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Exact Solution of the Six-Vertex Model with Domain Wall Boundary Conditions. Ferroelectric Phase

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Abstract: This is a continuation of the paper [4] of Bleher and Fokin, in which the large n asymptotics is obtained for the partition function Z_n of the six-vertex model with domain wall boundary conditions in the disordered phase. In the present paper we obtain the large n asymptotics of Z_n in the ferroelectric phase. We prove that for any $\varepsilon > 0$, as $n \to \infty$, $Z_n = CG^n F^{n^2}[1 + O(e^{-n^{1-\varepsilon}})]$, and we find the exact values of the constants C, G and F. The proof is based on the large n asymptotics for the underlying discrete orthogonal polynomials and on the Toda equation for the tau-function.

1. Introduction and Formulation of the Main Result

1.1. Definition of the model. The six-vertex model, or the model of two-dimensional ice, is stated on a square $n \times n$ lattice with arrows on edges. The arrows obey the rule that at every vertex there are two arrows pointing in and two arrows pointing out. Such rule is sometimes called the *ice-rule*. There are only six possible configurations of arrows at each vertex, hence the name of the model, see Fig. 1.

We will consider the *domain wall boundary conditions* (DWBC), in which the arrows on the upper and lower boundaries point in the square, and the ones on the left and right boundaries point out. One possible configuration with DWBC on the 4×4 lattice is shown on Fig. 2.

For each possible vertex state we assign a weight w_i, $i = 1, \ldots, 6$, and define, as usual, the partition function as a sum over all possible arrow configurations of the product of the vertex weights,

$$Z_n = \sum_{\text{arrow configurations } \sigma} w(\sigma), \qquad w(\sigma) = \prod_{x \in V_n} w_{t(x;\sigma)} = \prod_{i=1}^{6} w_i^{N_i(\sigma)}, \tag{1.1}$$

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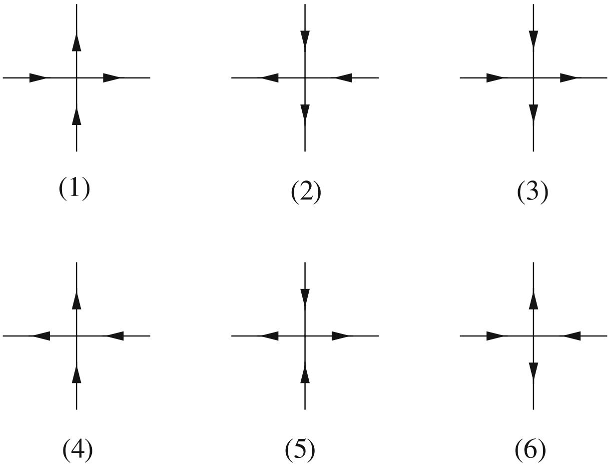


Fig. 1. The six arrow configurations allowed at a vertex

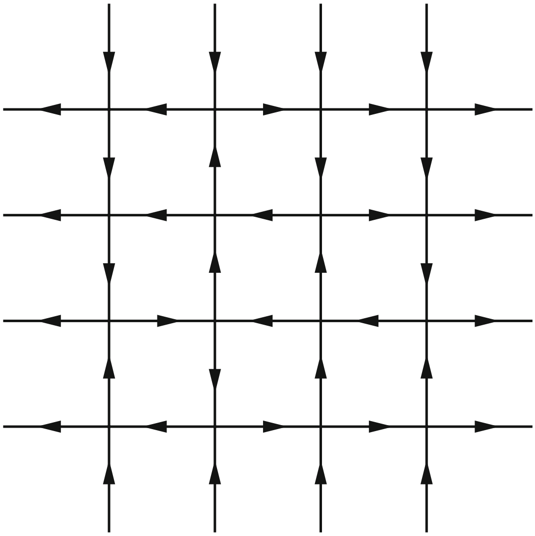


Fig. 2. An example of 4×4 configuration with DWBC

where V_n is the $n \times n$ set of vertices, $t(x;\sigma) \in \{1, \dots, 6\}$ is the type of configuration σ at vertex x according to Fig. 1, and $N_i(\sigma)$ is the number of vertices of type i in the configuration σ. The sum is taken over all possible configurations obeying the given boundary condition. The Gibbs measure is defined then as

$$\mu_n(\sigma) = \frac{w(\sigma)}{Z_n}. \tag{1.2}$$

Our main goal is to obtain the large n asymptotics of the partition function Z_n.

The six-vertex model has six parameters: the weights w_i. By using some conservation laws it can be reduced to only two parameters. It is convenient to derive the conservation laws from the *height function*.

1.2. Height function. Consider the dual lattice,

$$V' = \left\{ x = \left(i + \frac{1}{2}, j + \frac{1}{2} \right), \quad 0 \le i, j \le n \right\}. \tag{1.3}$$

Given a configuration σ on E, an integer-valued function $h = h_\sigma$ on V' is called a *height function* of σ, if for any two neighboring points $x, y \in V'$, $|x - y| = 1$, we have that

$$h(y) - h(x) = (-1)^s, \tag{1.4}$$

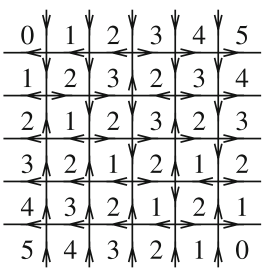
where $s = 0$ if the arrow σ_e on the edge $e \in E$, crossing the segment $[x, y]$, is oriented in such a way that it points from left to right with respect to the vector $\vec{xy}$, and $s = 1$ if σ_e is oriented from right to left with respect to $\vec{xy}$. The ice-rule ensures that the height function $h = h_\sigma$ exists for any configuration σ. It is unique up to addition of a constant. Figure 3 shows a 5×5 configuration with a height function, and the corresponding alternating sign matrix, which is obtained from the configuration by replacing the vertex (5) of Fig. 1 by 1, the vertex (6) by (-1), and all the other vertices by 0.

Observe that if $h(x_1), h(x_2), h(x_3), h(x_4)$ are the four values of the height function around a vertex $x = (j, k)$, enumerated in the positive direction around x starting from the first quadrant, then the value of the element a_{jk} of the ASM is equal to

$$a_{jk} = \frac{h(x_1) - h(x_2) + h(x_3) - h(x_4)}{2}. \tag{1.5}$$

1.3. Conservation laws. Conservation laws are obtained in the paper [12] of Ferrari and Spohn, as a corollary of a path representation of the six-vertex model. We will derive them from the height function representation. Consider the height function $h = h_\sigma$ on a diagonal sequence of points defined by the formula,

$$x_j = x_0 + (j, j), \quad 0 \le j \le k, \tag{1.6}$$



$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Fig. 3. A 5×5 configuration with a height function and the corresponding alternating sign matrix

where both x_0 and x_k lie on the boundary B' of the dual lattice V',

$$B' = \left\{x = \left(i + \frac{1}{2}, \frac{1}{2}\right),\ 0 \le i \le n\right\} \cup \left\{x = \left(n + \frac{1}{2}, j + \frac{1}{2}\right),\ 0 \le j \le n\right\}$$
$$\cup \left\{x = \left(i + \frac{1}{2}, n + \frac{1}{2}\right),\ 0 \le i \le n\right\} \cup \left\{x = \left(\frac{1}{2}, j + \frac{1}{2}\right),\ 0 \le j \le n\right\}. \tag{1.7}$$

Then it follows from the definition of the height function, that

$$h(x_j) - h(x_{j-1}) = \begin{cases} 2, & \text{if} \quad t(x;\sigma) = 3, \\ -2, & \text{if} \quad t(x;\sigma) = 4, \\ 0, & \text{if} \quad t(x;\sigma) = 1, 2, 5, 6, \end{cases} \tag{1.8}$$

where

$$x = \frac{x_j + x_{j-1}}{2}. \tag{1.9}$$

Hence

$$0 = h(x_k) - h(x_0) = 2N_3(\sigma; L) - 2N_4(\sigma; L), \tag{1.10}$$

where $N_i(\sigma; L)$ is the number of vertex states of type i in σ on the line

$$L = \{x = x_0 + (t, t),\ t \in \mathbb{R}\}. \tag{1.11}$$

The line L is parallel to the diagonal $y = x$. By summing up over all possible lines L, we obtain that

$$N_3(\sigma) - N_4(\sigma) = 0, \tag{1.12}$$

where $N_i(\sigma)$ is the total number of vertex states of the type i in the configuration σ.

