

Exact Solution of the Six-Vortex Model with Domain Wall Boundary Conditions: Antiferroelectric Phase

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Abstract

We obtain the large- n asymptotics of the partition function Z_n of the six-vertex model with domain wall boundary conditions in the antiferroelectric phase region, with the weights $a = \sinh(\gamma - t)$, $b = \sinh(\gamma + t)$, $c = \sinh(2\gamma)$, $|t| < \gamma$. We prove the conjecture of Zinn-Justin, that as $n \rightarrow \infty$, $Z_n = C \vartheta_4(n\omega) F^{n^2} [1 + O(n^{-1})]$, where ω and F are given by explicit expressions in γ and t , and $\vartheta_4(z)$ is the Jacobi theta function. The proof is based on the Riemann-Hilbert approach to the large- n asymptotic expansion of the underlying discrete orthogonal polynomials and on the Deift-Zhou nonlinear steepest-descent method. © 2009 Wiley Periodicals, Inc.

1 Introduction and Formulation of the Main Result

1.1 Definition of the Model

The six-vertex model, or the model of two-dimensional ice, is stated on a square $n \times n$ lattice with arrows on the edges. The arrows obey the rule that at every vertex there are two arrows pointing in and two arrows pointing out. Such a rule is sometimes called the *ice rule*. There are only six possible configurations of arrows at each vertex, hence the name of the model; see Figure 1.1.

We will consider the *domain wall boundary conditions* (DWBC), in which the arrows on the upper and lower boundaries point into the square, and the ones on the left and right boundaries point out. One possible configuration with DWBC on the 4×4 lattice is shown on Figure 1.2.

For each possible vertex state we assign a weight w_i , $i = 1, \dots, 6$, and define, as usual, the partition function as a sum over all possible arrow configurations of the product of the vertex weights,

$$(1.1) \quad Z_n = \sum_{\text{arrow configurations } \sigma} w(\sigma), \quad w(\sigma) = \prod_{x \in V_n} w_{t(x;\sigma)} = \prod_{i=1}^6 w_i^{N_i(\sigma)},$$

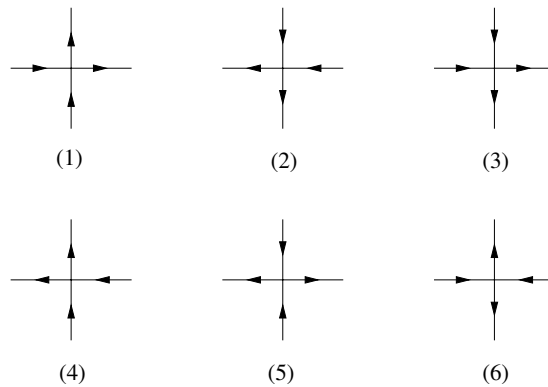
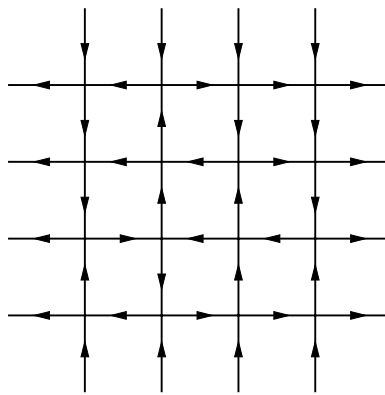


FIGURE 1.1. The six arrow configurations allowed at a vertex.

FIGURE 1.2. An example of a 4×4 configuration with domain wall boundary conditions (DWBC).

where V_n is the $n \times n$ set of vertices, $t(x; \sigma) \in \{1, \dots, 6\}$ is the vertex type of configuration σ at vertex x according to Figure 1.1, and $N_i(\sigma)$ is the number of vertices of type i in the configuration σ . The sum is taken over all possible configurations obeying the given boundary condition. The Gibbs measure is defined then as

$$(1.2) \quad \mu_n(\sigma) = \frac{w(\sigma)}{Z_n}.$$

Our main goal is to obtain the large- n asymptotics of the partition function Z_n .

The six-vertex model has six parameters: the weights w_i . By using some conservation laws it can be reduced to only two parameters (see, e.g., [1, 8, 14]). Namely, we have that

$$(1.3) \quad Z_n(w_1, w_2, w_3, w_4, w_5, w_6) = C(n)Z_n(a, a, b, b, c, c)$$

and

$$(1.4) \quad \mu_n(\sigma; w_1, w_2, w_3, w_4, w_5, w_6) = \mu_n(\sigma; a, a, b, b, c, c),$$

where

$$(1.5) \quad a = \sqrt{w_1 w_2}, \quad b = \sqrt{w_3 w_4}, \quad c = \sqrt{w_5 w_6},$$

and

$$(1.6) \quad C(n) = \left(\frac{w_5}{w_6} \right)^{n/2}.$$

Furthermore,

$$(1.7) \quad Z_n(a, a, b, b, c, c) = c^{n^2} Z_n \left(\frac{a}{c}, \frac{a}{c}, \frac{b}{c}, \frac{b}{c}, 1, 1 \right)$$

and

$$(1.8) \quad \mu_n(\sigma; a, a, b, b, c, c) = \mu_n \left(\sigma; \frac{a}{c}, \frac{a}{c}, \frac{b}{c}, \frac{b}{c}, 1, 1 \right),$$

so that a general weight reduces to the two parameters $\frac{a}{c}$ and $\frac{b}{c}$.

1.2 Exact Solution of the Six-Vortex Model for a Finite n

Introduce the parameter

$$(1.9) \quad \Delta = \frac{a^2 + b^2 - c^2}{2ab}.$$

The phase diagram of the six-vertex model consists of three phase regions: the ferroelectric phase region, $\Delta > 1$; the antiferroelectric phase region, $\Delta < -1$; and the disordered phase region, $-1 < \Delta < 1$. Observe that $|a - b| > c$ in the ferroelectric phase region and $c > a + b$ in the antiferroelectric phase region, while in the disordered phase region a, b, c satisfy the triangle inequalities. In the three phase regions we parametrize the weights in the standard way: for the ferroelectric phase,

$$(1.10) \quad a = \sinh(t - \gamma), \quad b = \sinh(t + \gamma), \quad c = \sinh(2|\gamma|), \quad 0 < |\gamma| < t;$$

for the antiferroelectric phase,

$$(1.11) \quad a = \sinh(\gamma - t), \quad b = \sinh(\gamma + t), \quad c = \sinh(2\gamma), \quad |t| < \gamma;$$

and for the disordered phase

$$(1.12) \quad a = \sin(\gamma - t), \quad b = \sin(\gamma + t), \quad c = \sin(2\gamma), \quad |t| < \gamma.$$

The phase diagram of the six-vertex model is shown in Figure 1.3. The phase diagram and the Bethe ansatz solution of the six-vertex model for periodic and antiperiodic boundary conditions are thoroughly discussed in the works of Lieb [21, 22, 23, 24], Lieb and Wu [25], Sutherland [28], Baxter [4], and Batchelor, Baxter, O'Rourke, and Yung [3]. See also the work of Wu and Lin [30], in which the Pfaffian solution for the six-vertex model with periodic boundary conditions is obtained on the free fermion line, $\Delta = 0$. Brascamp, Kunz, and Wu [10] prove the equality of the free energy with periodic and free boundary conditions under

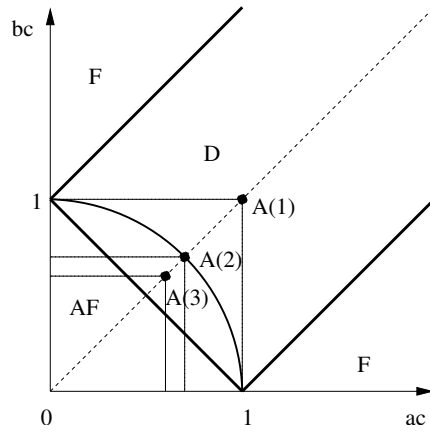


FIGURE 1.3. The phase diagram of the model, where F, AF, and D mark ferroelectric, antiferroelectric, and disordered phases, respectively. The circular arc corresponds to the so-called free fermion line, when $\Delta = 0$, and the three dots correspond to 1-, 2-, and 3-enumeration of alternating sign matrices.

various conditions on a, b, c , and they also prove the existence of the spontaneous staggered polarization for sufficiently small values of the parameters $\frac{a}{c}$ and $\frac{b}{c}$.

In this paper we will discuss the *antiferroelectric phase region*, and we will use parametrization (1.11). The parameter Δ in the antiferroelectric phase region reduces to

$$(1.13) \quad \Delta = -\cosh(2\gamma).$$

The six-vertex model with DWBC was introduced by Korepin [17], who derived important recursion relations for the partition function of the model. These recursion relations were solved by Izergin [15], and this led to a beautiful determinantal formula for the partition function with DWBC. A detailed proof of this formula, usually called the Izergin-Korepin formula, and its generalizations are given in the paper of Izergin, Coker, and Korepin [16]. When the weights are parametrized according to (1.11), the Izergin-Korepin formula is

$$(1.14) \quad Z_n = \frac{[\sinh(\gamma - t) \sinh(\gamma + t)]^{n^2} \tau_n}{(\prod_{j=0}^{n-1} j!)^2},$$

where τ_n is the Hankel determinant,

$$(1.15) \quad \tau_n = \det \left(\frac{d^{j+k-2} \phi}{dt^{j+k-2}} \right)_{1 \leq j, k \leq n},$$

and

$$(1.16) \quad \phi(t) = \frac{\sinh(2\gamma)}{\sinh(\gamma - t) \sinh(\gamma + t)}.$$

An elegant derivation of the Izergin-Korepin formula from the Yang-Baxter equations is given in the papers of Korepin and Zinn-Justin [18] and Kuperberg [20] (see also the book of Bressoud [11]).

One of the applications of the determinantal formula is that it implies that the function τ_n solves the Toda equation

$$(1.17) \quad \tau_n \tau_n'' - \tau_n'^2 = \tau_{n+1} \tau_{n-1}, \quad n \geq 1, \quad (\prime) = \frac{\partial}{\partial t};$$

compare the work of Sogo [27]. The Toda equation was used by Korepin and Zinn-Justin [18] to derive the free energy of the six-vertex model with DWBC, assuming some ansatz on the behavior of subdominant terms in the large- n asymptotics of the free energy.

Another application of the Izergin-Korepin formula is that τ_n can be expressed in terms of a partition function of a random matrix model; see the paper [31] of Zinn-Justin. Namely, let us write $\phi(t)$ in the form of the Laplace transform of a discrete measure,

$$(1.18) \quad \phi(t) = \frac{\sinh(2\gamma)}{\sinh(\gamma - t) \sinh(\gamma + t)} = 2 \sum_{l=-\infty}^{\infty} e^{2tl - 2\gamma|l|}.$$

Then

$$(1.19) \quad \tau_n = \frac{2^{n^2}}{n!} \sum_{l_1, \dots, l_n = -\infty}^{\infty} \Delta(l)^2 \prod_{i=1}^n e^{2tl_i - 2\gamma|l_i|},$$

where

$$(1.20) \quad \Delta(l) = \prod_{1 \leq i < j \leq n} (l_j - l_i)$$

is the Vandermonde determinant. For a proof see [31] or appendix A in [6]. We omit the proof of this and some other formulae in the paper, due to a publishing limitation on the length of the paper. For the proofs, see [6].

Introduce now discrete monic polynomials $P_j(x) = x^j + \dots$ orthogonal on the set \mathbb{Z} with respect to the weight,

$$(1.21) \quad w(l) = e^{2tl - 2\gamma|l|},$$

so that

$$(1.22) \quad \sum_{l=-\infty}^{\infty} P_j(l) P_k(l) w(l) = h_k \delta_{jk}.$$

Then it follows from (1.19) that

$$(1.23) \quad \tau_n = 2^{n^2} \prod_{k=0}^{n-1} h_k;$$

see appendix B of [6].

1.3 Rescaling of the Weight

Set

$$(1.24) \quad \Delta_n = \frac{2\gamma}{n}, \quad x = l\Delta_n, \quad w_n(x) = e^{-n(|x|-\xi x)}, \quad \xi = \frac{t}{\gamma},$$

and

$$(1.25) \quad P_{nk}(x) = \Delta_n^k P_k\left(\frac{x}{\Delta_n}\right).$$

Consider also the lattice

$$(1.26) \quad L_n = \left\{x = \frac{2\gamma k}{n}, k \in \mathbb{Z}\right\}.$$

Then from (1.22) we obtain that the monic polynomials $P_{nk}(x)$ satisfy the orthogonality condition,

$$(1.27) \quad \sum_{x \in L_n} P_{nj}(x) P_{nk}(x) w_n(x) \Delta_n = h_{nk} \delta_{jk}, \quad h_{nk} = h_k \Delta_n^{2k+1}.$$

We can then combine equations (1.14), (1.23), and (1.27) to obtain

$$(1.28) \quad Z_n = \left(\frac{nab}{\gamma}\right)^{n^2} \prod_{k=0}^{n-1} \frac{h_{nk}}{(k!)^2}, \quad a = \sinh(\gamma - t), \quad b = \sinh(\gamma + t).$$

For what follows we will need to extend the weight $w_n(x)$ to the complex plane. We do so by defining $w_n(z)$ on the complex plane as

$$(1.29) \quad w_n(z) = e^{-nV(z)}$$

where

$$(1.30) \quad V(z) = \begin{cases} z - \zeta z & \text{when } \operatorname{Re} z \geq 0, \\ -z - \zeta z & \text{when } \operatorname{Re} z \leq 0, \end{cases}$$

so that $V(z)$, and thus $w_n(z)$, is two-valued on the imaginary axis.

1.4 Main Result: Asymptotics of the Partition Function

This work is a continuation of the work [5] of the first author with Vladimir Fokin and [7, 8] by the authors of the present work. In [5] the authors obtain the large- n asymptotics of the partition function Z_n in the disordered phase. They prove the conjecture of Paul Zinn-Justin [31] that the large- n asymptotics of Z_n in the disordered phase has the following form: For some $\varepsilon > 0$,

$$(1.31) \quad Z_n = C n^\kappa F^{n^2} [1 + O(n^{-\varepsilon})].$$

Furthermore, they find the exact value of the exponent κ ,

$$(1.32) \quad \kappa = \frac{1}{12} - \frac{2\gamma^2}{3\pi(\pi - 2\gamma)}.$$

The value of F in the disordered phase is given by the formula

$$(1.33) \quad F = \frac{\pi ab}{2\gamma \cos \frac{\pi t}{2\gamma}}, \quad a = \sin(\gamma - t), \quad b = \sin(\gamma + t),$$

in parametrization (1.12).

