach*, Courant Lecture Notes in Mathematics, vol. 3, American Mathematical Society, Rhode Island, 1999.

[12] P. Deift, T. Kriecherbauer, K. T-R McLaughlin, S. Venakides, and X. Zhou, *Strong asymptotics of orthogonal polynomials with respect to exponential weights*, Comm. Pure Appl. Math. **52** (1999), no. 12, 1491–1552.

[13] ———, *Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory*, Comm. Pure Appl. Math. **52** (1999), no. 11, 1335–1425.

[14] A. S. Fokas, A. Its, and A. V. Kitaev, *The isomonodromy approach to matrix models in 2D quantum gravity*, Comm. Math. Phys. **147** (1992), no. 2, 395–430.

[15] Harish-Chandra, *Differential operators on a semisimple Lie algebra*, Amer. J. Math. **79** (1957), 87–120.

[16] C. Itzykson and J. B. Zuber, *The planar approximation. II*, J. Math. Phys. **21** (1980), no. 3, 411–421.

[17] M. L. Mehta, *Random Matrices*, 2nd ed., Academic Press, Massachusetts, 1991.

[18] L. A. Pastur, *The spectrum of random matrices*, Teoret. Mat. Fiz. **10** (1972), no. 1, 102–112 (Russian).

[19] W. Van Assche, J. S. Geronimo, and A. B. J. Kuijlaars, *Riemann-Hilbert problems for multiple orthogonal polynomials*, NATO ASI Special Functions 2000. Current Perspective and Future Directions (J. Bustoz, M. E. H. Ismail, and S. K. Suslov, eds.), NATO Science Series II, vol. 30, Kluwer Academic Publishers, Dordrecht, 2001, pp. 23–59.

[20] P. Zinn-Justin, *Random Hermitian matrices in an external field*, Nuclear Phys. B **497** (1997), no. 3, 725–732.

[21] ———, *Universality of correlation functions of Hermitian random matrices in an external field*, Comm. Math. Phys. **194** (1998), no. 3, 631–650.

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