Similarly, by considering lines L parallel to the diagonal $y = -x$, we obtain that

$$N_1(\sigma) - N_2(\sigma) = 0. \tag{1.13}$$

Also,

$$N_5(\sigma) - N_6(\sigma) = n, \tag{1.14}$$

which follows if we consider lines L parallel to the x-axis.

The conservation laws allow to reduce the weights $w_1, \dots, w_6$ to 3 parameters. Namely, we have that

$$w_1^{N_1} w_2^{N_2} w_3^{N_3} w_4^{N_4} w_5^{N_5} w_6^{N_6} = C(n) a^{N_1} a^{N_2} b^{N_3} b^{N_4} c^{N_5} c^{N_6}, \tag{1.15}$$

where

$$a = \sqrt{w_1 w_2}, \quad b = \sqrt{w_3 w_4}, \quad c = \sqrt{w_5 w_6}, \tag{1.16}$$

and the constant

$$C(n) = \left(\frac{w_5}{w_6}\right)^{\frac{n}{2}}. \tag{1.17}$$

This implies the relation between the partition functions,

$$Z_n(w_1, w_2, w_3, w_4, w_5, w_6) = C(n) Z_n(a, a, b, b, c, c), \tag{1.18}$$

and between the Gibbs measures,

$$\mu_n(\sigma; w_1, w_2, w_3, w_4, w_5, w_6) = \mu_n(\sigma; a, a, b, b, c, c). \tag{1.19}$$

Therefore, for fixed boundary conditions, like DWBC, the general weights are reduced to the case when

$$w_1 = w_2 = a, \quad w_3 = w_4 = b, \quad w_5 = w_6 = c. \tag{1.20}$$

Furthermore,

$$Z_n(a, a, b, b, c, c) = c^{n^2} Z_n\left(\frac{a}{c}, \frac{a}{c}, \frac{b}{c}, \frac{b}{c}, 1, 1\right) \tag{1.21}$$

and

$$\mu_n(\sigma; a, a, b, b, c, c) = \mu_n\left(\sigma; \frac{a}{c}, \frac{a}{c}, \frac{b}{c}, \frac{b}{c}, 1, 1\right), \tag{1.22}$$

so that a general weight reduces to the two parameters, $\frac{a}{c}, \frac{b}{c}$.

1.4. Exact solution of the six-vertex model for a finite n. Introduce the parameter

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}. \tag{1.23}$$

There are three physical phases in the six-vertex model: the ferroelectric phase, $\Delta > 1$; the anti-ferroelectric phase, $\Delta < -1$; and, the disordered phase, $-1 < \Delta < 1$. In the three phases we parametrize the weights in the standard way: for the ferroelectric phase,

$$a = \sinh(t - \gamma), \quad b = \sinh(t + \gamma), \quad c = \sinh(2|\gamma|), \quad 0 < |\gamma| < t, \tag{1.24}$$

for the anti-ferroelectric phase,

$$a = \sinh(\gamma - t), \quad b = \sinh(\gamma + t), \quad c = \sinh(2\gamma), \quad |t| < \gamma, \tag{1.25}$$

and for the disordered phase

$$a = \sin(\gamma - t), \quad b = \sin(\gamma + t), \quad c = \sin(2\gamma), \quad |t| < \gamma. \tag{1.26}$$

The phase diagram of the six-vertex model is shown on Fig. 4.

The phase diagram and the Bethe-Ansatz solution of the six-vertex model for periodic and anti-periodic boundary conditions are thoroughly discussed in the works of Lieb [21–24], Lieb, Wu [25], Sutherland [30], Baxter [2], Batchelor, Baxter, O'Rourke, Yung [3]. See also the work of Wu, Lin [31], in which the Pfaffian solution for the six-vertex model with periodic boundary conditions is obtained on the free fermion line, $\Delta = 0$.

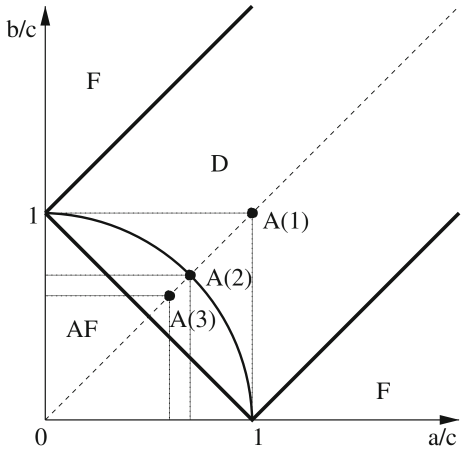


Fig. 4. The phase diagram of the model, where **F**, **AF** and **D** mark ferroelectric, antiferroelectric, and disordered phases, respectively. The circular arc corresponds to the so-called "free fermion" line, when $\Delta = 0$, and the three dots correspond to 1-, 2-, and 3-enumeration of alternating sign matrices

As concerns the six-vertex model with DWBC, it is noticed by Kuperberg [20], that on the diagonal,

$$\frac{a}{c} = \frac{b}{c} = x, \tag{1.27}$$

the six-vertex model with DWBC is equivalent to the s-enumeration of alternating sign matrices (ASM), in which the weight of each such matrix is equal to s^{N_-}, where N_- is the number of (-1)'s in the matrix and $s = \frac{1}{x^2}$. The exact solution for a finite n is known for 1-, 2-, and 3-enumerations of ASMs, see the works by Kuperberg [20] and Colomo-Pronko [9] for a solution based on the Izergin-Korepin formula. A fascinating story of the discovery of the ASM formula is presented in the book [7] of Bressoud. On the free fermion line, $\gamma = \frac{\pi}{4}$, the partition function of the six-vertex model with DWBC has a very simple form: $Z_n = 1$. For a nice short proof of this formula see the work [9] of Colomo-Pronko.

Here we will discuss the ferroelectric phase, and we will use parametrization (1.24). Without loss of generality we may assume that

$$\gamma > 0, \tag{1.28}$$

which corresponds to the region,

$$b > a + c. \tag{1.29}$$

The parameter Δ in the ferroelectric phase reduces to

$$\Delta = \cosh(2\gamma). \tag{1.30}$$

The six-vertex model with DWBC was introduced by Korepin in [16], who derived an important recursion relation for the partition function of the model. This lead to a beautiful determinantal formula of Izergin [13] for the partition function with DWBC. A detailed proof of this formula and its generalizations are given in the paper of Izergin, Coker, and Korepin [14]. When the weights are parameterized according to (1.24), the formula of Izergin is

$$Z_n = \frac{[\sinh(t-\gamma)\sinh(t+\gamma)]^{n^2}}{\left(\prod_{j=0}^{n-1} j!\right)^2}\,\tau_n, \tag{1.31}$$

where τ_n is the Hankel determinant,

$$\tau_n = \det\left(\frac{d^{j+k-2}\phi}{dt^{j+k-2}}\right)_{1\le j,k\le n}, \tag{1.32}$$

and

$$\phi(t) = \frac{\sinh(2\gamma)}{\sinh(t+\gamma)\sinh(t-\gamma)}. \tag{1.33}$$

An elegant derivation of the Izergin determinantal formula from the Yang-Baxter equation is given in the papers of Korepin, Zinn-Justin [19] and Kuperberg [20] (see also the book of Bressoud [7]).