In the work [8] we obtain the following large- n asymptotic formula for Z_n in the ferroelectric phase region: For any $\varepsilon > 0$,

$$(1.34) \quad Z_n = C G^n F^{n^2} [1 + O(e^{-n^{1-\varepsilon}})],$$

where $C = 1 - e^{-4\gamma}$, $G = e^{\gamma-t}$, and $F = \sinh(t + \gamma)$ in parametrization (1.10).

Finally, in the work [7] we obtain the following large- n asymptotic formula for Z_n on the borderline between the ferroelectric and disordered phase regions:

$$(1.35) \quad Z_n(a, a, a + 1, a + 1, 1, 1) = C n^\kappa G^{\sqrt{n}} F^{n^2} [1 + O(n^{-1/2})],$$

where $C > 0$,

$$(1.36) \quad \kappa = \frac{1}{4}, \quad G = \exp \left[-\zeta \left(\frac{3}{2} \right) \sqrt{\frac{a}{\pi}} \right], \quad F = a + 1,$$

and ζ is the Riemann zeta function.

In the present paper we obtain the large- n asymptotic formula for Z_n in the anti-ferroelectric phase region. The formulation of the main result of the present paper and the proofs involve the Jacobi theta functions. Let us review their definition and basic properties.

There are four Jacobi theta functions:

$$(1.37) \quad \begin{aligned} \vartheta_1(z) &= 2 \sum_{n=0}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} \sin((2n+1)z), \\ \vartheta_2(z) &= 2 \sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2} \cos((2n+1)z), \\ \vartheta_3(z) &= 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos(2nz), \\ \vartheta_4(z) &= 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos(2nz), \end{aligned}$$

where q is the *elliptic nome*. We will assume that $1 > q > 0$. Figure 1.4 shows the graphs of ϑ_1 and ϑ_2 (left figure) and ϑ_3 and ϑ_4 (right figure) on the interval $[0, \pi]$ for $q = 0.5$. Observe that ϑ_1 and ϑ_4 are increasing on $[0, \frac{\pi}{2}]$ while ϑ_2 and ϑ_3 are decreasing on this interval.

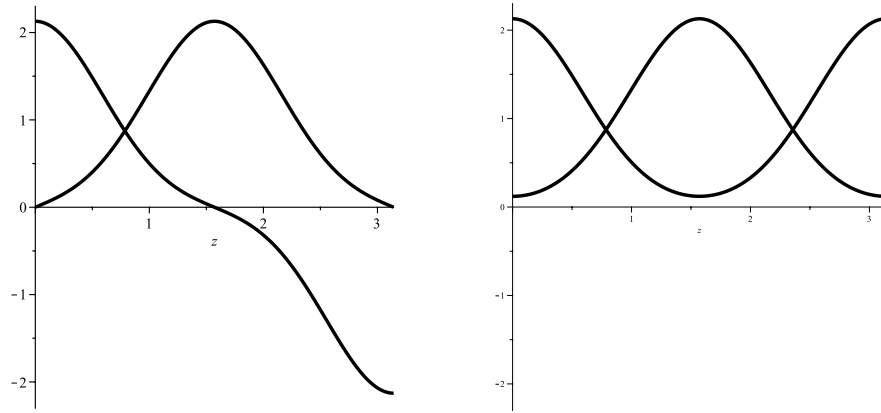


FIGURE 1.4. The graphs of ϑ_1 and ϑ_2 (left figure) and ϑ_3 and ϑ_4 (right figure) on the interval $[0, \pi]$ for $q = 0.5$.

The Jacobi theta functions satisfy the following periodicity conditions:

$$(1.38) \quad \begin{aligned} \vartheta_1(z + \pi) &= -\vartheta_1(z), & \vartheta_1(z + \pi\tau) &= -e^{-2iz}q^{-1}\vartheta_1(z), \\ \vartheta_2(z + \pi) &= -\vartheta_2(z), & \vartheta_2(z + \pi\tau) &= e^{-2iz}q^{-1}\vartheta_2(z), \\ \vartheta_3(z + \pi) &= \vartheta_3(z), & \vartheta_3(z + \pi\tau) &= e^{-2iz}q^{-1}\vartheta_3(z), \\ \vartheta_4(z + \pi) &= \vartheta_4(z), & \vartheta_4(z + \pi\tau) &= -e^{-2iz}q^{-1}\vartheta_4(z), \end{aligned}$$

where τ is a pure imaginary number related to q by the equation

$$(1.39) \quad q = e^{i\pi\tau}.$$

The theta functions also satisfy the symmetry conditions

$$(1.40) \quad \begin{aligned} \vartheta_1(-z) &= -\vartheta_1(z), & \vartheta_2(-z) &= \vartheta_2(z), \\ \vartheta_3(-z) &= \vartheta_3(z), & \vartheta_4(-z) &= \vartheta_4(z), \end{aligned}$$

and the equations

$$(1.41) \quad \begin{aligned} \vartheta_1(z) &= \vartheta_2\left(z - \frac{\pi}{2}\right), & \vartheta_3(z) &= \vartheta_4\left(z + \frac{\pi}{2}\right), \\ \vartheta_1(z) &= -ie^{iz + \frac{i\pi\tau}{4}}\vartheta_4\left(z + \frac{\pi\tau}{2}\right). \end{aligned}$$

The only zeroes of the theta functions are

$$(1.42) \quad \vartheta_1(0) = 0, \quad \vartheta_2\left(\frac{\pi}{2}\right) = 0, \quad \vartheta_3\left(\frac{\pi}{2} + \frac{\pi\tau}{2}\right) = 0, \quad \vartheta_4\left(\frac{\pi\tau}{2}\right) = 0,$$

and their shifts by $m\pi + n\pi\tau$, $m, n \in \mathbb{Z}$.

In the antiferroelectric phase region we use parametrization (1.11), with two parameters t and γ such that $|t| < \gamma$. In what follows we will also use the following

parameters:

$$(1.43) \quad \zeta = \frac{t}{\gamma} \in (-1, 1), \quad \omega = \frac{\pi(1 + \zeta)}{2} \in (0, \pi).$$

The elliptic nome for all Jacobi theta functions in this paper will be equal to

$$(1.44) \quad q = e^{-\pi^2/2\gamma}.$$

Our main result in the present paper is the following asymptotic formula for Z_n :

THEOREM 1.1 *As $n \rightarrow \infty$,*

$$(1.45) \quad Z_n = C \vartheta_4(n\omega) F^{n^2} (1 + O(n^{-1})),$$

where $C > 0$ is a constant, and

$$(1.46) \quad F = \frac{\pi \sinh(\gamma - t) \sinh(\gamma + t) \vartheta_1'(0)}{2\gamma \vartheta_1(\omega)}.$$

The asymptotic formula (1.45) proves the conjecture of Zinn-Justin in [31]. The proof of Theorem 1.1 will be based on the Riemann-Hilbert approach to discrete orthogonal polynomials. An important first step in this approach is constructing the equilibrium measure.

2 Equilibrium Measure

2.1 Heuristic Motivation and Definition of the Equilibrium Measure

If we scale the variables in (1.19) as $\mu_i = 2\gamma l_i/n$, then we can rewrite formula (1.19) as

$$(2.1) \quad \tau_n = \frac{2^{n^2}}{n!} \sum_{\mu \in \frac{2\gamma}{n} \mathbb{Z}^n} e^{-n^2 H_n(v_\mu)},$$

where

$$(2.2) \quad dv_\mu(x) = \frac{1}{n} \sum_{j=1}^n \delta(x - \mu_j),$$

and

$$(2.3) \quad H(v) = \iint_{x \neq y} \log \frac{1}{|x - y|} dv(x)dv(y) + \int (|x| - \zeta x) dv(x),$$

where all integrals are over \mathbb{R} .

Due to the factor $(-n^2)$ in the exponent of (2.1), we expect the sum, in the large- n limit, to be focused in a neighborhood of a global minimum of the functional H . Clearly, we have that v_μ is a probability measure and

$$(2.4) \quad v_\mu(a, b) \leq \frac{b - a}{2\gamma} \quad \text{for any } -\infty < a < b < \infty,$$

because in (2.2), $\mu_j \in \frac{2\gamma}{n}\mathbb{Z}$ and $\mu_j \neq \mu_k$ if $j \neq k$. With these constraints in mind, we define

$$(2.5) \quad E_0 = \inf_{\nu} H(\nu)$$

where the infimum is taken over all probability measures satisfying (2.4). It is possible to prove that there exists a unique minimizer ν_0 so that

$$(2.6) \quad E_0 = H(\nu_0);$$

see, e.g., the works of Saff and Totik [26], Dragnev and Saff [13], and Kuijlaars [19]. Furthermore, ν_0 has support on a finite number of intervals and is absolutely continuous with respect to the Lebesgue measure. The minimizer ν_0 is called the *equilibrium measure*.

Denote the density function of the equilibrium measure as $\rho(x)$ and its resolvent as ω , so we have

$$(2.7) \quad \frac{d\nu_0}{dx} = \rho(x), \quad \omega(z) = \int_{\mathbb{R}} \frac{\rho(x)dx}{z-x},$$

and

$$(2.8) \quad \rho(x) = \frac{1}{2\pi i}(\omega(x-i0) - \omega(x+i0)).$$

The structure of the equilibrium measure ν_0 is studied in the paper of Zinn-Justin [31], who shows that ν_0 has support on an interval $[\alpha, \beta]$, with a saturated region $[\alpha', \beta']$ in which

$$(2.9) \quad \rho(x) = \frac{1}{2\gamma}, \quad x \in [\alpha', \beta'],$$

and two unsaturated regions, $[\alpha, \alpha']$ and $[\beta', \beta]$, in which

$$(2.10) \quad 0 < \rho(x) < \frac{1}{2\gamma}, \quad x \in (\alpha, \alpha') \cup (\beta', \beta);$$

see Figure 2.1. We also have that

$$(2.11) \quad \alpha < \alpha' < 0 < \beta' < \beta,$$

so that the origin, which is a singular point of the potential $V(x) = |x| - \zeta x$, lies inside the saturated region $[\alpha', \beta']$.

The measure ν_0 is uniquely determined by the Euler-Lagrange variational conditions

$$(2.12) \quad 2 \int \log|x-y|d\nu_0(y) - (|x| - \zeta x) \begin{cases} = l & \text{for } x \in [\alpha, \alpha'] \cup [\beta', \beta], \\ \geq l & \text{for } x \in [\alpha', \beta'], \\ \leq l & \text{for } x \notin [\alpha, \beta], \end{cases}$$

where l is the Lagrange multiplier. The Euler-Lagrange variational conditions imply

$$(2.13) \quad \omega(x-i0) + \omega(x+i0) = -\zeta + \operatorname{sgn}(x) \quad \text{for } x \in [\alpha, \alpha'] \cup [\beta', \beta],$$

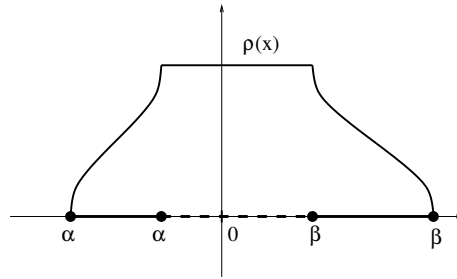


FIGURE 2.1. The equilibrium density function $\rho(x)$.

whereas in the saturated region, we have

$$(2.14) \quad \rho(x) = \frac{1}{2\pi i}(\omega(x - i0) - \omega(x + i0)) = \frac{1}{2\gamma} \quad \text{for } x \in [\alpha', \beta'].$$

Now we will give a detailed description of the equilibrium measure. We begin with explicit formulae for the endpoints of the support of the equilibrium measure.

2.2 Explicit Formulae for the Endpoints

PROPOSITION 2.1 *The endpoints of the support of the equilibrium measure ν_0 are equal to*

$$(2.15) \quad \begin{aligned} \alpha &= -\pi \frac{\vartheta'_1(\frac{\omega}{2})}{\vartheta_1(\frac{\omega}{2})}, & \alpha' &= -\pi \frac{\vartheta'_4(\frac{\omega}{2})}{\vartheta_4(\frac{\omega}{2})}, \\ \beta' &= -\pi \frac{\vartheta'_3(\frac{\omega}{2})}{\vartheta_3(\frac{\omega}{2})}, & \beta &= -\pi \frac{\vartheta'_2(\frac{\omega}{2})}{\vartheta_2(\frac{\omega}{2})}. \end{aligned}$$

The differences between the endpoints are equal to

$$(2.16) \quad \begin{aligned} \alpha' - \alpha &= \pi \vartheta_4^2(0) \frac{\vartheta_2(\frac{\omega}{2})\vartheta_3(\frac{\omega}{2})}{\vartheta_1(\frac{\omega}{2})\vartheta_4(\frac{\omega}{2})}, & \beta' - \alpha' &= \pi \vartheta_2^2(0) \frac{\vartheta_1(\frac{\omega}{2})\vartheta_2(\frac{\omega}{2})}{\vartheta_3(\frac{\omega}{2})\vartheta_4(\frac{\omega}{2})}, \\ \beta - \beta' &= \pi \vartheta_4(0)^2 \frac{\vartheta_1(\frac{\omega}{2})\vartheta_4(\frac{\omega}{2})}{\vartheta_2(\frac{\omega}{2})\vartheta_3(\frac{\omega}{2})}. \end{aligned}$$

and

$$(2.17) \quad \begin{aligned} \beta - \alpha &= \pi \vartheta_2^2(0) \frac{\vartheta_3(\frac{\omega}{2})\vartheta_4(\frac{\omega}{2})}{\vartheta_1(\frac{\omega}{2})\vartheta_2(\frac{\omega}{2})}, & \beta - \alpha' &= \pi \vartheta_3(0)^2 \frac{\vartheta_1(\frac{\omega}{2})\vartheta_3(\frac{\omega}{2})}{\vartheta_2(\frac{\omega}{2})\vartheta_4(\frac{\omega}{2})}, \\ \beta' - \alpha &= \pi \vartheta_3^2(0) \frac{\vartheta_2(\frac{\omega}{2})\vartheta_4(\frac{\omega}{2})}{\vartheta_1(\frac{\omega}{2})\vartheta_3(\frac{\omega}{2})}. \end{aligned}$$

For a proof of Proposition 2.1, see the next section.