One of the applications of the determinantal formula is that it implies that the partition function τ_n solves the Toda equation

$$\tau_n\tau_n'' - \tau_n'^2 = \tau_{n+1}\tau_{n-1}, \qquad n \ge 1, \qquad (') = \frac{\partial}{\partial t}, \tag{1.34}$$

cf. the work of Sogo, [27]. The Toda equation was used by Korepin and Zinn-Justin [19] to derive the free energy of the six-vertex model with DWBC, assuming some Ansatz on the behavior of subdominant terms in the large N asymptotics of the free energy.

Another application of the Izergin determinantal formula is that τ_N can be expressed in terms of a partition function of a random matrix model and also in terms of related orthogonal polynomials, see the paper [32] of Zinn-Justin. In the ferroelectric phase the expression in terms of orthogonal polynomials can be obtained as follows. For the evaluation of the Hankel determinant, let us write $\phi(t)$ in the form of the Laplace transform of a discrete measure,

$$\phi(t) = \frac{\sinh(2\gamma)}{\sinh(t+\gamma)\sinh(t-\gamma)} = 4\sum_{l=1}^{\infty} e^{-2tl}\sinh(2\gamma l). \tag{1.35}$$

Then

$$\tau_n = \frac{2^{n^2}}{n!}\sum_{l_1,\ldots,l_n=1}^{\infty} \Delta(l)^2 \prod_{i=1}^{n}\left[2e^{-2tl_i}\sinh(2\gamma l_i)\right], \tag{1.36}$$

where

$$\Delta(l) = \prod_{1\le i<j\le n}(l_j - l_i) \tag{1.37}$$

is the Vandermonde determinant.

Introduce now discrete monic polynomials $P_j(x) = x^j + \cdots$ orthogonal on the set $\mathbb{N} = \{l = 1, 2, \ldots\}$ with respect to the weight,

$$w(l) = 2e^{-2tl} \sinh(2\gamma l) = e^{-2tl+2\gamma l} - e^{-2tl-2\gamma l}, \tag{1.38}$$

so that

$$\sum_{l=1}^{\infty} P_j(l) P_k(l) w(l) = h_k \delta_{jk}. \tag{1.39}$$

Then it follows from (1.36) that

$$\tau_n = 2^{n^2} \prod_{k=0}^{n-1} h_k, \tag{1.40}$$

see the Appendix in the end of the paper. We will prove the following asymptotics of h_k.

Theorem 1.1. *For any $\varepsilon > 0$, as $k \to \infty$,*

$$h_k = \frac{(k!)^2 q^{k+1}}{(1-q)^{2k+1}} \left(1 + O(e^{-k^{1-\varepsilon}})\right), \tag{1.41}$$

where

$$q = e^{2\gamma - 2t}. \tag{1.42}$$

The error term in (1.41) is uniform on any compact subset of the set

$$\{(t, \gamma) : \ 0 < \gamma < t\}. \tag{1.43}$$

1.5. Main result: Asymptotics of the partition function. This work is a continuation of the work [4] of the first author with Vladimir Fokin. In [4] the authors obtain the large n asymptotics of the partition function Z_n in the disordered phase. They prove the conjecture of Paul Zinn-Justin [32] that the large n asymptotics of Z_n in the disordered phase has the following form: for some $\varepsilon > 0$,

$$Z_n = C n^{\kappa} F^{n^2} [1 + O(n^{-\varepsilon})], \tag{1.44}$$

and they find the exact value of the exponent κ,

$$\kappa = \frac{1}{12} - \frac{2\gamma^2}{3\pi(\pi - 2\gamma)}. \tag{1.45}$$

The value of F in the disordered phase is given by the formula,

$$F = \frac{\pi [\sin(\gamma + t) \sin(\gamma - t)]}{2\gamma \cos \frac{\pi t}{2\gamma}}, \tag{1.46}$$

and the exact value of constant $C > 0$ is not yet known.

Our main result in the present paper is the following theorem.

Theorem 1.2. *In the ferroelectric phase with $t > \gamma > 0$, for any $\varepsilon > 0$, as $n \to \infty$,*

$$Z_n = CG^n F^{n^2} \left[1 + O\left(e^{-n^{1-\varepsilon}}\right)\right], \tag{1.47}$$

where $C = 1 - e^{-4\gamma}$, $G = e^{\gamma - t}$, and $F = \sinh(t+\gamma)$. The error term in (1.41) is uniform on any compact subset of the set (1.43).

Up to a constant factor this result will follow from Theorem 1.1. To find the constant factor C we will use the Toda equation, combined with the asymptotics of C as $t \to \infty$. The proof of Theorems 1.1 and 1.2 will be given below in Sects. 2–6. Here we would like to make some remarks concerning asymptotics (1.47).

1.6. Ground state configuration of the ferroelectric phase. Let us compare asymptotics (1.47) with the energy of the ground state. The ground state is the configuration

$$\sigma^{\mathrm{gs}}(x) = \begin{cases} \sigma_5 & \text{if } x \text{ is on the diagonal,} \\ \sigma_3 & \text{if } x \text{ is above the diagonal,} \\ \sigma_4 & \text{if } x \text{ is below the diagonal,} \end{cases} \tag{1.48}$$

see Fig. 5. The weight of the ground state configuration is

$$w(\sigma^{\mathrm{gs}}) = b^{n^2} \left(\frac{c}{b}\right)^n = F^{n^2} G_0^n, \tag{1.49}$$

where

$$F = \sinh(t+\gamma), \qquad G_0 = \frac{\sinh(2\gamma)}{\sinh(t+\gamma)}. \tag{1.50}$$

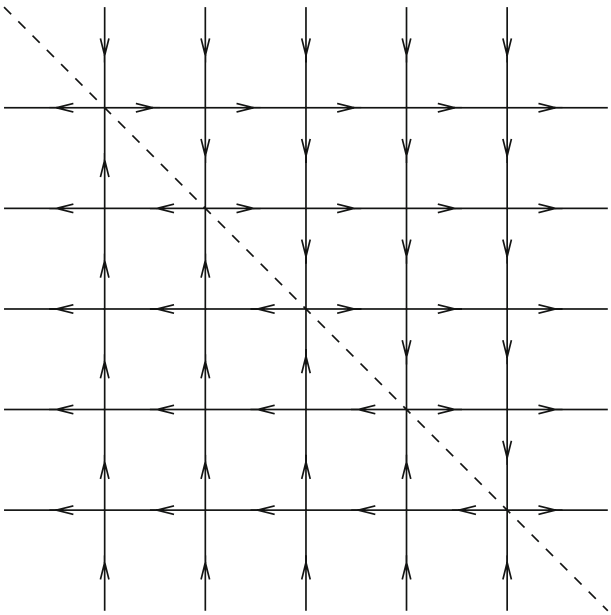


Fig. 5. A ground state configuration

By (1.47) the ratio $Z_n/w(\sigma^{\mathrm{gs}})$ is evaluated as

$$\frac{Z_n}{w(\sigma^{\mathrm{gs}})} = CG_1^n\left[1+O\left(e^{-n^{1-\varepsilon}}\right)\right], \tag{1.51}$$

where

$$G_1 = \frac{G}{G_0} = \frac{e^{\gamma-t}\sinh(t+\gamma)}{\sinh 2\gamma} = \frac{e^{2\gamma}-e^{-2t}}{e^{2\gamma}-e^{-2\gamma}} > 1. \tag{1.52}$$

Observe that "volume contribution", F^{n^2}, to the partition function coincides with the one to the energy of the ground state configuration, but the "surface contributions", G^n and G_0^n, are different. This indicates that low energy excited states in the ferroelectric phase are local perturbations of the ground state around the diagonal. Namely, it is impossible to create a new configuration by perturbing the ground state locally away of the diagonal: the conservation law $N_3(\sigma) = N_4(\sigma)$ forbids such a configuration, and a typical configuration of the six-vertex model in the ferroelectric phase is frozen outside of a relatively small neighborhood of the diagonal.

This behavior of typical configurations in the ferroelectric phase is in a big contrast with the situation in the disordered and anti-ferroelectric phases. Extensive rigorous, theoretical and numerical studies, see, e.g., the works of Cohn, Elkies, Propp [8], Eloranta [11], Syljuasen, Zvonarev [28], Allison, Reshetikhin [1], Kenyon, Okounkov [15], Kenyon, Okounkov, Sheffield [17], Sheffield [26], Ferrari, Spohn [12], Colomo, Pronko [10], Zinn-Justin [33], and references therein, show that in the disordered and anti-ferroelectric phases the "arctic circle" phenomenon persists, so that there are macroscopically big frozen and random domains in typical configurations, separated in the limit $n \to \infty$ by an "arctic curve".

It is worth noticing a different structure of the subleading terms in asymptotic formulae (1.44) and (1.47), which correspond to the disordered and ferroelectric phase regions, respectively. The presence of the pre-exponential, power-like term n^κ in formula (1.44) is an indication of the criticality of the disordered phase. The criticality of the disordered phase in the six-vertex model is also observed by Baxter [2], who relates it to an infinite degeneracy of the ground state of the transfer-matrix with periodic boundary conditions in the thermodynamic limit. In contrast, there is no power-like term in formula (1.47), which suggests that the ferroelectric phase is not critical. On the other hand, the presence of the surface term, G^n, in (1.47) shows the existence of a surface tension (under the domain wall boundary conditions) in the ferroelectric phase region, while (1.44) exhibits no surface tension in the disordered phase region. To obtain the exact value of the constant factor in the asymptotics of the partition function is usually a very difficult problem. As mentioned above, the exact value of the constant C in (1.47) does not follow from the large k asymptotics of h_k in (1.41), and it requires an additional study (see Sects. 5 and 6 below). The exact value of C in (1.44) is still not known. Finally, there is a noticeable difference in the asymptotic behavior of the error terms in formulae (1.44) and (1.47). Namely, as shown in [4], in formula (1.44), which corresponds to the disordered phase region, the error term is expanded in an asymptotic series in fractional powers of n, while the error term in (1.47) is (almost) exponentially small. This is also an indicator of a very different statistical behavior of typical configurations in the disordered and ferroelectric phases.

1.7. Order of the phase transition between the ferroelectric and disordered phases. We would like to compare the free energy in the disordered phase and in the ferroelectric phase when we approach a point of phase transition. Consider first the ferroelectric phase. Observe that $t, \gamma \to 0$ as we approach the line of phase transition,

$$\frac{b}{c} = \frac{a}{c} + 1, \quad a > 0, \tag{1.53}$$

hence $a, b, c \to 0$ in parametrization (1.24). Consider the regime,

$$t, \gamma \to +0, \qquad \frac{t}{\gamma} \to \alpha > 1. \tag{1.54}$$

In this regime,

$$\lim_{\gamma\to 0} \frac{b}{c} = \lim_{\gamma\to 0} \frac{\sinh(t+\gamma)}{\sinh(2\gamma)} = \frac{\alpha+1}{2}, \qquad \lim_{\gamma\to 0} \frac{a}{c} = \lim_{\gamma\to 0} \frac{\sinh(t-\gamma)}{\sinh(2\gamma)} = \frac{\alpha-1}{2}. \tag{1.55}$$

We have to rescale formula (1.47) according to (1.21),

$$Z_n\left(\frac{a}{c}, \frac{a}{c}, \frac{b}{c}, \frac{b}{c}, 1, 1\right) = c^{-n^2} Z_n(a, a, b, b, c, c) = CG^n F_0^{n^2}\left[1 + O\left(e^{-n^{1-\varepsilon}}\right)\right], \tag{1.56}$$

in the ferroelectric phase, where

$$F_0 = \frac{F}{c} = \frac{\sinh(t+\gamma)}{\sinh(2\gamma)}. \tag{1.57}$$

Similarly, in the disordered phase,

$$Z_n\left(\frac{a}{c}, \frac{a}{c}, \frac{b}{c}, \frac{b}{c}, 1, 1\right) = Cn^\kappa F_0^{n^2}[1 + O(n^{-\varepsilon})], \tag{1.58}$$

where

$$F_0 = \frac{F}{c} = \frac{\pi \sin(\gamma - t)\sin(\gamma + t)}{2\gamma \sin(2\gamma)\cos\frac{\pi t}{2\gamma}}. \tag{1.59}$$

Observe that parametrization (1.26) in the disordered phase is not convenient as we approach critical line (1.53). Namely, it corresponds to the limit when

$$t, \gamma \to \frac{\pi}{2} - 0, \qquad \frac{\frac{\pi}{2} - t}{\frac{\pi}{2} - \gamma} \to \alpha > 1. \tag{1.60}$$

Therefore, we replace t for $\frac{\pi}{2} - t$ and γ for $\frac{\pi}{2} - \gamma$. This gives the parametrization,

$$a = \sin(t - \gamma), \quad b = \sin(t + \gamma), \quad c = \sin(2|\gamma|), \quad |\gamma| < t. \tag{1.61}$$

The approach to critical line (1.53) is described by regime (1.54). Formula (1.59) reads in the new t, γ as

$$F_0 = \frac{\pi \sin(t - \gamma)\sin(t + \gamma)}{(\pi - 2\gamma)\sin(2\gamma)\cos\left[\frac{\pi(\frac{\pi}{2} - t)}{2(\frac{\pi}{2} - \gamma)}\right]}. \tag{1.62}$$

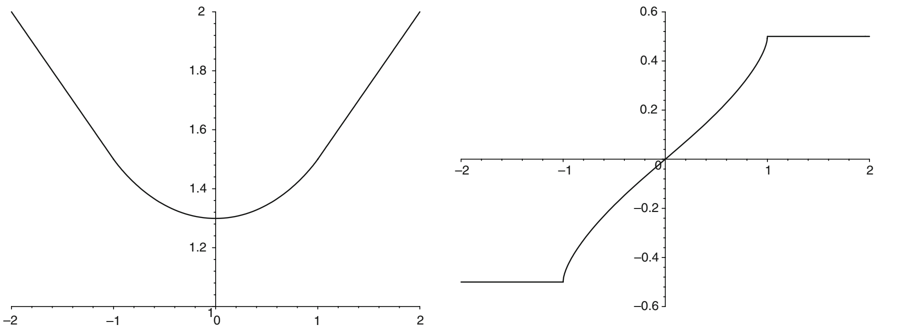


Fig. 6. Free energy $F_0 = F_0(\beta)$ (the left graph) and its derivative (the right graph), as functions of $\beta = \frac{b-a}{c}$ on the line $\frac{b+a}{c} = 2$

We consider F_0 on the line

$$\frac{a+b}{c} = \alpha, \tag{1.63}$$

and we use the parameter

$$\beta = \frac{b-a}{c} \tag{1.64}$$

on this line. In variables α, β,

$$F_0 = \frac{\alpha+\beta}{2} \quad \text{in the ferroelectric phase,} \tag{1.65}$$

and

$$F_0 = \frac{(\alpha+\beta)g(t,\gamma)}{2} \quad \text{in the disordered phase,} \tag{1.66}$$

where

$$g(t,\gamma) = \frac{\pi \sin(t-\gamma)}{(\pi - 2\gamma)\sin\left[\frac{\pi(t-\gamma)}{(\pi-2\gamma)}\right]}. \tag{1.67}$$