2.3 Equilibrium Density Function

PROPOSITION 2.2 *The equilibrium density function $\rho(x)$ is given by the formulae*

$$(2.18) \quad \rho(x) = \begin{cases} \frac{1}{\pi} \int_{\alpha}^x \frac{dx'}{\sqrt{(x' - \alpha)(\alpha' - x')(\beta' - x')(\beta - x')}}}, & \alpha \leq x \leq \alpha', \\ \frac{1}{2\gamma}, & \alpha' \leq x \leq \beta', \\ \frac{1}{\pi} \int_x^{\beta} \frac{dx'}{\sqrt{(x' - \alpha)(x' - \alpha')(x' - \beta')(\beta - x')}}}, & \beta' \leq x \leq \beta. \end{cases}$$

Also,

$$(2.19) \quad \int_0^{\beta} \rho(x) dx = \frac{1 + \zeta}{2}.$$

The resolvent $\omega(z)$ of the equilibrium measure is given as

$$(2.20) \quad \omega(z) = \int_z^{\infty} \frac{dz'}{\sqrt{(z' - \alpha)(z' - \alpha')(z' - \beta')(z' - \beta)}},$$

where integration takes place on the sheet of

$$\sqrt{R(z')} \equiv \sqrt{(z' - \alpha)(z' - \alpha')(z' - \beta')(z' - \beta)}$$

for which $\sqrt{R(z')} > 0$ for $z' > \beta$, with cuts on $[\alpha, \alpha']$ and $[\beta', \beta]$.

For a proof of this proposition, see the next section.

2.4 g -function

Define the g -function on $\mathbb{C} \setminus [-\infty, \beta]$ as

$$(2.21) \quad g(z) = \int_{\alpha}^{\beta} \log(z - x) dv_0(x)$$

where we take the principal branch for logarithm.

Properties of $g(z)$.

- (1) $g(z)$ is analytic in $\mathbb{C} \setminus (-\infty, \beta]$.
- (2) For large z ,

$$(2.22) \quad g(z) = \log z - \sum_{j=1}^{\infty} \frac{g_j}{z^j}, \quad g_j = \int_{\alpha}^{\beta} \frac{x^j}{j} dv_0(x).$$

- (3) $g'(z) = \omega(z)$.
- (4) From the first relation in (2.12) we have that

$$(2.23) \quad g_+(x) + g_-(x) = |x| - \zeta x + l \quad \text{for } x \in [\alpha, \alpha'] \cup [\beta', \beta],$$

where g_+ and g_- refer to the limiting values of g from the upper and lower half-planes, respectively. By differentiating this equation we obtain that

$$(2.24) \quad \omega_+(x) + \omega_-(x) = g'_+(x) + g'_-(x) = \operatorname{sgn} x - \zeta \quad \text{for } x \in [\alpha, \alpha'] \cup [\beta', \beta].$$

Consider the function

$$(2.25) \quad f(x) = g_+(x) + g_-(x) - (|x| - \zeta x + l).$$

We have from (2.23) and (2.24) that

$$(2.26) \quad f(x) = f'(x) = 0 \quad \text{for } x = \alpha, \alpha', \beta', \beta,$$

and from (2.20) that

$$(2.27) \quad f''(x) = -\frac{1}{\sqrt{(x - \alpha)(x - \alpha')(x - \beta')(x - \beta)}} \quad \text{for } x \in (-\infty, \alpha) \cup (\alpha', \beta') \cup (\beta, \infty).$$

Since

$$(2.28) \quad f''(x) < 0 \quad \text{for } x \in (-\infty, \alpha) \cup (\beta, \infty)$$

and

$$(2.29) \quad f''(x) > 0 \quad \text{for } x \in (\alpha', \beta'), x \neq 0,$$

we obtain that

$$(2.30) \quad g_+(x) + g_-(x) \begin{cases} = |x| - \zeta x + l & \text{for } x \in [\alpha, \alpha'] \cup [\beta', \beta], \\ > |x| - \zeta x + l & \text{for } x \in (\alpha', \beta'), \\ < |x| - \zeta x + l & \text{for } x \in \mathbb{R} \setminus [\alpha, \beta]. \end{cases}$$

(5) Equation (2.21) implies that the function

$$(2.31) \quad G(x) \equiv g_+(x) - g_-(x)$$

is pure imaginary for all real x , and

$$(2.32) \quad G(x) = \begin{cases} 2\pi i & \text{for } -\infty < x \leq \alpha, \\ 2\pi i - 2\pi i \int_{\alpha}^x \rho(s) ds & \text{for } \alpha \leq x \leq \alpha', \\ 2\pi i \left(\frac{1+\zeta}{2} - \frac{x}{2\gamma} \right) & \text{for } \alpha' \leq x \leq \beta', \\ 2\pi i \int_x^{\beta} \rho(s) ds & \text{for } \beta' \leq x \leq \beta, \\ 0 & \text{for } \beta \leq x < \infty. \end{cases}$$

From (2.30) and (2.32) we obtain that

$$(2.33) \quad 2g_{\pm}(x) = \begin{cases} |x| - \zeta x + l \pm [2\pi i - 2\pi i \int_{\alpha}^x \rho(s) ds] & \text{for } \alpha \leq x \leq \alpha', \\ |x| - \zeta x + l \pm 2\pi i \int_x^{\beta} \rho(s) ds & \text{for } \beta' \leq x \leq \beta, \end{cases}$$

(6) Also, from (2.32)

$$(2.34) \quad \left. \frac{dG(x+iy)}{dy} \right|_{y=0} = 2\pi\rho(x) > 0, \quad x \in (\alpha, \beta).$$

Observe that from (2.23) we have that

$$(2.35) \quad G(x) = 2g_+(x) - V(x) - l = -[2g_-(x) - V(x) - l], \quad x \in [\alpha, \alpha'] \cup [\beta', \beta],$$

where $V(x) \equiv |x| - \zeta x$.

2.5 Evaluation of the Lagrange Multiplier l

PROPOSITION 2.3 *The Lagrange multiplier l solves the equation*

$$(2.36) \quad e^{l/2} = \frac{\pi\vartheta_1'(0)}{2e\vartheta_1(\omega)}.$$

For a proof of this proposition, see the next section.

3 Proofs of Propositions 2.1, 2.2, and 2.3

PROOF OF PROPOSITION 2.1: Following Zinn-Justin [31], we make the following elliptic change of variables:

$$(3.1) \quad u(z) = \frac{1}{2} \sqrt{(\beta' - \alpha)(\beta - \alpha')} \int_{\beta}^z \frac{dz'}{\sqrt{(z' - \alpha)(z' - \alpha')(z' - \beta')(z' - \beta)}},$$

where integration takes place on the sheet on $\sqrt{R(z')}$ specified in Proposition 2.2. To understand this integral in terms of the usual elliptic integrals, we first make the change of variables

$$(3.2) \quad v(z') = \frac{(\beta - z')(\beta' - \alpha)}{(\beta' - z')(\beta - \alpha)},$$

so that

$$(3.3) \quad z' = \frac{\beta'(\beta - \alpha)v - \beta(\beta' - \alpha)}{(\beta - \alpha)v - (\beta' - \alpha)}.$$

Note that $v(\beta) = 0$, $v(\beta') = \infty$, and $v(\alpha) = 1$. When we substitute v into equation (3.1), we have

$$(3.4) \quad u(z) = \frac{1}{2k} \int_0^{v(z)} \frac{dv}{\sqrt{v(v-1)(v-1/k^2)}},$$

where

$$(3.5) \quad k = \sqrt{\frac{(\beta - \alpha)(\beta' - \alpha')}{(\beta' - \alpha)(\beta - \alpha')}}.$$

We next take $v' = \sqrt{v}$, obtaining

$$(3.6) \quad u(z) = \int_0^{\sqrt{v(z)}} \frac{dv'}{\sqrt{(1-v'^2)(1-k^2v'^2)}},$$

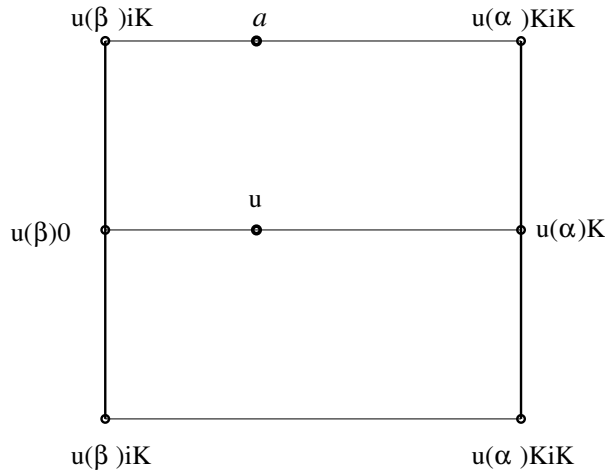


FIGURE 3.1. The u -plane. Here $u_* = u_\infty \equiv u(\infty)$ and $a = u_* + iK'$.

which corresponds to $\sqrt{v(z)} = \text{sn}(u, k)$, so that

$$(3.7) \quad \frac{(\beta - z)(\beta' - \alpha)}{(\beta' - z)(\beta - \alpha)} = \text{sn}^2(u), \quad \text{sn}(u) = \text{sn}(u, k).$$

Notice that u maps the upper z -plane conformally and bijectively onto the rectangle $[0, K] \times [0, iK']$, and the lower z -plane conformally and bijectively onto the rectangle $[0, K] \times [-iK', 0]$, where

$$(3.8) \quad \begin{aligned} K &= u(\alpha) = \int_0^1 \frac{dv'}{\sqrt{(1-v'^2)(1-k^2v'^2)}}, \\ K' &= -iu(\beta') = \int_1^{1/k} \frac{dv'}{\sqrt{(v'^2-1)(1-k^2v'^2)}} \end{aligned}$$

are the usual complete integrals of the first kind. More specifically (see Figure 3.1),

- (1) The upper (respectively, lower) cusp of the interval $[\beta, \beta']$ is mapped onto the interval $[0, iK']$ (respectively, $[0, -iK']$).
- (2) The upper (respectively, lower) cusp of the interval $[\alpha, \alpha']$ is mapped onto the interval $[K, K + iK']$ (respectively, $[K, K - iK']$).
- (3) The interval $[\beta', \alpha']$ is mapped onto the interval $[iK', K + iK']$ or the interval $[-iK', K - iK']$, depending on the path of integration.
- (4) The remaining part of the real axis, $[-\infty, \alpha] \cup [\beta, \infty]$, is mapped onto the interval $[0, K]$, with $u(\infty) = u_* = u_\infty$.

We will denote the rectangle $[0, K] \times [-iK', iK']$ as R , the *fundamental domain* of the function $z(u)$. We can now define

$$(3.9) \quad \tilde{\omega}(u) = \omega(z(u)) \quad \text{for } u \in R.$$

The Euler-Lagrange equation (2.13) and the equation (2.14) then become

$$\begin{aligned}
 & \tilde{\omega}(u) + \tilde{\omega}(-u) = 1 - \zeta \quad \text{for } u \in [-iK', iK'], \\
 (3.10) \quad & \tilde{\omega}(u) + \tilde{\omega}(-u + 2K) = -1 - \zeta \quad \text{for } u \in [K - iK', K + iK'], \\
 & \tilde{\omega}(u + 2iK') - \tilde{\omega}(u) = -\frac{i\pi}{\gamma} \quad \text{for } u \in [-iK', K - iK'].
 \end{aligned}$$

The function $\omega(z)$ is analytic in $\mathbb{C} \setminus [\alpha, \beta]$ but can be analytically continued from either above or below through any of the cuts $[\alpha, \alpha']$, $[\alpha', \beta']$, and $[\beta', \beta]$. These analytic continuations in the z -plane give an analytic continuation of $\tilde{\omega}$ in the u -plane into a neighborhood of R , which can then be continued by equations (3.10) to the entire u -plane. We therefore have that $\tilde{\omega}$ is analytic and satisfies equations (3.10) throughout the u -plane. The first two equations of (3.10) can be combined as

$$(3.11) \quad \tilde{\omega}(u + 2K) = \tilde{\omega}(u) - 2.$$

It therefore follows that $\tilde{\omega}$ is a linear function of u , as its derivative is a doubly periodic entire function. We also know from the fact that $\omega(z) \sim \frac{1}{z}$ at infinity that

$$(3.12) \quad \tilde{\omega}(u) = -\frac{2}{\sqrt{(\beta' - \alpha)(\beta - \alpha')}}(u - u_\infty) + O(u - u_\infty)^2$$

in some neighborhood of u_∞ , where u_∞ is the image of infinity under the map $u(z)$. It thus follows from (3.10), (3.11), and (3.12) that

$$(3.13) \quad \tilde{\omega}(u) = -\frac{1}{K}(u - u_\infty)$$

and that

$$(3.14) \quad \frac{K'}{K} = \frac{\pi}{2\gamma},$$

$$(3.15) \quad \sqrt{(\beta' - \alpha)(\beta - \alpha')} = 2K,$$

$$(3.16) \quad \frac{u_\infty}{K} = \frac{1 - \zeta}{2}.$$

From (3.7) we obtain that

$$(3.17) \quad \frac{\beta' - \alpha}{\beta - \alpha} = \text{sn}^2(u_\infty).$$

This implies that

$$\begin{aligned}
 \text{cn}^2(u_\infty) &= 1 - \text{sn}^2(u_\infty) = 1 - \frac{\beta' - \alpha}{\beta - \alpha} = \frac{\beta - \beta'}{\beta - \alpha}, \\
 (3.18) \quad \text{dn}^2(u_\infty) &= 1 - k^2 \text{sn}^2(u_\infty) \\
 &= 1 - \frac{(\beta - \alpha)(\beta' - \alpha')}{(\beta' - \alpha)(\beta - \alpha')} \frac{(\beta' - \alpha)}{(\beta - \alpha)} = \frac{\beta - \beta'}{\beta - \alpha'}.
 \end{aligned}$$

From equations (3.15), (3.17), and (3.18) we obtain the distances between the turning points in terms of u_∞ :

$$(3.19) \quad \begin{aligned} \beta - \alpha &= 2K \frac{\operatorname{dn}(u_\infty)}{\operatorname{sn}(u_\infty) \operatorname{cn}(u_\infty)}, & \beta - \alpha' &= 2K \frac{\operatorname{cn}(u_\infty)}{\operatorname{sn}(u_\infty) \operatorname{dn}(u_\infty)}, \\ \beta - \beta' &= 2K \frac{\operatorname{cn}(u_\infty) \operatorname{dn}(u_\infty)}{\operatorname{sn}(u_\infty)}. \end{aligned}$$

The functions sn , cn , and dn are expressed in terms of Jacobi theta functions as follows (see, e.g., [29]),

$$(3.20) \quad \begin{aligned} \operatorname{sn}(u) &= \frac{\vartheta_3(0) \vartheta_1(\frac{\pi u}{2K})}{\vartheta_2(0) \vartheta_4(\frac{\pi u}{2K})}, & \operatorname{cn}(u) &= \frac{\vartheta_4(0) \vartheta_2(\frac{\pi u}{2K})}{\vartheta_2(0) \vartheta_4(\frac{\pi u}{2K})}, \\ \operatorname{dn}(u) &= \frac{\vartheta_4(0) \vartheta_3(\frac{\pi u}{2K})}{\vartheta_3(0) \vartheta_4(\frac{\pi u}{2K})}. \end{aligned}$$

By (3.14), the half-period ratio τ and the elliptic nome q of the theta functions are

$$(3.21) \quad \tau = \frac{iK'}{K} = \frac{i\pi}{2\gamma} \quad \text{and} \quad q = e^{-\frac{\pi K'}{K}} = e^{-\frac{\pi^2}{2\gamma}}.$$

If we take into account the fact that

$$(3.22) \quad \vartheta_3(0)^2 = \frac{2K}{\pi}$$

along with equation (3.16), we can write equations for the distances between the turning points that involve only the original parameters:

$$(3.23) \quad \begin{aligned} \beta - \alpha &= \pi \vartheta_2^2(0) \frac{\vartheta_3(\frac{\omega}{2}) \vartheta_4(\frac{\omega}{2})}{\vartheta_1(\frac{\omega}{2}) \vartheta_2(\frac{\omega}{2})}, & \beta - \alpha' &= \pi \vartheta_3(0)^2 \frac{\vartheta_1(\frac{\omega}{2}) \vartheta_3(\frac{\omega}{2})}{\vartheta_2(\frac{\omega}{2}) \vartheta_4(\frac{\omega}{2})}, \\ \beta' - \alpha &= \pi \vartheta_3^2(0) \frac{\vartheta_2(\frac{\omega}{2}) \vartheta_4(\frac{\omega}{2})}{\vartheta_1(\frac{\omega}{2}) \vartheta_3(\frac{\omega}{2})}, \end{aligned}$$

giving (2.17). These equations determine the endpoints $\alpha, \alpha', \beta', \beta$ up to a shift. To fix the shift we use equation (2.12) at the points α' and β' to obtain

$$(3.24) \quad \int_{\alpha'}^{\beta'} (\omega(z + i0) + \omega(z - i0)) dz = (1 - \zeta)\beta' + (1 + \zeta)\alpha'.$$

Writing this integral in terms of u gives

$$(3.25) \quad \begin{aligned} \int_{iK'}^{K+iK'} \frac{1}{K} (u - u_\infty) r'(u) du \\ + \int_{-iK'}^{K-iK'} \frac{1}{K} (u - u_\infty) r'(u) du = (1 - \zeta)\beta' + (1 + \zeta)\alpha' \end{aligned}$$

where

$$(3.26) \quad r(u) = \frac{\beta'(\beta - \alpha) \operatorname{sn}^2(u) - \beta(\beta' - \alpha)}{(\beta - \alpha) \operatorname{sn}^2(u) - (\beta' - \alpha)} = \frac{\beta - \frac{\beta' \operatorname{sn}^2(u)}{\operatorname{sn}^2(u_\infty)}}{1 - \frac{\operatorname{sn}^2(u)}{\operatorname{sn}^2(u_\infty)}},$$

$$r'(u) = \frac{d}{du} r(u).$$

Note that $r(\pm iK') = \beta'$ and $r(K \pm iK') = \alpha'$.

Integrating by parts gives

$$(3.27) \quad \frac{2}{K}((K - u_\infty)\alpha' + \beta'u_\infty) - \int_{iK'}^{K+iK'} \frac{r(u)}{K} du - \int_{-iK'}^{K-iK'} \frac{r(u)}{K} du = (1 - \zeta)\beta' + (1 + \zeta)\alpha'$$

or equivalently

$$(3.28) \quad \int_{iK'}^{K+iK'} r(u) du + \int_{-iK'}^{K-iK'} r(u) du = 0.$$

We can evaluate these integrals by first writing $r(u)$ in the form

$$(3.29) \quad r(u) = \beta + \left(\frac{\beta - \beta'}{\operatorname{sn}^2(u_\infty)} \right) \frac{\operatorname{sn}^2(u)}{1 - \frac{\operatorname{sn}^2(u)}{\operatorname{sn}^2(u_\infty)}}$$

and using the functions

$$(3.30) \quad \Theta(u) = \vartheta_4\left(\frac{\pi u}{2K}\right), \quad Z(u) = \frac{\Theta'(u)}{\Theta(u)}.$$

The addition formulae for the sn and Z functions are (see [29])

$$(3.31) \quad \operatorname{sn}(u \pm a) = \frac{\operatorname{sn}(u) \operatorname{cn}(a) \operatorname{dn}(a) \pm \operatorname{sn}(a) \operatorname{cn}(u) \operatorname{dn}(u)}{1 - k^2 \operatorname{sn}^2(a) \operatorname{sn}^2(u)},$$

$$Z(u \pm a) = Z(u) \pm Z(a) \mp k^2 \operatorname{sn}(u) \operatorname{sn}(a) \operatorname{sn}(u \pm a).$$

Thus we have

$$(3.32) \quad \begin{aligned} & Z(u - a) - Z(u + a) + 2Z(a) \\ &= k^2 \operatorname{sn}(u) \operatorname{sn}(a) (\operatorname{sn}(u + a) + \operatorname{sn}(u - a)) \\ &= \frac{k^2 \operatorname{sn}(u) \operatorname{sn}(a) [2 \operatorname{sn}(u) \operatorname{cn}(a) \operatorname{dn}(a)]}{1 - k^2 \operatorname{sn}^2(a) \operatorname{sn}^2(u)} \\ &= \frac{2k^2 \operatorname{sn}(a) \operatorname{cn}(a) \operatorname{dn}(a) \operatorname{sn}^2(u)}{1 - k^2 \operatorname{sn}^2(a) \operatorname{sn}^2(u)}. \end{aligned}$$

We also have the half- and quarter-period identities

$$(3.33) \quad \begin{aligned} \operatorname{sn}(u + iK') &= \frac{1}{k \operatorname{sn}(u)}, & \operatorname{cn}(u + iK') &= \frac{-i \operatorname{dn}(u)}{k \operatorname{sn}(u)}, \\ \operatorname{dn}(u + iK') &= -i \frac{\operatorname{cn}(u)}{\operatorname{sn}(u)}. \end{aligned}$$

In particular, notice that $\frac{1}{\operatorname{sn}(u_\infty)} = k \operatorname{sn}(u_\infty + iK')$. Using the addition formulae (3.32), we can write $r(u)$ as

$$(3.34) \quad \begin{aligned} r(u) &= \beta + \left(\frac{\beta - \beta'}{2k^2 \operatorname{sn}(a) \operatorname{cn}(a) \operatorname{dn}(a) \operatorname{sn}^2(u_\infty)} \right) \\ &\quad \times (Z(u - a) - Z(u + a) + 2Z(a)), \end{aligned}$$

where $a = u_\infty + iK'$ (see Figure 3.1). From (3.33) and (3.19), it follows that

$$(3.35) \quad \frac{\beta - \beta'}{2k^2 \operatorname{sn}(u_\infty + iK') \operatorname{cn}(u_\infty + iK') \operatorname{dn}(u_\infty + iK') \operatorname{sn}^2(u_\infty)} = -K.$$

Thus we can write (3.34) as

$$(3.36) \quad \begin{aligned} r(u) &= \beta - K [Z(u - u_\infty - iK') \\ &\quad - Z(u + u_\infty + iK') + 2Z(u_\infty + iK')]. \end{aligned}$$

If we write $u = x + iK'$ in the first integral of (3.28) and $u = x - iK'$ in the second, we obtain

$$(3.37) \quad \int_0^K [2\beta - 4KZ(u_\infty + iK') - K[Z(x - u_\infty) - Z(x + u_\infty + 2iK') \\ + Z(x - u_\infty - 2iK') - Z(x + u_\infty)]] dx = 0.$$

From the periodic properties of ϑ_4 , it follows that

$$(3.38) \quad Z(u \pm 2iK') = Z(u) \mp \frac{\pi i}{K},$$

so we can write (3.37) as

$$(3.39) \quad \int_0^K (2\beta - 4KZ(u_\infty + iK') - 2\pi i \\ + 2K[Z(x + u_\infty) - Z(x - u_\infty)]) dx = 0.$$

This equation is readily integrated, as Z is the logarithmic derivative of the Θ function. Integrating gives

$$(3.40) \quad \begin{aligned} 0 &= \left[(2\beta - 4KZ(u_\infty + iK') - 2\pi i)x + 2K \log \frac{\Theta(x + u_\infty)}{\Theta(x - u_\infty)} \right]_{x=0}^K \\ &= 2K\beta - 4K^2Z(u_\infty + iK') - 2K\pi i + 2K \log \left(\frac{\Theta(K + u_\infty)}{\Theta(K - u_\infty)} \frac{\Theta(-u_\infty)}{\Theta(u_\infty)} \right). \end{aligned}$$

The logarithmic term in this equation is 0 due to the evenness and periodicity (period $2K$) of the Θ -function and the fact that the relevant term in the integration is real on the entire contour of integration. Thus we have that

$$(3.41) \quad \beta = 2KZ(u_\infty + iK') + \pi i.$$

From (1.41), we can deduce that

$$(3.42) \quad Z(u_\infty + iK') = -\frac{\pi}{2K} \left(\frac{\vartheta_2'(\frac{\omega}{2})}{\vartheta_2(\frac{\omega}{2})} + i \right)$$

and write (3.41) as

$$(3.43) \quad \beta = -\pi \frac{\vartheta_2'(\frac{\omega}{2})}{\vartheta_2(\frac{\omega}{2})}.$$

This equation, together with equations (3.23) and (3.21), determine the endpoints $\alpha, \alpha', \beta', \beta$. In fact, similar to (3.43) we have the following explicit formulae for the other endpoints:

$$(3.44) \quad \alpha = -\pi \frac{\vartheta_1'(\frac{\omega}{2})}{\vartheta_1(\frac{\omega}{2})}, \quad \alpha' = -\pi \frac{\vartheta_4'(\frac{\omega}{2})}{\vartheta_4(\frac{\omega}{2})}, \quad \beta' = -\pi \frac{\vartheta_3'(\frac{\omega}{2})}{\vartheta_3(\frac{\omega}{2})},$$

which follow from (3.23), (3.43), and the identities (see [29]),

$$(3.45) \quad \begin{aligned} \vartheta_4'(z) &= \frac{\vartheta_1'(z)\vartheta_4(z) - \vartheta_4(0)^2\vartheta_2(z)\vartheta_3(z)}{\vartheta_1(z)}, \\ \vartheta_2'(z) &= \frac{\vartheta_1'(z)\vartheta_2(z) - \vartheta_2(0)^3\vartheta_3(z)\vartheta_4(z)}{\vartheta_1(z)}, \\ \vartheta_3'(z) &= \frac{\vartheta_1'(z)\vartheta_3(z) - \vartheta_3(0)^2\vartheta_2(z)\vartheta_4(z)}{\vartheta_1(z)}. \end{aligned}$$

Similarly, in addition to the formulae (3.23) for distances between turning points, we get (2.16). \square

PROOF OF PROPOSITION 2.2: From equations (3.13), (3.1), and (3.15) we obtain formula (2.20); compare [31]. From formula (2.20) and equations (2.8) and (2.14), we obtain that the equilibrium density function $\rho(x)$ is given by formulae (2.18). We are left to prove formula (2.19).

By (3.1), (3.15), and (2.18), on the interval $[\beta', \beta]$,

$$(3.46) \quad \rho(x) = \frac{1}{iK\pi} u_+(x) \quad \text{for } x \in [\beta', \beta].$$

It follows that

$$(3.47) \quad \int_{\beta'}^{\beta} \rho(x) dx = \frac{1}{iK\pi} \int_{\beta'}^{\beta} u_+(x) dx = \frac{1}{iK\pi} \int_{iK'}^0 ur'(u) du,$$

where $r(u)$ is defined in (3.26). If we use equation (3.36) together with formula (3.41), we can write $r(u)$ as

$$(3.48) \quad r(u) = i\pi - K[Z(u - u_\infty - iK') - Z(u + u_\infty + iK')].$$

Integrating (3.47) by parts, we get

$$(3.49) \quad \int_{iK'}^0 ur'(u)du = -\beta'iK' - \pi K' - K \left[\log \left(\frac{\Theta(-u_\infty - iK')\Theta(u_\infty + 2iK')}{\Theta(-u_\infty)\Theta(u_\infty + iK')} \right) \right].$$

Using the fact that Θ is an even function and the identity

$$(3.50) \quad \Theta(u + 2iK') = e^{i\pi} e^{-i\pi\tau} e^{\frac{-i\pi u}{K}} \Theta(u),$$

we can write (3.49) as

$$(3.51) \quad \int_{iK'}^0 ur'(u)du = -\beta'iK' - \pi K' + K \left(i\pi + \pi \frac{K'}{K} - \frac{i\pi u_\infty}{K} \right) = i(K\pi - \beta'K' - \pi u_\infty).$$

Remark: There is some question here as to which branch of the logarithm to take, but it is clear that we have chosen the correct branch, as it is the only one that gives $0 < \int_{\beta'}^\beta \rho(x)dx < 1$.