A straightforward calculation shows that on the line $\frac{a+b}{c} = \alpha$ in the disordered phase, as $\beta \to 1-0$,

$$g(t,\gamma) = 1 + \frac{2(\alpha-1)^{3/2}(1-\beta)^{3/2}}{3\pi(\alpha+1)^{1/2}} + O((1-\beta)^2). \tag{1.68}$$

By (1.65), $g(t,\gamma) = 1$ in the ferroelectric phase. This implies that the free energy F_0 exhibits a phase transition of the order $\frac{3}{2}$ with respect to the parameter β at the point $\beta = 1$. Figure 6 depicts the graph of $F_0 = F_0(\beta)$ (the left graph) and its derivative, $F_0'(\beta)$ (the right graph), as a function of $\beta = \frac{b-a}{c}$ on the line $\frac{b+a}{c} = 2$. Observe the square root singularities of F_0' at $\beta = \pm 1$, which correspond to the phase transition of order $\frac{3}{2}$. Since

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab} = \frac{\alpha^2 + \beta^2 - 2}{\alpha^2 - \beta^2} = 1 + \frac{4(\beta - 1)}{\alpha^2 - 1} + O((\beta - 1)^2), \qquad (1.69)$$

it is a phase transition of the order $\frac{3}{2}$ with respect to the parameter Δ as well, at the point $\Delta = 1$.

The set-up for the remainder of the article is the following. In Section 2 we will discuss the Meixner polynomials, which will serve as a good approximation to the polynomials $P_n(z)$. In Section 3 we will discuss the Riemann-Hilbert approach to discrete orthogonal polynomials, and we will derive a basic identity, which will be used in the proof of Theorem 1.1. In Section 4 we will prove Theorem 1.1. Then, in Sections 5 and 6 we will obtain an explicit formula for the constant factor C, and we will finish the proof of Theorem 1.2.

2. Meixner Polynomials

We will use the two weights: the weight $w(l)$ defined in (1.38) and the exponential weight on $\mathbb{N}$,

$$w^{\mathrm{Q}}(l) = q^l, \quad l \in \mathbb{N}; \qquad q = e^{2\gamma - 2t} < 1, \qquad (2.1)$$

which can be viewed as an approximation to $w(l)$ for large l. The orthogonal polynomials with the weight $w^{\mathrm{Q}}(l)$ are expressed in terms of the Meixner polynomials with $\beta = 1$, which are defined by the formula,

$$\begin{aligned} M_k(z;q) &= {}_2F_1\left(\begin{matrix} -k, -z \\ 1 \end{matrix}; 1 - q^{-1}\right) = \sum_{j=0}^{\infty} \frac{(-k)_j(-z)_j}{(1)_j} \frac{(1-q^{-1})^j}{j!} \\ &= \sum_{j=0}^{k} \frac{(1-q^{-1})^j \prod_{i=0}^{j-1}(k-i) \prod_{i=0}^{j-1}(z-i)}{(j!)^2}. \end{aligned} \qquad (2.2)$$

They satisfy the orthogonality condition,

$$\sum_{l=0}^{\infty} M_j(l;q) M_k(l;q) q^l = \frac{q^{-k}\delta_{jk}}{1-q}, \qquad (2.3)$$

see, e.g. [18]. For the corresponding monic polynomials,

$$P_k^{\mathrm{M}}(z) = \frac{k!}{(1-q^{-1})^k} M_k(z;q) \qquad (2.4)$$

(M in P_k^{M} stands for Meixner), the orthogonality condition reads

$$\sum_{l=0}^{\infty} P_j^{\mathrm{M}}(l) P_k^{\mathrm{M}}(l) q^l = h_k^{\mathrm{M}} \delta_{jk}, \qquad h_k^{\mathrm{M}} = \frac{(k!)^2 q^k}{(1-q)^{2k+1}}. \qquad (2.5)$$

They satisfy the three term recurrence relation,

$$zP_k^{\mathrm{M}}(z) = P_{k+1}^{\mathrm{M}}(z) + \frac{kq+k+q}{1-q}\, P_k^{\mathrm{M}}(z) + \frac{k^2 q}{(1-q)^2}\, P_{k-1}^{\mathrm{M}}(z), \tag{2.6}$$

see [18]. According to (2.1), we take $q = e^{2\gamma-2t}$.

For our purposes it is convenient to introduce a shifted Meixner polynomial,

$$Q_k(z) = P_k^{\mathrm{M}}(z-1) = \frac{(-1)^k k! q^k}{(1-q)^k} M_k(z-1;q), \tag{2.7}$$

which is a monic polynomial as well. Equation (2.5) implies the orthogonality condition,

$$\sum_{l=1}^{\infty} Q_j(l)Q_k(l)q^l = h_k^{\mathrm{Q}}\delta_{jk}, \qquad h_k^{\mathrm{Q}} = \frac{(k!)^2 q^{k+1}}{(1-q)^{2k+1}}. \tag{2.8}$$

By analogy with (1.40), define

$$\tau_n^{\mathrm{Q}} = 2^{n^2} \prod_{k=0}^{n-1} h_k^{\mathrm{Q}}. \tag{2.9}$$

From (2.8) we obtain that

$$\tau_n^{\mathrm{Q}} = 2^{n^2} \prod_{k=0}^{n-1} \frac{(k!)^2 q^{k+1}}{(1-q)^{2k+1}} = \frac{2^{n^2} q^{(n+1)n/2}}{(1-q)^{n^2}} \prod_{k=0}^{n-1} (k!)^2. \tag{2.10}$$

By analogy with (1.31), define also

$$Z_n^{\mathrm{Q}} = \frac{[\sinh(\gamma+t)\sinh(\gamma-t)]^{n^2}}{\prod_{k=0}^{n-1}(k!)^2}\, \tau_n^{\mathrm{Q}}. \tag{2.11}$$

Then from (2.10) we obtain that

$$Z_n^{\mathrm{Q}} = F^{n^2} G^n, \tag{2.12}$$

where

$$F = \frac{2\sinh(t-\gamma)\sinh(t+\gamma)q^{1/2}}{1-q} = \frac{2\sinh(t-\gamma)\sinh(t+\gamma)e^{\gamma-t}}{1-e^{2\gamma-2t}} = \sinh(t+\gamma), \tag{2.13}$$

and

$$G = q^{1/2} = e^{\gamma-t}. \tag{2.14}$$

Our goal will be to compare the normalizing constants for orthogonal polynomials with the weights w and w^{Q}. To this end let us discuss the Riemann-Hilbert approach to discrete orthogonal polynomials.

3. Riemann Hilbert Approach: Interpolation Problem

The Riemann-Hilbert approach to discrete orthogonal polynomials is based on the following Interpolation Problem (IP), which was introduced in the paper [6] of Borodin and Boyarchenko under the name of the discrete Riemann-Hilbert problem. See also the monograph [5] of Baik, Kriecherbauer, McLaughlin, and Miller, in which it is called the Interpolation Problem. Let $w(l) \geq 0$ be a weight function on $\mathbb{N}$ (it can be a more general discrete set, as discussed in [6 and 5], but we will need $\mathbb{N}$ in our problem).

Interpolation Problem. *For a given* $k = 0, 1, \ldots,$ *find a* 2×2 *matrix-valued function* $Y(z; k) = (Y_{ij}(z; k))_{1\leq i,j\leq 2}$ *with the following properties*:

(1) **Analyticity**: $Y(z; k)$ *is an analytic function of* z *for* $z \in \mathbb{C}\setminus\mathbb{N}$.