Thus, from (3.47) and (3.51), we have

$$(3.52) \quad \int_{\beta'}^\beta \rho(x)dx = \frac{1}{iK\pi} i(K\pi - \beta'K' - \pi u_\infty) = 1 - \frac{\beta'K'}{\pi K} - \frac{u_\infty}{K} = 1 - \frac{\beta'}{2\gamma} - \frac{1-\zeta}{2};$$

hence by (2.18),

$$(3.53) \quad \int_0^\beta \rho(x)dx = \int_0^{\beta'} \rho(x)dx + \int_{\beta'}^\beta \rho(x)dx = \frac{\beta'}{2\gamma} + 1 - \frac{\beta'}{2\gamma} - \frac{1-\zeta}{2} = \frac{1+\zeta}{2},$$

which proves formula (2.19). □

PROOF OF PROPOSITION 2.3: By taking $x = \beta$ we obtain from (2.30) that

$$(3.54) \quad l = 2g(\beta) - V(\beta) = 2g(\beta) - (1 - \zeta)\beta.$$

We also have that

$$(3.55) \quad \lim_{A \rightarrow \infty} [g(A) - \log A] = 0;$$

hence

$$\begin{aligned}
 (3.56) \quad l &= -2 \lim_{A \rightarrow \infty} [g(A) - g(\beta) - \log A] - (1 - \zeta)\beta \\
 &= -2 \lim_{A \rightarrow \infty} \left[\int_{\beta}^A \omega(z) dz - \log A \right] - (1 - \zeta)\beta.
 \end{aligned}$$

Writing this integral in terms of u (so $z = r(u)$) gives

$$\begin{aligned}
 (3.57) \quad l &= 2 \lim_{A \rightarrow \infty} \left[\int_0^B \frac{1}{K} (u - u_{\infty}) r'(u) du + \log A \right] - (1 - \zeta)\beta \\
 &= 2 \lim_{B \rightarrow u_{\infty}} \left[\int_0^B \frac{1}{K} (u - u_{\infty}) r'(u) du + \log r(B) \right] - (1 - \zeta)\beta,
 \end{aligned}$$

where $A = r(B)$. Integrating by parts gives

$$\begin{aligned}
 (3.58) \quad l &= 2 \lim_{B \rightarrow u_{\infty}} \left[\frac{1}{K} (u - u_{\infty}) r(u) \Big|_{u=0}^B \right. \\
 &\quad \left. - \frac{1}{K} \int_0^B r(u) du + \log r(B) \right] - (1 - \zeta)\beta.
 \end{aligned}$$

From (3.26) we obtain that $r(0) = \beta$ and

$$\begin{aligned}
 (3.59) \quad \lim_{B \rightarrow u_{\infty}} (B - u_{\infty}) r(B) &= - \frac{(\beta - \beta') \operatorname{sn}(u_{\infty})}{2 \operatorname{sn}'(u_{\infty})} \\
 &= - \frac{(\beta - \beta') \operatorname{sn}(u_{\infty})}{2 \operatorname{cn}(u_{\infty}) \operatorname{dn}(u_{\infty})} = -K;
 \end{aligned}$$

hence

$$\begin{aligned}
 (3.60) \quad l &= 2 \left[-1 + \frac{\beta(10 - \zeta)}{2} \right] \\
 &\quad - 2 \lim_{B \rightarrow u_{\infty}} \left[\frac{1}{K} \int_0^B r(u) du - \log r(B) \right] - (1 - \zeta)\beta \\
 &= -2 - 2 \lim_{B \rightarrow u_{\infty}} \left[\frac{1}{K} \int_0^B r(u) du - \log r(B) \right].
 \end{aligned}$$

Using equation (3.48) for $r(u)$, we immediately get that

$$(3.61) \quad \frac{1}{K} \int_0^B r(u) du = \frac{B\pi i}{K} + \log \left[\frac{\Theta(B + u_{\infty} + iK')}{\Theta(B - u_{\infty} - iK')} \right].$$

Now using equation (3.26) for $r(u)$, we have

$$\begin{aligned}
 & \lim_{B \rightarrow u_\infty} \left[\frac{1}{K} \int_0^B r(u) du - \log r(B) \right] \\
 &= \frac{u_\infty \pi i}{K} + \lim_{B \rightarrow u_\infty} \log \left[\frac{\Theta(B + u_\infty + iK')(\operatorname{sn}^2(u_\infty) - \operatorname{sn}^2(B))}{\Theta(B - u_\infty - iK')(\beta \operatorname{sn}^2(u_\infty) - \beta' \operatorname{sn}^2(B))} \right] \\
 (3.62) \quad &= \frac{u_\infty \pi i}{K} + \log \left[\frac{\Theta(2u_\infty + iK') 2 \operatorname{sn}(u_\infty) \operatorname{sn}'(u_\infty)}{\Theta'(iK')(\beta - \beta') \operatorname{sn}^2(u_\infty)} \right] \\
 &= \frac{u_\infty \pi i}{K} + \log \left[\frac{2e^{-\frac{i\pi u_\infty}{K}} \vartheta_1\left(\frac{\pi u_\infty}{K}\right)}{\pi \vartheta_1'(0)} \right] \\
 &= \log \left[\frac{2\vartheta_1(\omega)}{\pi \vartheta_1'(0)} \right].
 \end{aligned}$$

Plugging this into (3.60) gives

$$(3.63) \quad l = -2 + 2 \log \left(\frac{\pi \vartheta_1'(0)}{2\vartheta_1(\omega)} \right),$$

and thus we obtain that

$$(3.64) \quad e^{l/2} = \frac{\pi \vartheta_1'(0)}{2e \vartheta_1(\omega)}.$$

□

4 Riemann-Hilbert Approach: Interpolation Problem

The Riemann-Hilbert approach to discrete orthogonal polynomials is based on the following interpolation problem (IP), which was introduced by Borodin and Boyarchenko [9] as the “discrete Riemann-Hilbert problem.” See also the monograph [2] of Baik, Kriecherbauer, McLaughlin, and Miller, in which it is called the “interpolation problem.”

We will consider the lattice L_n defined in (1.26) and the weight $w_n(x)$ defined in (1.24).

INTERPOLATION PROBLEM. For a given $n = 0, 1, \dots$, find a 2×2 matrix-valued function $\mathbf{P}_n(z) = (\mathbf{P}_{nij}(z))_{1 \leq i, j \leq 2}$ with the following properties:

- (1) *Analyticity.* $\mathbf{P}_n(z)$ is an analytic function of z for $z \in \mathbb{C} \setminus L_n$.
- (2) *Residues at poles.* At each node $x \in L_n$, the elements $\mathbf{P}_{n11}(z)$ and $\mathbf{P}_{n21}(z)$ of the matrix $\mathbf{P}_n(z)$ are analytic functions of z , and the elements $\mathbf{P}_{n12}(z)$ and $\mathbf{P}_{n22}(z)$ have a simple pole with the residues

$$(4.1) \quad \operatorname{Res}_{z=x} \mathbf{P}_{nj2}(z) = w_n(x) \mathbf{P}_{nj1}(x), \quad j = 1, 2.$$

- (3) *Asymptotics at infinity.* There exists a function $r(x) > 0$ on L_n such that

$$(4.2) \quad \lim_{x \rightarrow \infty} r(x) = 0,$$

and such that as $z \rightarrow \infty$, $\mathbf{P}_n(z)$ admits the asymptotic expansion

$$(4.3) \quad \mathbf{P}_n(z) \sim \left(\mathbf{I} + \frac{\mathbf{P}_1}{z} + \frac{\mathbf{P}_2}{z^2} + \cdots \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix},$$

$$z \in \mathbb{C} \setminus \left[\bigcup_{x \in L_n}^{\infty} D(x, r(x)) \right],$$

where $D(x, r(x))$ denotes a disk of radius $r(x) > 0$ centered at x and \mathbf{I} is the identity matrix.

It is not difficult to see (see [2, 9]) that the IP has a unique solution, which is

$$(4.4) \quad \mathbf{P}_n(z) = \begin{pmatrix} P_{nn}(z) & C(w_n P_{nn})(z) \\ (h_{n,n-1})^{-1} P_{n,n-1}(z) & (h_{n,n-1})^{-1} C(w_n P_{n,n-1})(z) \end{pmatrix}$$

where the Cauchy transformation C is defined by the formula

$$(4.5) \quad C(f)(z) = \sum_{x \in L_n} \frac{f(x)}{z-x},$$

and $P_{nk}(z) = z^k + \cdots$ are the orthogonal polynomials defined in (1.27). Because of the orthogonality condition, as $z \rightarrow \infty$,

$$(4.6) \quad C(w_n P_{nk})(z) = \sum_{x \in L_n} \frac{w_n(x) P_{nk}(x)}{z-x} \sim \sum_{x \in L_n} w_n(x) P_{nk}(x) \sum_{j=0}^{\infty} \frac{x^j}{z^{j+1}}$$

$$= \frac{h_{nk}}{z^{k+1}} + \sum_{j=k+2}^{\infty} \frac{a_j}{z^j},$$

which justifies asymptotic expansion (4.3). We have that

$$(4.7) \quad h_{nn} = [\mathbf{P}_1]_{12}, \quad h_{n,n-1}^{-1} = [\mathbf{P}_1]_{21}.$$

5 Reduction of IP to RHP

5.1 Preliminary Considerations

We would like to reduce the interpolation problem to a Riemann-Hilbert problem (RHP). Introduce the function

$$(5.1) \quad \Pi(z) = \frac{2\gamma}{n\pi} \sin \frac{n\pi z}{2\gamma}.$$

Observe that

$$(5.2) \quad \Pi(x_k) = 0, \quad \Pi'(x_k) = (-1)^k, \quad \exp\left(\frac{in\pi x_k}{2\gamma}\right) = (-1)^k$$

$$\text{for } x_k = \frac{2\gamma k}{n} \in L_n.$$

Introduce the upper-triangular matrices

$$(5.3) \quad \mathbf{D}_{\pm}^u(z) = \begin{pmatrix} 1 & -\frac{w_n(z)}{\Pi(z)} e^{\pm \frac{in\pi z}{2\gamma}} \\ 0 & 1 \end{pmatrix}$$

and the lower-triangular matrices

$$(5.4) \quad \begin{aligned} \mathbf{D}_{\pm}^l &= \begin{pmatrix} \Pi(z)^{-1} & 0 \\ -\frac{1}{w_n(z)} e^{\pm \frac{in\pi z}{2\gamma}} & \Pi(z) \end{pmatrix} \\ &= \begin{pmatrix} \Pi(z)^{-1} & 0 \\ 0 & \Pi(z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{\Pi(z)w_n(z)} e^{\pm \frac{in\pi z}{2\gamma}} & 1 \end{pmatrix}. \end{aligned}$$

Define the matrix-valued functions,

$$(5.5) \quad \mathbf{R}_n^u = \mathbf{P}_n(z) \times \begin{cases} \mathbf{D}_+^u(z) & \text{when } \text{Im } z \geq 0, \\ \mathbf{D}_-^u(z) & \text{when } \text{Im } z \leq 0, \end{cases}$$

and

$$(5.6) \quad \mathbf{R}_n^l = \mathbf{P}_n(z) \times \begin{cases} \mathbf{D}_+^l(z) & \text{when } \text{Im } z \geq 0, \\ \mathbf{D}_-^l(z) & \text{when } \text{Im } z \leq 0. \end{cases}$$

From (4.4) we have that

$$\begin{aligned} &\mathbf{R}_n^u(z) \\ &= \begin{pmatrix} P_{nn}(z) & -\frac{w_n(z)P_{nn}(z)}{\Pi(z)} e^{\pm \frac{in\pi z}{2\gamma}} + C(w_n P_{nn})(z) \\ h_{n,n-1}^{-1} P_{n,n-1}(z) & -\frac{w_n(z)h_{n,n-1}^{-1} P_{n,n-1}(z)}{\Pi(z)} e^{\pm \frac{in\pi z}{2\gamma}} + h_{n,n-1}^{-1} C(w_n P_{n,n-1})(z) \end{pmatrix} \\ &\text{when } \pm \text{Im } z \geq 0, \end{aligned}$$

and

$$\begin{aligned} &\mathbf{R}_n^l(z) \\ &= \begin{pmatrix} \frac{P_{nn}(z)}{\Pi(z)} - \frac{C(w_n P_{nn})(z)}{w_n(z)} e^{\pm \frac{in\pi z}{2\gamma}} & \Pi(z)C(w_n P_{nn})(z) \\ \frac{h_{n,n-1}^{-1} P_{n,n-1}(z)}{\Pi(z)} - \frac{h_{n,n-1}^{-1} C(w_n P_{n,n-1})(z)}{w_n(z)} e^{\pm \frac{in\pi z}{2\gamma}} & \Pi(z)h_{n,n-1}^{-1} C(w_n P_{n,n-1})(z) \end{pmatrix} \\ &\text{when } \pm \text{Im } z \geq 0. \end{aligned}$$

Observe that the functions $\mathbf{R}_n^u(z)$ and $\mathbf{R}_n^l(z)$ are meromorphic on the closed quadrants of the complex plane, and they are two-valued on the real and imaginary axes. Their possible poles are located on the lattice L_n . An important result is that, in fact, due to some cancellations, they do not have any poles at all. We have the following proposition:

PROPOSITION 5.1 *The matrix-valued functions $\mathbf{R}_n^u(z)$ and $\mathbf{R}_n^l(z)$ have no poles, and on the real line they satisfy the following jump conditions at $x \in \mathbb{R}$:*

$$(5.7) \quad \mathbf{R}_{n+}^u(x) = \mathbf{R}_{n-}^u(x) j_R^u(x), \quad j_R^u(x) = \begin{pmatrix} 1 & -\frac{n\pi i w_n(x)}{\gamma} \\ 0 & 1 \end{pmatrix},$$

and

$$(5.8) \quad \mathbf{R}_{n+}^l(x) = \mathbf{R}_{n-}^l(x) j_R^l(x), \quad j_R^l(x) = \begin{pmatrix} 1 & 0 \\ -\frac{n\pi i}{\gamma w_n(x)} & 1 \end{pmatrix}.$$

PROOF: It follows from the definition of $\mathbf{R}_n^u(z)$ that all possible poles of $\mathbf{R}_n^u(z)$ are located on the lattice L_n . Let us show that the residue of all these poles is equal to 0. Consider any $x_k \in L_n$. The residue of the matrix element $\mathbf{R}_{n,12}^u(z)$ at x_k is equal to

$$(5.9) \quad \operatorname{Res}_{z=x_k} \mathbf{R}_{n,12}^u(z) = -\frac{w_n(x_k) P_{nn}(x_k)}{(-1)^k} (-1)^k + w_n(x_k) P_{nn}(x_k) = 0.$$

Similarly, we get that

$$(5.10) \quad \operatorname{Res}_{z=x_k} \mathbf{R}_{n,22}(z) = 0;$$

hence $\mathbf{R}_n^u(z)$ has no pole at x_k .