(2) **Residues at poles**: *At each node* $l \in \mathbb{N}$, *the elements* $Y_{11}(z; k)$ *and* $Y_{21}(z; k)$ *of the matrix* $Y(z; k)$ *are analytic functions of* z, *and the elements* $Y_{12}(z; k)$ *and* $Y_{22}(z; k)$ *have a simple pole with the residues*,

$$\operatorname*{Res}_{z=l} Y_{j2}(z; k) = w(l) Y_{j1}(l; k), \quad j = 1, 2. \tag{3.1}$$

(3) **Asymptotics at infinity**: *There exists a sequence* $\{r_l > 0,\ l = 1, 2, \ldots\}$ *such that*

$$\lim_{l\to\infty} r_l = 0. \tag{3.2}$$

and such that if $z \to \infty$ *outside of the set* $\bigcup_{l=1}^{\infty} D(l, r_l)$, *where* $D(a, r)$ *is a disk of radius* $r > 0$ *centered at* $a \in \mathbb{C}$, *then* $Y(z; k)$ *admits the asymptotic expansion*,

$$Y(z; k) \sim \left(I + \frac{Y_1}{z} + \frac{Y_2}{z^2} + \cdots\right) \begin{pmatrix} z^k & 0 \\ 0 & z^{-k} \end{pmatrix}. \tag{3.3}$$

It is not difficult to see (see [6] and [5]) that under some conditions on $w(l)$, the IP has a unique solution, which is

$$Y(z; k) = \begin{pmatrix} P_k(z) & C(wP_k)(z) \\ (h_{k-1})^{-1} P_{k-1}(z) & (h_{k-1})^{-1} C(wP_{k-1})(z) \end{pmatrix}, \tag{3.4}$$

where the Cauchy transformation C is defined by the formula,

$$C(f)(z) = \sum_{l=1}^{\infty} \frac{f(l)}{z - l}, \tag{3.5}$$

and $P_k(z) = z^k + \cdots$ are monic polynomials orthogonal with the weight $w(l)$, so that

$$\sum_{l=1}^{\infty} P_j(l) P_k(l) w(l) = h_j \delta_{jk}. \tag{3.6}$$

It follows from (3.4), that

$$h_k = [Y_1]_{12}, \tag{3.7}$$

where $[Y_1]_{12}$ is the (12)-element of the matrix Y_1, which is the coefficient at $\frac{1}{z}$ in asymptotic expansion (3.3) (see [6 and 5]). In what follows we will consider the solution $Y(z; k)$ for the weight $w(l)$, introduced in (1.38).

Let Y^{Q} be a solution to the IP with the exponential weight w^{Q},

$$Y^{\mathrm{Q}}(z;k) = \begin{pmatrix} Q_k(z) & C(w^{\mathrm{Q}}Q_k)(z) \\ (h^{\mathrm{Q}}_{k-1})^{-1}Q_{k-1}(z) & (h^{\mathrm{Q}}_{k-1})^{-1}C(w^{\mathrm{Q}}Q_{k-1})(z) \end{pmatrix}. \tag{3.8}$$

Consider the quotient matrix,

$$X(z;k) = Y(z;k)[Y^{\mathrm{Q}}(z;k)]^{-1}. \tag{3.9}$$

Observe that det $Y^{\mathrm{Q}}(z;k)$ has no poles and it approaches 1 as $z \to \infty$ outside of the disks $D(l,r_l)$, $l = 1,2,\ldots$, hence

$$\det Y^{\mathrm{Q}}(z;k) = 1. \tag{3.10}$$

Also,

$$X(z;k) \to I \quad \text{as } z \to \infty \text{ outside of the disks } D(l,r_l),\ l = 1,2,\ldots \tag{3.11}$$

This implies that the matrix X can be written as

$$X(z;k) = I + C[(w^{\mathrm{Q}} - w)R], \tag{3.12}$$

where

$$R(z) = \begin{pmatrix} (h^{\mathrm{Q}}_{k-1})^{-1}P_k(z)Q_{k-1}(z) & -P_k(z)Q_k(z) \\ (h_{k-1}h^{\mathrm{Q}}_{k-1})^{-1}P_{k-1}(z)Q_{k-1}(z) & -(h_{k-1})^{-1}P_{k-1}(z)Q_k(z) \end{pmatrix}. \tag{3.13}$$

From formula (3.7) and (3.12) we obtain that

$$h_k - h^{\mathrm{Q}}_k = -\sum_{l=1}^{\infty} P_k(l)Q_k(l)\,[w^{\mathrm{Q}}(l) - w(l)]. \tag{3.14}$$

We will use this identity to estimate $|h_k - h^{\mathrm{Q}}_k|$. Observe that formula (3.12) can be further used to evaluate the large n asymptotics of the orthogonal polynomials $P_n(z)$, but we will not pursue it here.

We would like to remark that identity (3.14) can be also derived as follows. Observe that since P_k and Q_k are monic polynomials, the difference, $P_k - Q_k$, is a polynomial of degree less than k, hence

$$\sum_{l=1}^{\infty} P_k(l)[Q_k(l) - P_k(l)]w(l) = 0. \tag{3.15}$$

By adding this to equation (3.6) with $j = k$, we obtain that

$$h_k = \sum_{l=1}^{\infty} P_k(l)Q_k(l)w(l). \tag{3.16}$$

Similarly, from (2.8) we obtain that

$$h^{\mathrm{Q}}_k = \sum_{l=1}^{\infty} P_k(l)Q_k(l)w^{\mathrm{Q}}(l). \tag{3.17}$$

By subtracting the last two equations, we obtain identity (3.14).

4. Evaluation of the Ratio h_k/h_k^{Q}

In this section we will prove Theorem 1.1. By applying the Cauchy-Schwarz inequality to identity (3.14), we obtain that

$$|h_k - h_k^{\mathrm{Q}}| \le \left[\sum_{l=1}^{\infty} P_k(l)^2\,|w(l) - w^{\mathrm{Q}}(l)|\right]^{1/2} \left[\sum_{l=1}^{\infty} Q_k(l)^2\,|w(l) - w^{\mathrm{Q}}(l)|\right]^{1/2}, \quad (4.1)$$

so that

$$\left|\frac{h_k}{h_k^{\mathrm{Q}}} - 1\right| \le \left[\frac{1}{h_k^{\mathrm{Q}}}\sum_{l=1}^{\infty} P_k(l)^2\,|w(l) - w^{\mathrm{Q}}(l)|\right]^{1/2} \left[\frac{1}{h_k^{\mathrm{Q}}}\sum_{l=1}^{\infty} Q_k(l)^2\,|w(l) - w^{\mathrm{Q}}(l)|\right]^{1/2}. \quad (4.2)$$

Since $0 < \gamma < t$, we obtain from (1.38) and (2.1) that

$$|w(l) - w^{\mathrm{Q}}(l)| = e^{-(2t+2\gamma)l} \le C_0 w(l), \quad l \ge 1; \qquad C_0 = \frac{1}{e^{4\gamma} - 1}, \quad (4.3)$$

hence

$$\frac{1}{h_k^{\mathrm{Q}}}\sum_{l=1}^{\infty} P_k(l)^2\,|w(l) - w^{\mathrm{Q}}(l)| \le C_0 \frac{1}{h_k^{\mathrm{Q}}}\sum_{l=1}^{\infty} P_k(l)^2 w(l) = \frac{C_0 h_k}{h_k^{\mathrm{Q}}} \le C_0(1 + \varepsilon_k), \quad (4.4)$$

where

$$\varepsilon_k = \left|\frac{h_k}{h_k^{\mathrm{Q}}} - 1\right|. \quad (4.5)$$

Thus, by (4.2),

$$\varepsilon_k^2 \le C_0(1 + \varepsilon_k)\delta_k, \quad (4.6)$$

where

$$\delta_k = \frac{1}{h_k^{\mathrm{Q}}}\sum_{l=1}^{\infty} Q_k(l)^2\,|w(l) - w^{\mathrm{Q}}(l)|. \quad (4.7)$$

By (4.3),

$$\delta_k = \frac{1}{h_k^{\mathrm{Q}}}\sum_{l=1}^{\infty} Q_k(l)^2\, q_0^l, \qquad q_0 = e^{-2(t+\gamma)}. \quad (4.8)$$

Let us evaluate δ_k.