Likewise, the residue of the matrix element $\mathbf{R}_{n,11}^l(z)$ at x_k is equal to

$$(5.11) \quad \operatorname{Res}_{z=x_k} \mathbf{R}_{n,11}^l(z) = \frac{P_{nn}(x_k)}{(-1)^k} - \frac{w_n(x_k) P_{nn}(x_k) (-1)^k}{w_n(x_k)} = 0.$$

In the same way we obtain that

$$(5.12) \quad \operatorname{Res}_{z=x_k} \mathbf{R}_{n,21}(z) = 0.$$

In the entry $\mathbf{R}_{n,21}^l(z)$, the pole of the function $C(w_n P_n)(z)$ at $z = x_k$ is cancelled by the zero of the function $\Pi(z)$; hence $\mathbf{R}_{n,21}^l(z)$ has no pole at x_k . Similarly, $\mathbf{R}_{n,22}^l(z)$ has no pole at x_k as well; hence $\mathbf{R}_n^l(z)$ has no pole at x_k .

Let us evaluate the jump matrices at $x \in \mathbb{R}$. From (5.5) we have that

$$(5.13) \quad j_R^u(x) = \mathbf{D}_-(x)^{-1} \mathbf{D}_+^u(x) = \begin{pmatrix} 1 & -\frac{w_n(x)}{\Pi(x)} 2i \sin \frac{n\pi x}{2\gamma} \\ 0 & 1 \end{pmatrix} \\ = \begin{pmatrix} 1 & -\frac{n\pi i w_n(x)}{\gamma} \\ 0 & 1 \end{pmatrix},$$

which proves (5.7). Similarly,

$$(5.14) \quad j_R^l(x) = \mathbf{D}_-(x)^{-1} \mathbf{D}_+^l(x) = \begin{pmatrix} 1 & 0 \\ -\frac{1}{\Pi(x)w_n(x)} 2i \sin \frac{n\pi x}{2\gamma} & 1 \end{pmatrix} \\ = \begin{pmatrix} 1 & 0 \\ -\frac{n\pi i}{\gamma w_n(x)} & 1 \end{pmatrix},$$

which proves (5.8). □

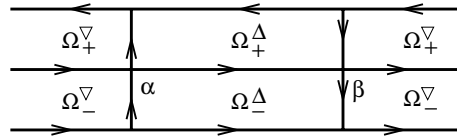


FIGURE 5.1. The contour Σ .

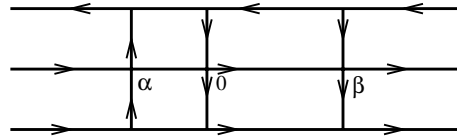


FIGURE 5.2. The contour Σ_R .

5.2 Reduction of IP to RHP

Let us discuss how to reduce the interpolation problem to a Riemann-Hilbert problem. We follow the work [2] with some modifications. Denote

$$(5.15) \quad \Delta = L_n \cap [\alpha', \beta'], \quad \nabla = L_n \setminus \Delta.$$

Consider the oriented contour Σ on the complex plane depicted in Figure 5.1, in which the horizontal lines are $\text{Im } z = \varepsilon, 0, -\varepsilon$, where $\varepsilon > 0$ is a small positive constant that will be determined later, and the vertical segments pass through the points $z = \alpha'$ and $z = \beta'$. Also consider the regions Ω_{\pm}^{Δ} and Ω_{\pm}^{∇} bounded by the contour Σ ; see Figure 5.1. Observe that the regions Ω_{\pm}^{∇} consist of two connected components, to the left and to the right of Ω_{\pm}^{Δ} .

Define

$$(5.16) \quad \mathbf{R}_n(z) = \begin{cases} \mathbf{K}_n \mathbf{R}_n^u(z) \mathbf{K}_n^{-1} & \text{for } z \in \Omega_{\pm}^{\nabla}, \\ \mathbf{K}_n \mathbf{R}_n^l(z) \mathbf{K}_n^{-1} & \text{for } z \in \Omega_{\pm}^{\Delta}, \\ \mathbf{K}_n \mathbf{P}_n(z) \mathbf{K}_n^{-1} & \text{otherwise.} \end{cases}$$

where $\mathbf{K}_n = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{n\pi i}{\gamma} \end{pmatrix}$.

Define a contour Σ_R by adding to the contour Σ a vertical segment $[i\varepsilon, -i\varepsilon]$; see Figure 5.2. If $A \subset \mathbb{C}$ is a set on the complex plane and $b \in \mathbb{C}$, then, as usual, we denote

$$(5.17) \quad A + b = \{z = a + b, a \in A\}.$$

PROPOSITION 5.2 *The matrix-valued function $\mathbf{R}_n(z)$ has the following jumps on the contour Σ_R :*

$$(5.18) \quad \mathbf{R}_{n+}(z) = \mathbf{R}_{n-}(z) j_R(z),$$

where

$$(5.19) \quad j_R(z) = \begin{cases} \begin{pmatrix} 1 & w_n(z) \\ 0 & 1 \end{pmatrix} & \text{when } z \in (-\infty, \alpha') \cup (\beta', \infty), \\ \begin{pmatrix} 1 & 0 \\ -(\frac{n\pi}{\gamma})^2 w_n(z)^{-1} & 1 \end{pmatrix} & \text{when } z \in [\alpha', \beta'], \\ \mathbf{K}_n \mathbf{D}_{\pm}^u(z) \mathbf{K}_n^{-1} = \begin{pmatrix} 1 & -\frac{i\gamma}{n\pi} \frac{w_n(z) e^{\pm \frac{i n \pi z}{2\gamma}}}{\Pi(z)} \\ 0 & 1 \end{pmatrix} & \text{when } z \in (-\infty, \alpha') \cup (\beta', \infty) \pm i\varepsilon, \\ \mathbf{K}_n \mathbf{D}_{\pm}^l(z) \mathbf{K}_n^{-1} = \begin{pmatrix} \Pi(z)^{-1} & 0 \\ \frac{i n \pi}{\gamma} \frac{e^{\pm \frac{i n \pi z}{2\gamma}}}{w_n(z)} & \Pi(z) \end{pmatrix} & \text{when } z \in (\alpha', \beta') \pm i\varepsilon, \\ \mathbf{K}_n \mathbf{D}_{\pm}^l(z)^{-1} \mathbf{D}_{\pm}^u(z) \mathbf{K}_n^{-1} = \begin{pmatrix} \Pi(z) & \frac{\gamma}{n\pi i} w_n(z) e^{\pm \frac{i n \pi z}{2\gamma}} \\ -\frac{n\pi i}{\gamma} w_n(z)^{-1} e^{\pm \frac{i n \pi z}{2\gamma}} & \mp \frac{n\pi i}{\gamma} e^{\pm \frac{i n \pi z}{2\gamma}} \end{pmatrix} & \text{when } z \in (0, \pm i\varepsilon) + \alpha' \text{ or } z \in (0, \pm i\varepsilon) + \beta', \\ \mathbf{K}_n \mathbf{D}_{\pm}^0(z) \mathbf{K}_n^{-1} & \text{when } z \in (0, \pm i\varepsilon), \end{cases}$$

and

$$(5.20) \quad \mathbf{D}_{\pm}^0(z) = \begin{pmatrix} 1 & 0 \\ -\frac{2 \sinh(nz) e^{-n\xi z} e^{\pm \frac{i n \pi z}{2\gamma}}}{\Pi(z)} & 1 \end{pmatrix}.$$

Notice that the jumps on vertical contours close to the origin, $\mathbf{D}_{\pm}^0(z)$, are exponentially close to the identity matrix.

6 First Transformation of the RHP

Define the matrix function $\mathbf{T}_n(z)$ as follows from the equation

$$(6.1) \quad \mathbf{R}_n(z) = e^{\frac{n l}{2} \sigma_3} \mathbf{T}_n(z) e^{n(g(z) - \frac{l}{2}) \sigma_3},$$

where the Lagrange multiplier l and the function $g(z)$ are as described in Section 2 and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the third Pauli matrix. Then $\mathbf{T}_n(z)$ satisfies the following Riemann-Hilbert problem:

- (1) $\mathbf{T}_n(z)$ is analytic in $\mathbb{C} \setminus \Sigma_R$.
 - (2) $\mathbf{T}_{n+}(z) = \mathbf{T}_{n-}(z) j_T(z)$ for $z \in \Sigma_R$, where
- $$(6.2) \quad j_T(z) = \begin{cases} e^{n(g_-(z) - \frac{l}{2}) \sigma_3} j_R(z) e^{-n(g_+(z) - \frac{l}{2}) \sigma_3} & \text{for } z \in \mathbb{R} \\ e^{n(g(z) - \frac{l}{2}) \sigma_3} j_R(z) e^{-n(g(z) - \frac{l}{2}) \sigma_3} & \text{for } z \in \Sigma_R \setminus \mathbb{R}. \end{cases}$$

$$(3) \mathbf{T}_n(z) \sim \mathbf{I} + \frac{\mathbf{T}_1}{z} + \frac{\mathbf{T}_2}{z^2} + \dots \text{ as } z \rightarrow \infty.$$

From (2.22) we have that

$$(6.3) \quad g(z) = \log z + O(z^{-1}) \text{ as } z \rightarrow \infty.$$

This implies that

$$(6.4) \quad [\mathbf{T}_1]_{12} = e^{-nI} [\mathbf{R}_1]_{12}.$$

Let's take a closer look at the behavior of the jump matrix j_T described in (6.2) on the horizontal segments of Σ_R . We have that

$$(6.5) \quad j_T(z) = \begin{cases} \begin{pmatrix} e^{-nG(z)} & e^{n(g_+(z)+g_-(z)-V(z)-I)} \\ 0 & e^{nG(z)} \end{pmatrix} & \text{when } z \in (-\infty, \alpha') \cup (\beta', \infty), \\ \begin{pmatrix} e^{-nG(z)} & 0 \\ -(\frac{n\pi}{\gamma})^2 e^{-n(g_+(z)+g_-(z)-V(z)-I)} & e^{nG(z)} \end{pmatrix} & \text{when } z \in (\alpha', \beta'), \\ \begin{pmatrix} 1 & \pm \frac{e^{\pm nG(z)}}{1 - e^{\pm \frac{i n \pi x}{\gamma}} e^{\pm \frac{\epsilon n \pi}{\gamma}}} \\ 0 & 1 \end{pmatrix} & \text{when } z = x + i\epsilon \in (\alpha, \alpha') \cup (\beta', \beta) \pm i\epsilon, \\ \begin{pmatrix} 1 & \pm \frac{e^{n(2g(z)-I-V(z))}}{1 - e^{\pm \frac{i n \pi x}{\gamma}} e^{\pm \frac{\epsilon n \pi}{\gamma}}} \\ 0 & 1 \end{pmatrix} & \text{when } z = x \pm i\epsilon \in (-\infty, \alpha) \cup (\beta, \infty) \pm i\epsilon, \\ \begin{pmatrix} \Pi(z)^{-1} & 0 \\ \frac{i n \pi}{\gamma} e^{\pm \frac{i n \pi x}{2\gamma}} e^{-n(2g(z)-I-V(z))} & \Pi(z) \end{pmatrix} & \text{when } z = x \pm i\epsilon \in (\alpha', \beta') \pm i\epsilon. \end{cases}$$

According to the properties of the g -function, we have the following proposition:

PROPOSITION 6.1 *The jump function j_T has the following large- n asymptotics on the real axis:*

$$(6.6) \quad j_T(z) = \begin{cases} \begin{pmatrix} e^{-nG(z)} & 0 \\ O(e^{-nC(z)}) & e^{nG(z)} \end{pmatrix} & \text{for } z \in (\alpha', \beta'), \\ \begin{pmatrix} e^{-nG(z)} & 1 \\ 0 & e^{nG(z)} \end{pmatrix} & \text{for } z \in (\alpha, \alpha') \cup (\beta', \beta), \\ \begin{pmatrix} 1 & O(e^{-nC(z)}) \\ 0 & 1 \end{pmatrix} & \text{for } z \in (-\infty, \alpha) \cup (\beta, \infty), \\ \begin{pmatrix} 1 & e^{\pm nG(z)} O(e^{-\frac{\epsilon n \pi}{\gamma}}) \\ 0 & 1 \end{pmatrix} & \text{for } z \in (\alpha, \alpha') \cup (\beta', \beta) \pm i\epsilon, \end{cases}$$

where $C(z)$ is a positive continuous function on any subset of the given interval that is bounded away from the endpoints of each interval and satisfies

$$(6.7) \quad C(z) > c|z + 1| \text{ for some } c > 0.$$

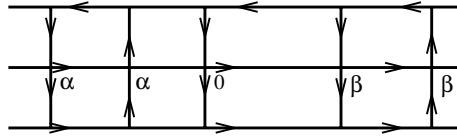


FIGURE 7.1. The contour Σ_S .

7 Second Transformation of the RHP

The second transformation is based on two observations. The first is the well-known “opening of the lenses” in a neighborhood of the unconstrained support of the equilibrium measure. Namely, notice that, for $x \in (\alpha, \alpha') \cup (\beta', \beta)$, the jump matrix $j_T(x)$ factors as

$$(7.1) \quad j_T(x) = \begin{pmatrix} e^{-nG(x)} & 1 \\ 0 & e^{nG(x)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{nG(x)} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-nG(x)} & 1 \end{pmatrix} = j_-(x)j_M j_+(x),$$

which allows us to reduce the jump matrix j_T to the one j_M plus asymptotically small jumps on the lens boundaries. The second observation consists of two facts. First, the jumps on the segments $[\alpha', \beta'] \pm i\varepsilon$ behave, for large n , as $\pm e^{\pm in\pi z/(2\gamma)}$. Second, note that, for $x \in [\alpha', \beta']$, $G(x)$ is a linear function with slope $-\frac{\pi i}{\gamma}$. With these facts in mind, we make the second transformation of the RHP. Let

$$(7.2) \quad \mathbf{S}_n(z) = \begin{cases} \mathbf{T}_n(z)j_+(z)^{-1} & \text{for } z \in \{(\alpha, \alpha') \cup (\beta', \beta)\} \times (0, i\varepsilon), \\ \mathbf{T}_n(z)j_-(z) & \text{for } z \in \{(\alpha, \alpha') \cup (\beta', \beta)\} \times (0, -i\varepsilon), \\ \mathbf{T}_n(z) \begin{pmatrix} -\frac{\gamma}{n\pi i} e^{-\frac{in\pi z}{2\gamma}} & 0 \\ 0 & -\frac{n\pi i}{\gamma} e^{\frac{in\pi z}{2\gamma}} \end{pmatrix} & \text{for } z \in (\alpha', \beta') \times (0, i\varepsilon), \\ \mathbf{T}_n(z) \begin{pmatrix} \frac{\gamma}{n\pi i} e^{\frac{in\pi z}{2\gamma}} & 0 \\ 0 & \frac{n\pi i}{\gamma} e^{-\frac{in\pi z}{2\gamma}} \end{pmatrix} & \text{for } z \in (\alpha', \beta') \times (0, -i\varepsilon), \\ \mathbf{T}_n(z) & \text{otherwise.} \end{cases}$$

This function satisfies a similar RHP to \mathbf{T} , but jumps now occur on a new contour, Σ_S , which is obtained from Σ_R by adding the two segments $(\alpha - i\varepsilon, \alpha + i\varepsilon)$ and $(\beta - i\varepsilon, \beta + i\varepsilon)$; see Figure 7.1.

On the horizontal segments for which the jump function j_S differs from j_T , we have that, as $n \rightarrow \infty$,

$$(7.3) \quad j_S(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } z \in (\alpha, \alpha') \cup (\beta', \beta), \\ \begin{pmatrix} 1 + O(e^{-\epsilon n \pi / \gamma}) & O(e^{n[G(z) - \epsilon \pi / \gamma]}) \\ -e^{-nG(z)} & 1 \end{pmatrix} & \text{for } z - i\epsilon \in (\alpha, \alpha') \cup (\beta', \beta), \\ \begin{pmatrix} 1 + O(e^{-\epsilon n \pi / \gamma}) & O(e^{n[-G(z) - \epsilon \pi / \gamma]}) \\ e^{nG(z)} & 1 \end{pmatrix} & \text{for } z + i\epsilon \in (\alpha, \alpha') \cup (\beta', \beta), \\ \begin{pmatrix} 1 + O(e^{-\epsilon n \pi / \gamma}) & 0 \\ \frac{n\pi i}{\gamma} e^{-n(2g(z) - l - V(z))} & 1 + O(e^{-\epsilon n \pi / \gamma}) \end{pmatrix} & \text{for } z \in \{[\alpha', \beta'] + i\epsilon\}, \\ \begin{pmatrix} 1 + O(e^{-\epsilon n \pi / \gamma}) & 0 \\ \frac{n\pi i}{\gamma} e^{-n(2g(z) - l - V(z))} & 1 + O(e^{-\epsilon n \pi / \gamma}) \end{pmatrix} & \text{for } z \in \{[\alpha', \beta'] - i\epsilon\}, \\ \begin{pmatrix} -e^{-n\pi i(1+\zeta)} & 0 \\ e^{-n(g_+(z) + g_-(z) - l - V(z))} & -e^{n\pi i(1+\zeta)} \end{pmatrix} & \text{for } z \in [\alpha', \beta']. \end{cases}$$

By formula (2.32) for the G -function and the upper constraint (2.10) on the density ρ , we obtain that, for sufficiently small $\epsilon > 0$,

$$0 < \mp \operatorname{Re} G(x \pm i\epsilon) = 2\pi\epsilon\rho(x) + O(\epsilon^2) < \frac{\pi\epsilon}{\gamma} + O(\epsilon^2) \quad \text{for } x \in (\alpha, \alpha') \cup (\beta', \beta).$$

This, combined with property (2.30) of the g -function, implies that all jumps on horizontal segments are exponentially close to the identity matrix, provided that they are bounded away from the segment $[\alpha, \beta]$. For what follows we denote

$$(7.4) \quad \Omega_n = \pi + n2\pi \int_0^\beta \rho(x)dx = \pi + n\pi(1 + \zeta),$$

so that

$$(7.5) \quad -e^{-n\pi i(1+\zeta)} = e^{-i\Omega_n}.$$

8 Model RHP

The model RHP appears when we drop in the jump matrix $j_S(z)$ the terms that vanish as $n \rightarrow \infty$:

- (1) $\mathbf{M}(z)$ is analytic in $\mathbb{C} \setminus [\alpha, \beta]$.
- (2) $\mathbf{M}_+(z) = \mathbf{M}_-(z)j_M(z)$ for $z \in [\alpha, \beta]$, where

$$(8.1) \quad j_M(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } z \in [\alpha, \alpha'] \cup [\beta', \beta], \\ e^{-i\Omega_n\sigma_3} & \text{for } z \in [\alpha', \beta']. \end{cases}$$

(3) As $z \rightarrow \infty$,

$$(8.2) \quad \mathbf{M}(z) \sim \mathbf{I} + \frac{\mathbf{M}_1}{z} + \frac{\mathbf{M}_2}{z^2} + \dots$$

This model problem was first solved, in the general multicut case, in [12], and is solved in two steps. In the first step, we solve the following auxiliary RHP:

- (1) $\mathbf{Q}(z)$ is analytic in $\mathbb{C} \setminus [\alpha, \alpha'] \cup [\beta', \beta]$.
- (2) $\mathbf{Q}_+(z) = \mathbf{Q}_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for $z \in [\alpha, \alpha'] \cup [\beta', \beta]$.
- (3) $\mathbf{Q}(z) = \mathbf{I} + O(z^{-1})$ as $z \rightarrow \infty$.

This RHP has the unique solution (see [12])

$$(8.3) \quad \mathbf{Q}(z) = \begin{pmatrix} \frac{\gamma(z) + \gamma^{-1}(z)}{2} & \frac{\gamma(z) - \gamma^{-1}(z)}{-2i} \\ \frac{\gamma(z) - \gamma^{-1}(z)}{2i} & \frac{\gamma(z) + \gamma^{-1}(z)}{2} \end{pmatrix}$$

where

$$(8.4) \quad \gamma(z) = \left(\frac{(z - \alpha)(z - \beta')}{(z - \alpha')(z - \beta)} \right)^{1/4}$$

with cuts on $[\alpha, \alpha'] \cup [\beta', \beta]$, taking the branch such that $\gamma(z) \sim 1$ as $z \rightarrow \infty$.

To solve the model RHP described in (8.1) and (8.2), we again use elliptic functions. Define the function

$$(8.5) \quad f(s) = \frac{\vartheta_3(s + d + c)}{\vartheta_3(s + d)}$$

where ϑ_3 is as defined in (1.37) with elliptic nome

$$q = e^{i\pi\tau} = e^{\frac{-\pi^2}{2\gamma}} \quad \left(\tau = \frac{i\pi}{2\gamma} \right),$$

and d and c are arbitrary complex numbers. Notice that f has the periodic properties

$$(8.6) \quad f(s + \pi) = f(s), \quad f(s + \pi\tau) = e^{-2ic} f(s),$$

and that f is an even function. Now let

$$(8.7) \quad \tilde{u}(z) = \frac{\pi}{2K} u(z) = \frac{\pi}{2} \int_{\beta}^z \frac{dz'}{\sqrt{R(z')}}$$

where u is as defined in (3.1). Then \tilde{u} is two-valued on $[\alpha, \beta]$ and satisfies

$$(8.8) \quad \tilde{u}_+(x) - \tilde{u}_-(x) = \pi\tau \quad \text{for } x \in [\alpha', \beta'].$$

Also,

$$(8.9) \quad \tilde{u}_{\pm}(\alpha) = \frac{\pi}{2}, \quad \tilde{u}_{\pm}(\alpha') = \frac{\pi}{2} \pm \frac{\pi\tau}{2}, \quad \tilde{u}_{\pm}(\beta') = \pm \frac{\pi\tau}{2}, \quad \tilde{u}_{\pm}(\beta) = 0;$$

compare Figure 3.1. Because $\sqrt{R(x)}_+ = -\sqrt{R(x)}_-$ for $x \in [\alpha, \alpha'] \cup [\beta', \beta]$, it immediately follows that

$$(8.10) \quad \tilde{u}_+(x) + \tilde{u}_-(x) = 0 \quad \text{for } x \in [\beta', \beta],$$

and that

$$(8.11) \quad \begin{aligned} \tilde{u}_+(x) + \tilde{u}_-(x) &= \tilde{u}_+(\alpha') - \tilde{u}_+(\beta') + \tilde{u}_-(\alpha') - \tilde{u}_-(\beta') \\ &= \pi \quad \text{for } x \in [\alpha, \alpha']. \end{aligned}$$

We now define

$$(8.12) \quad \begin{aligned} f_1(z) &= \frac{\vartheta_3(\tilde{u}(z) + d + \frac{\Omega_n}{2})}{\vartheta_3(\tilde{u}(z) + d)}, \\ f_2(z) &= \frac{\vartheta_3(-\tilde{u}(z) + d + \frac{\Omega_n}{2})}{\vartheta_3(-\tilde{u}(z) + d)}, \end{aligned} \quad \text{for } z \in \mathbb{C} \setminus [\alpha, \beta],$$

where d is an arbitrary complex number. It then follows from (8.6) and (8.8) that

$$(8.13) \quad \begin{aligned} f_{1+}(x) &= e^{-i\Omega_n} f_{1-}(x), \\ f_{2+}(x) &= e^{i\Omega_n} f_{2-}(x), \end{aligned} \quad \text{for } x \in [\alpha', \beta'],$$

and from (8.6), (8.10), and (8.11) that

$$(8.14) \quad f_{1+}(x) = f_{2-}(x) \quad \text{and} \quad f_{2+}(x) = f_{1-}(x) \quad \text{for } x \in [\alpha, \alpha'] \cup [\beta', \beta].$$

Define the matrix-valued function

$$(8.15) \quad \mathbf{F}(z) = \begin{pmatrix} \frac{\vartheta_3(\tilde{u}(z) + d_1 + \frac{\Omega_n}{2})}{\vartheta_3(\tilde{u}(z) + d_1)} & \frac{\vartheta_3(-\tilde{u}(z) + d_1 + \frac{\Omega_n}{2})}{\vartheta_3(-\tilde{u}(z) + d_1)} \\ \frac{\vartheta_3(\tilde{u}(z) + d_2 + \frac{\Omega_n}{2})}{\vartheta_3(\tilde{u}(z) + d_2)} & \frac{\vartheta_3(-\tilde{u}(z) + d_2 + \frac{\Omega_n}{2})}{\vartheta_3(-\tilde{u}(z) + d_2)} \end{pmatrix}$$

where d_1 and d_2 are yet undetermined complex constants. Then, from (8.13) and (8.14) we have that

$$(8.16) \quad \begin{aligned} \mathbf{F}_+(x) &= \mathbf{F}_-(x) \begin{pmatrix} e^{-i\Omega_n} & 0 \\ 0 & e^{i\Omega_n} \end{pmatrix} \quad \text{for } x \in [\alpha', \beta'], \\ \mathbf{F}_+(x) &= \mathbf{F}_-(x) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{for } x \in [\alpha, \alpha'] \cup [\beta', \beta]. \end{aligned}$$

We can now combine (8.3) and (8.15) to obtain

$$(8.17) \quad \mathbf{M}(z) = \mathbf{F}(\infty)^{-1} \begin{pmatrix} \frac{\gamma(z) + \gamma^{-1}(z)}{2} \frac{\vartheta_3(\tilde{u}(z) + d_1 + \frac{\Omega_n}{2})}{\vartheta_3(\tilde{u}(z) + d_1)} & \frac{\gamma(z) - \gamma^{-1}(z)}{-2i} \frac{\vartheta_3(-\tilde{u}(z) + d_1 + \frac{\Omega_n}{2})}{\vartheta_3(-\tilde{u}(z) + d_1)} \\ \frac{\gamma(z) - \gamma^{-1}(z)}{2i} \frac{\vartheta_3(\tilde{u}(z) + d_2 + \frac{\Omega_n}{2})}{\vartheta_3(\tilde{u}(z) + d_2)} & \frac{\gamma(z) + \gamma^{-1}(z)}{2} \frac{\vartheta_3(-\tilde{u}(z) + d_2 + \frac{\Omega_n}{2})}{\vartheta_3(-\tilde{u}(z) + d_2)} \end{pmatrix}$$

where

$$(8.18) \quad \mathbf{F}(\infty) = \begin{pmatrix} \frac{\vartheta_3(\tilde{u}_\infty + d_1 + \frac{\Omega_n}{2})}{\vartheta_3(\tilde{u}_\infty + d_1)} & 0 \\ 0 & \frac{\vartheta_3(-\tilde{u}_\infty + d_2 + \frac{\Omega_n}{2})}{\vartheta_3(-\tilde{u}_\infty + d_2)} \end{pmatrix}$$

and $\tilde{u}_\infty \equiv \tilde{u}(\infty)$. This matrix satisfies conditions (8.1) and (8.2) of the model RHP, but may not be analytic on $\mathbb{C} \setminus [\alpha, \beta]$, as it may have some poles at the zeroes of $\vartheta_3(\pm\tilde{u}(z) + d_{1,2})$. However, we can choose the constants d_1 and d_2 such that

these zeroes coincide with the zeroes of $\gamma(z) \pm \gamma^{-1}(z)$ and are thus cancelled in the product.