We partition the sum in (4.8) into two parts:

$$\delta_k' = \frac{1}{h_k^Q} \sum_{l=1}^{L} Q_k(l)^2\, q_0^l, \tag{4.9}$$

and

$$\delta_k'' = \frac{1}{h_k^Q} \sum_{l=L+1}^{\infty} Q_k(l)^2\, q_0^l, \tag{4.10}$$

where

$$L = [k^\lambda], \qquad 0 < \lambda < 1. \tag{4.11}$$

Let us estimate first δ_k'. We have from (2.7), (2.8) that

$$\frac{Q_k(l)}{(h_k^Q)^{1/2}} = \frac{(-1)^k(1-q)^{1/2}q^{k/2}}{q^{1/2}} M_k(l-1;q). \tag{4.12}$$

By (2.2),

$$\begin{aligned} M_k(l-1;q) &= 1 + (1-q^{-1})k(l-1) + (1-q^{-1})^2\frac{k(k-1)(l-1)(l-2)}{(2!)^2} \\ &\quad + (1-q^{-1})^3\frac{k(k-1)(k-2)(l-1)(l-2)(l-3)}{(3!)^2} + \cdots . \end{aligned} \tag{4.13}$$

If $l < k$, then the latter sum consists of l nonzero terms. For $l \le L$ it is estimated as

$$M_k(l-1;q) = O(k^L L^{L+1}) = O(e^{L\ln k + (L+1)\ln L}), \tag{4.14}$$

hence

$$\frac{Q_k(l)}{(h_k^Q)^{1/2}} = O(e^{\frac{k\ln q}{2} + L\ln k + (L+1)\ln L}). \tag{4.15}$$

Due to our choice of L in (4.11), this implies the estimate,

$$\frac{Q_k(l)}{(h_k^Q)^{1/2}} = O(e^{\frac{k\ln q}{2} + 2k^\lambda \ln k}). \tag{4.16}$$

Since $0 < q < 1$ and $0 < \lambda < 1$, the expression on the right is exponentially small as $k \to \infty$. From (4.9) we obtain now that

$$\delta_k' = O(e^{k\ln q + 4k^\lambda \ln k}). \tag{4.17}$$

Since $\lambda < 1$ and $q < 1$, we obtain that

$$\delta_k' = O(e^{-c_0 k}), \qquad c_0 = -\frac{\ln q}{2} > 0. \tag{4.18}$$

Let us estimate δ_k''.

By (2.8),

$$\frac{1}{h_k^{\mathrm{Q}}} \sum_{l=1}^{\infty} Q_k(l)^2 q^l = 1, \tag{4.19}$$

hence

$$\delta_k'' = \frac{1}{h_k^{\mathrm{Q}}} \sum_{l=L+1}^{\infty} Q_k(l)^2 \, q_0^l < \left(\frac{q_0}{q}\right)^L \frac{1}{h_k^{\mathrm{Q}}} \sum_{l=L+1}^{\infty} Q_k(l)^2 \, q^l < \left(\frac{q_0}{q}\right)^L = e^{-4\gamma L}. \tag{4.20}$$

Thus,

$$\delta_k'' < e^{-4\gamma(k^\lambda - 1)}. \tag{4.21}$$

Since $0 < \lambda < 1$ is an arbitrary number, we obtain from (4.18) and (4.21) that for any $\eta > 0$,

$$\delta_k = O\left(e^{-k^{1-\eta}}\right). \tag{4.22}$$

Let us return back to inequality (4.6). Consider two cases: (1) $\varepsilon_k > 1$ and (2) $\varepsilon_k \le 1$. In the first case (4.6) implies that

$$\varepsilon_k \le 2C_0\delta_k, \tag{4.23}$$

which is impossible, because of (4.22). Hence $\varepsilon_k \le 1$, in which case (4.6) gives that

$$\varepsilon_k^2 \le 2C_0\delta_k. \tag{4.24}$$

Estimate (4.22) implies now that for any $\eta > 0$,

$$\varepsilon_k = O\left(e^{-k^{1-\eta}}\right), \tag{4.25}$$

so that as $k \to \infty$,

$$h_k = h_k^{\mathrm{Q}}(1 + \tilde{\varepsilon}_k), \qquad |\tilde{\varepsilon}_k| = \varepsilon_k = O\left(e^{-k^{1-\eta}}\right). \tag{4.26}$$

This proves Theorem 1.1.

From (4.26) we obtain that for any $\eta > 0$,

$$Z_n = Z_n^{\mathrm{Q}} \prod_{k=0}^{n-1} (1 + \tilde{\varepsilon}_k) = C Z_n^{\mathrm{Q}} \left[1 + O\left(e^{-n^{1-\eta}}\right)\right], \tag{4.27}$$

where

$$\infty > C = \prod_{k=0}^{\infty} (1 + \tilde{\varepsilon}_k) > 0. \tag{4.28}$$

Thus, we have proved the following result.

Proposition 4.1. *For any $\varepsilon > 0$, as $n \to \infty$,*

$$Z_n = CF^{n^2}G^n\left[1 + O\left(e^{-n^{1-\varepsilon}}\right)\right], \tag{4.29}$$

where $C > 0$, $F = \sinh(t + \gamma)$, and $G = e^{\gamma - t}$.

To finish the proof of Theorem 1.2, it remains to find the constant C.

5. Evaluation of the Constant Factor

In the next two sections we will find the exact value of the constant C in formula (4.29). This will be done in two steps: first, with the help of the Toda equation, we will find the form of the dependence of C on t, and second, we will find the large t asymptotics of C. By combining these two steps, we will obtain the exact value of C. In this section we will carry out the first step of our program.

By dividing the Toda equation, (1.34), by τ_n^2, we obtain that

$$\frac{\tau_n \tau_n'' - \tau_n'^2}{\tau_n^2} = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}, \qquad (') = \frac{\partial}{\partial t}. \tag{5.1}$$

The left-hand side can be written as

$$\frac{\tau_n \tau_n'' - \tau_n'^2}{\tau_n^2} = \left(\frac{\tau_n'}{\tau_n}\right)' = (\ln \tau_n)''. \tag{5.2}$$

From (1.40) we obtain that

$$\frac{\tau_{n+1}}{\tau_n} = 2^{2n+1}h_n, \tag{5.3}$$

hence Eq. (5.1) implies that

$$(\ln \tau_n)'' = \frac{4h_n}{h_{n-1}}. \tag{5.4}$$

From (1.41) we obtain that

$$\frac{4h_n}{h_{n-1}} = \frac{4n^2 q}{(1-q)^2} + O\left(e^{-n^{1-\varepsilon}}\right). \tag{5.5}$$

We have that

$$\frac{4q}{(1-q)^2} = \frac{4e^{2\gamma-2t}}{(1-e^{2\gamma-2t})^2} = \left[\frac{(-2)}{1-e^{2\gamma-2t}}\right]' = \left[-\ln(1-e^{2\gamma-2t})\right]'', \tag{5.6}$$

hence from (5.4), (5.5) we obtain that

$$(\ln \tau_n)'' = n^2\left[-\ln(1-e^{2\gamma-2t})\right]'' + O\left(e^{-n^{1-\varepsilon}}\right). \tag{5.7}$$

By (1.31) this implies that

$$(\ln Z_n)'' = n^2\left[\ln \frac{\sinh(t-\gamma)\sinh(t+\gamma)}{1-e^{2\gamma-2t}}\right]'' + O\left(e^{-n^{1-\varepsilon}}\right). \tag{5.8}$$

Since

$$\ln \frac{\sinh(t-\gamma)\sinh(t+\gamma)}{1-e^{2\gamma-2t}} = \ln[\sinh(t+\gamma)] + (t-\gamma) - \ln 2, \tag{5.9}$$

we can simplify (5.8) to

$$(\ln Z_n)'' = n^2 \left[\ln \sinh(t+\gamma)\right]'' + O\left(e^{-n^{1-\varepsilon}}\right). \tag{5.10}$$

Observe that the error term in the last formula is uniform when t belongs to a compact set on (γ, ∞), hence by integrating it we obtain that

$$\ln Z_n = n^2 \ln \sinh(t+\gamma) + c_1 t + c_0 + O\left(e^{-n^{1-\varepsilon}}\right), \tag{5.11}$$

where c_0, c_1 do not depend on t. In general, c_0, c_1 depend on γ and n. By substituting formula (4.29) into the preceding equation, we obtain that

$$\ln C + n(\gamma - t) = c_1 t + c_0 + O\left(e^{-n^{1-\varepsilon}}\right). \tag{5.12}$$

Denote

$$d_0 = c_0 - n\gamma, \qquad d_1 = c_1 + n. \tag{5.13}$$

Then Eq. (5.12) reads

$$\ln C = d_1 t + d_0 + O\left(e^{-n^{1-\varepsilon}}\right). \tag{5.14}$$

Observe that $C = C(\gamma, t)$ does not depend on n, while $d_j = d_j(\gamma, n)$ does not depend on t, $j = 1, 2$. Take any $0 < \gamma < t_1 < t_2$. Then

$$\ln C(\gamma, t_2) - \ln C(\gamma, t_1) = d_1(t_2 - t_1) + O\left(e^{-n^{1-\varepsilon}}\right). \tag{5.15}$$

From this formula we obtain that the limit,

$$\lim_{n\to\infty} d_1(\gamma, n) = d_1(\gamma), \tag{5.16}$$

exists. This in turn implies that the limit,

$$\lim_{n\to\infty} d_2(\gamma, n) = d_2(\gamma), \tag{5.17}$$

exists. By taking the limit $n \to \infty$ in (5.14), we obtain that

$$\ln C = d_1(\gamma)t + d_0(\gamma). \tag{5.18}$$

Thus we have proved the following result.

Proposition 5.1. *The constant C in asymptotic formula (4.29) has the form*

$$C = e^{d_1(\gamma)t + d_0(\gamma)}. \tag{5.19}$$

6. Explicit Formula for C

In this section we will find the exact value of C, and by doing this we will finish the proof of Theorem 1.2. Let us consider the following regime:

$$\gamma > 0 \text{ is fixed}, \quad t \to \infty, \tag{6.1}$$

and let us evaluate the asymptotics of C in this regime. By (3.6) and (1.38) we have that

$$h_0 = \sum_{l=1}^{\infty} w(l) = \sum_{l=1}^{\infty} \left(e^{-2tl+2\gamma l} - e^{-2tl-2\gamma l} \right) = \frac{e^{-2t+2\gamma}}{1 - e^{-2t+2\gamma}} - \frac{e^{-2t-2\gamma}}{1 - e^{-2t-2\gamma}}. \tag{6.2}$$

Similarly, by (2.8),

$$h_0^{\mathrm{Q}} = \frac{e^{-2t+2\gamma}}{1 - e^{-2t+2\gamma}}, \tag{6.3}$$

hence

$$\frac{h_0}{h_0^{\mathrm{Q}}} = 1 - e^{-4\gamma} + O(e^{-2t}), \qquad t \to \infty. \tag{6.4}$$

Let us evaluate $\varepsilon_k = \left| \frac{h_k}{h_k^{\mathrm{Q}}} - 1 \right|$ for $k \geq 1$.

By (4.6),

$$\varepsilon_k^2 \leq C_0(1 + \varepsilon_k)\delta_k, \qquad C_0 = \frac{1}{e^{4\gamma} - 1}. \tag{6.5}$$

In the partition of δ_k as $\delta_k' + \delta_k''$ in (4.9), (4.10), let us choose

$$L = [k^{2/3} + t^{2/3}]. \tag{6.6}$$

From (4.12), (4.13) we obtain that for $l \leq L$,

$$\frac{|Q_k(l)|}{(h_k^{\mathrm{Q}})^{1/2}} \leq q^{(k-1)/2} k^L L^{L+1}, \qquad q = e^{2\gamma - 2t}, \tag{6.7}$$

hence

$$\delta_k' \leq \frac{q_0 q^{k-1} k^L L^{L+1}}{1 - q_0} \leq \frac{q^k k^L L^{L+1}}{1 - q_0}, \qquad q_0 = e^{-2\gamma - 2t}. \tag{6.8}$$

In addition, by (4.20),

$$\delta_k'' \le e^{-4\gamma L}. \tag{6.9}$$

Our choice of L in (6.6) ensures that there exists $t_0 > 0$ such that for any $t \ge t_0$ and any $k \ge 1$,

$$\delta_k = \delta_k' + \delta_k'' \le e^{-k^{1/2} - t^{1/2}}. \tag{6.10}$$

From (6.5) we obtain now that for $k \ge 1$ and large t,

$$\varepsilon_k \le C_1 e^{-\frac{k^{1/2}}{2} - \frac{t^{1/2}}{2}}, \qquad C_1 = (2C_0)^{1/2}. \tag{6.11}$$

By (4.28),

$$\ln C = \sum_{k=0}^{\infty} \ln(1 + \tilde{\varepsilon}_k), \qquad |\tilde{\varepsilon}_k| = \varepsilon_k. \tag{6.12}$$

From Eqs. (6.4) and (6.11) we obtain now that

$$\ln C = \ln(1 - e^{-4\gamma}) + O(e^{-\frac{t^{1/2}}{2}}), \qquad t \to \infty. \tag{6.13}$$

On the other hand, by (5.14)

$$\ln C = d_1(\gamma)t + d_0(\gamma). \tag{6.14}$$

This implies that

$$d_1(\gamma) = 0, \qquad d_0(\gamma) = \ln(1 - e^{-4\gamma}), \tag{6.15}$$

so that

$$C = 1 - e^{-4\gamma}. \tag{6.16}$$

By substituting expression (6.16) into formula (4.29), we prove Theorem 1.2.

Appendix A. Derivation of Formula (1.40)

Multilinearity of the determinant function, combined with the form of the Vandermonde matrix, allows us to replace $\Delta(l)$ with

$$\det \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ P_1(l_1) & P_1(l_2) & P_1(l_3) & \cdots & P_1(l_n) \\ P_2(l_1) & P_2(l_2) & P_2(l_3) & \cdots & P_2(l_n) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{n-1}(l_1) & P_{n-1}(l_2) & P_{n-1}(l_3) & \cdots & P_{n-1}(l_n) \end{pmatrix}, \tag{A.1}$$

where $\{P_j(x)\}_{j=0}^{\infty}$ is the system of monic polynomials orthogonal with respect to the weight $w(l)$. Then (1.36) becomes

$$\tau_n = \frac{2^{n^2}}{n!} \sum_{l_1,\ldots,l_n=1}^{\infty} \left(\sum_{\pi \in S_n} (-1)^{\pi} \prod_{k=1}^{n} P_{\pi(k)-1}(l_k) \right)^2 \prod_{k=1}^{n} w(l_k). \tag{A.2}$$

Note that the orthogonality condition ensures that, after summing, only diagonal terms are non-zero, so we get

$$\tau_n = \frac{2^{n^2}}{n!} \sum_{l_1,\ldots,l_n=1}^{\infty} \left(\sum_{\pi \in S_n} \prod_{k=1}^{n} P^2_{\pi(k)-1}(l_k) \right) \prod_{k=1}^{n} w(l_k) = 2^{n^2} \prod_{k=0}^{n-1} h_k. \tag{A.3}$$

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