First consider the zeroes of $\gamma(z) \pm \gamma^{-1}(z)$. These are the zeroes of $\gamma^2(z) \pm 1$ and thus of $\gamma^4(z) - 1$; thus there is only one zero, which uniquely solves the equation

$$(8.19) \quad p(z) \equiv \frac{(z - \alpha)(z - \beta')}{(z - \alpha')(z - \beta)} = 1,$$

which is

$$(8.20) \quad x_0 = \frac{\beta\alpha' - \alpha\beta'}{(\alpha' - \alpha) + (\beta - \beta')} \in (\alpha', \beta').$$

It is easy to check that $\gamma(x_0) = 1$; thus x_0 is the unique zero of $\gamma(z) - \gamma^{-1}(z)$, whereas there are no zeroes of $\gamma(z) + \gamma^{-1}(z)$ on the specified sheet. We use here the change of variables v defined in (3.2). Notice that, by (3.19),

$$(8.21) \quad v(x_0) = \frac{\beta' - \alpha}{\beta' - \alpha'} = \frac{\operatorname{dn}^2(u_\infty)}{k^2 \operatorname{cn}^2(u_\infty)},$$

implying that

$$(8.22) \quad \operatorname{sn}^2(u(x_0)) = \frac{\operatorname{dn}^2(u_\infty)}{k^2 \operatorname{cn}^2(u_\infty)}.$$

Since $x_0 \in (\alpha', \beta')$, we must have $u(x_0) \in (iK', K + iK')$ (if we choose to take u_+). Since sn^2 is a one-to-one function on this interval there is a unique point $u_0 \in (iK', K + iK')$ such that $\operatorname{sn}^2(u_0) = \operatorname{dn}^2(u_\infty)/k^2 \operatorname{cn}^2(u_\infty)$. The simple period identity

$$(8.23) \quad \operatorname{sn}(u + K + iK') = \frac{\operatorname{dn}(u)}{k \operatorname{cn}(u)}$$

along with (8.22) gives that we must have

$$(8.24) \quad u_0 = u(x_0) = K - u_\infty + iK';$$

thus

$$(8.25) \quad \tilde{u}(x_0) = \frac{\pi}{2K}(K - u_\infty + iK') = \frac{\tau\pi}{2} + \frac{\pi}{2} - \tilde{u}_\infty.$$

We now consider zeroes of the function $\vartheta_3(\tilde{u}(z) - d) \equiv \vartheta_3(-\tilde{u}(z) + d)$. The zeroes of this function are the solutions to the equation

$$(8.26) \quad \tilde{u}(z) - d = (2m + 1)\frac{\pi}{2} + (2k + 1)\frac{\tau\pi}{2}$$

for any $m, k \in \mathbb{Z}$. Because \tilde{u} maps the first sheet of X to the rectangular domain $[0, \frac{\pi}{2}] \times [-\frac{\tau\pi}{2}, \frac{\tau\pi}{2}]$, it is clear that this equation can have at most one solution, and without any loss of generality we may take $m = k = 0$. Then, if we want the solution of this equation to be x_0 , we need to let

$$(8.27) \quad d = \tilde{u}(x_0) - \frac{\pi}{2}(1 + \tau) = -\tilde{u}_\infty.$$

This choice of d also ensures that $\vartheta_3(\tilde{u}(z) + d) \equiv \vartheta_3(-\tilde{u}(z) - d)$ has no zeroes on the first sheet of X . We can then let

$$(8.28) \quad d_1 = d, \quad d_2 = -d,$$

so that (8.17) and (8.18) become

$$(8.29) \quad \begin{aligned} \mathbf{M}(z) &= \mathbf{F}(\infty)^{-1} \begin{pmatrix} \frac{\gamma(z)+\gamma^{-1}(z)}{2} \frac{\vartheta_3(\tilde{u}(z)+d+\frac{\Omega n}{2})}{\vartheta_3(\tilde{u}(z)+d)} & \frac{\gamma(z)-\gamma^{-1}(z)}{-2i} \frac{\vartheta_3(-\tilde{u}(z)+d+\frac{\Omega n}{2})}{\vartheta_3(-\tilde{u}(z)+d)} \\ \frac{\gamma(z)-\gamma^{-1}(z)}{2i} \frac{\vartheta_3(\tilde{u}(z)-d+\frac{\Omega n}{2})}{\vartheta_3(\tilde{u}(z)-d)} & \frac{\gamma(z)+\gamma^{-1}(z)}{2} \frac{\vartheta_3(-\tilde{u}(z)-d+\frac{\Omega n}{2})}{\vartheta_3(-\tilde{u}(z)-d)} \end{pmatrix} \\ &= \mathbf{F}(\infty)^{-1} \begin{pmatrix} \frac{\gamma(z)+\gamma^{-1}(z)}{2} \frac{\vartheta_3(\tilde{u}(z)+(n+\frac{1}{2})\omega)}{\vartheta_4(\tilde{u}(z)+\frac{\omega}{2})} & \frac{\gamma(z)-\gamma^{-1}(z)}{-2i} \frac{\vartheta_3(\tilde{u}(z)-(n+\frac{1}{2})\omega)}{\vartheta_4(\tilde{u}(z)-\frac{\omega}{2})} \\ \frac{\gamma(z)-\gamma^{-1}(z)}{2i} \frac{\vartheta_3(\tilde{u}(z)+(n-\frac{1}{2})\omega)}{\vartheta_4(\tilde{u}(z)-\frac{\omega}{2})} & \frac{\gamma(z)+\gamma^{-1}(z)}{2} \frac{\vartheta_3(\tilde{u}(z)-(n-\frac{1}{2})\omega)}{\vartheta_4(\tilde{u}(z)+\frac{\omega}{2})} \end{pmatrix} \end{aligned}$$

where

$$(8.30) \quad \mathbf{F}(\infty) = \begin{pmatrix} \frac{\vartheta_3(\frac{\Omega n}{2})}{\vartheta_3(0)} & 0 \\ 0 & \frac{\vartheta_3(\frac{\Omega n}{2})}{\vartheta_3(0)} \end{pmatrix} = \begin{pmatrix} \frac{\vartheta_4(n\omega)}{\vartheta_3(0)} & 0 \\ 0 & \frac{\vartheta_4(n\omega)}{\vartheta_3(0)} \end{pmatrix},$$

solving the model RHP. The asymptotics at infinity are

$$(8.31) \quad \mathbf{M}(z) = I + \frac{\mathbf{M}_1}{z} + O(z^{-2})$$

where the matrix \mathbf{M}_1 has the form

$$(8.32) \quad \mathbf{M}_1 = \begin{pmatrix} * & \frac{\vartheta_3(-\tilde{u}_\infty+d+\frac{\Omega n}{2})\vartheta_3(\tilde{u}_\infty+d)}{\vartheta_3(\tilde{u}_\infty+d+\frac{\Omega n}{2})\vartheta_3(-\tilde{u}_\infty+d)} \frac{(\beta-\beta')+(\alpha'-\alpha)}{-4i} \\ \frac{\vartheta_3(\tilde{u}_\infty-d+\frac{\Omega n}{2})\vartheta_3(-\tilde{u}_\infty-d)}{\vartheta_3(-\tilde{u}_\infty-d+\frac{\Omega n}{2})\vartheta_3(\tilde{u}_\infty-d)} \frac{(\beta-\beta')+(\alpha'-\alpha)}{4i} & * \end{pmatrix}.$$

The matrix M_1 can be written in a cleaner fashion and in terms of the original parameters as follows:

PROPOSITION 8.1 *We have that*

$$(8.33) \quad [\mathbf{M}_1]_{12} = \frac{iA(\omega)\vartheta_4((n+1)\omega)}{\vartheta_4(n\omega)}, \quad [\mathbf{M}_1]_{21} = \frac{A(\omega)\vartheta_4(n\omega)}{i\vartheta_4((n-1)\omega)},$$

where

$$(8.34) \quad \omega = \frac{\pi(1+\zeta)}{2}, \quad A(\omega) = \frac{\pi\vartheta_1'(0)}{2\vartheta_1(\omega)}.$$

For a proof of this proposition, see appendix D of [6]. Notice that since \mathbf{M} solves the model RHP, we have that

$$(8.35) \quad \det \mathbf{M}(z) = 1, \quad z \in \mathbb{C}.$$

9 Parametrix at Outer Turning Points

We now consider small disks $D(\alpha, \varepsilon)$ and $D(\beta, \varepsilon)$ centered at the outer turning points. Denote $D = D(\alpha, \varepsilon) \cup D(\beta, \varepsilon)$. We will seek a local parametrix $\mathbf{U}_n(z)$ defined on D such that

$$(9.1) \quad \mathbf{U}_n(z) \text{ is analytic on } D \setminus \Sigma_S,$$

$$(9.2) \quad \mathbf{U}_{n+}(z) = \mathbf{U}_{n-}(z)j_S(z) \quad \text{for } z \in D \cap \Sigma_S,$$

$$(9.3) \quad \mathbf{U}_n(z) = \mathbf{M}(z)(I + O(n^{-1})) \quad \text{uniformly for } z \in \partial D.$$

We first construct the parametrix near β . The jumps j_S are given by

$$(9.4) \quad j_S(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } z \in (\beta - \varepsilon, \beta), \\ \begin{pmatrix} 1 & 0 \\ -e^{-nG(z)} & 1 \end{pmatrix} & \text{for } z \in (\beta, \beta + i\varepsilon), \\ \begin{pmatrix} 1 & 0 \\ e^{nG(z)} & 1 \end{pmatrix} & \text{for } z \in (\beta, \beta - i\varepsilon), \\ \begin{pmatrix} e^{-nG(z)} & e^{n(g_+(z)+g_-(z)-V(z)-l)} \\ 0 & e^{nG(z)} \end{pmatrix} & \text{for } z \in (\beta, \beta + \varepsilon). \end{cases}$$

If we let

$$(9.5) \quad \mathbf{U}_n(z) = \mathbf{Q}_n(z)e^{-n(g(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3},$$

then the jump conditions on \mathbf{Q}_n become

$$(9.6) \quad \mathbf{Q}_{n+}(z) = \mathbf{Q}_{n-}(z)j_Q(z)$$

where

$$(9.7) \quad j_Q(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } z \in (\beta - \varepsilon, \beta), \\ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & \text{for } z \in (\beta, \beta + i\varepsilon), \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \text{for } z \in (\beta, \beta - i\varepsilon), \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{for } z \in (\beta, \beta + \varepsilon), \end{cases}$$

where orientation is from left to right on horizontal contours, and down to up on vertical contours, according to Figure 7.1.

\mathbf{Q}_n can be constructed using Airy functions. The Airy function solves the differential equation $y'' = zy$ and has the following asymptotics at infinity:

$$(9.8) \quad \begin{aligned} \text{Ai}(z) &= \frac{1}{2\sqrt{\pi}z^{1/4}} e^{-\frac{2}{3}z^{3/2}} \left(1 - \frac{5}{48}z^{-3/2} + O(z^{-3}) \right), \\ \text{Ai}'(z) &= -\frac{1}{2\sqrt{\pi}} z^{1/4} e^{-\frac{2}{3}z^{3/2}} \left(1 + \frac{7}{48}z^{-3/2} + O(z^{-3}) \right), \end{aligned}$$

as $z \rightarrow \infty$ with $\arg z \in (-\pi + \varepsilon, \pi - \varepsilon)$ for any $\varepsilon > 0$. If we let

$$(9.9) \quad y_0(z) = \text{Ai}(z), \quad y_1(z) = \omega \text{Ai}(\omega z), \quad y_2(z) = \omega^2 \text{Ai}(\omega^2 z),$$

where $\omega = e^{2\pi i/3}$, then the functions $y_0, y_1,$ and y_2 satisfy the relation

$$(9.10) \quad y_0(z) + y_1(z) + y_2(z) = 0.$$

If we take

$$(9.11) \quad \Phi_\beta(z) = \begin{cases} \begin{pmatrix} y_0(z) & -y_2(z) \\ y_0'(z) & -y_2'(z) \end{pmatrix} & \text{for } \arg z \in (0, \frac{\pi}{2}), \\ \begin{pmatrix} -y_1(z) & -y_2(z) \\ -y_1'(z) & -y_2'(z) \end{pmatrix} & \text{for } \arg z \in (\frac{\pi}{2}, \pi), \\ \begin{pmatrix} -y_2(z) & y_1(z) \\ -y_2'(z) & y_1'(z) \end{pmatrix} & \text{for } \arg z \in (-\pi, -\frac{\pi}{2}), \\ \begin{pmatrix} y_0(z) & y_1(z) \\ y_0'(z) & y_1'(z) \end{pmatrix} & \text{for } \arg z \in (-\frac{\pi}{2}, 0), \end{cases}$$

then Φ_β satisfies jump conditions similar to (9.7), but for jumps on rays emanating from the origin rather than from β . We thus need to map the disk $D(\beta, \varepsilon)$ onto some convex neighborhood of the origin in order to take advantage of the function Φ_β . Our mapping should match the asymptotics of the Airy function in order to have the matching property (9.3).

To this end notice that, by (2.18), for $t \in [\beta', \beta]$, as $t \rightarrow \beta$,

$$(9.12) \quad \rho(t) = C(\beta - t)^{1/2} + O((\beta - t)^{3/2}), \quad C > 0.$$

It follows that, as $z \rightarrow \beta$ for $z \in [\beta', \beta]$,

$$(9.13) \quad \int_z^\beta \rho(t) dt = C_0(\beta - z)^{3/2} + O((\beta - z)^{5/2}), \quad C_0 = \frac{2}{3}C.$$

Thus,

$$(9.14) \quad \psi_\beta(z) = -\left\{ \frac{3\pi}{2} \int_z^\beta \rho(t) dt \right\}^{2/3}$$

is analytic at β and so extends to a conformal map from $D(\beta, \varepsilon)$ (for small enough ε) onto a convex neighborhood of the origin. Furthermore,

$$(9.15) \quad \psi_\beta(\beta) = 0, \quad \psi'_\beta(\beta) > 0;$$

therefore ψ_β is real negative on $(\beta - \varepsilon, \beta)$ and real positive on $(\beta, \beta + \varepsilon)$. Also, we can slightly deform the vertical pieces of the contour Σ_S close to β so that

$$(9.16) \quad \psi_\beta \{D(\beta, \varepsilon) \cap \Sigma_S\} = (-\varepsilon, \varepsilon) \cup (-i\varepsilon, i\varepsilon).$$

We now set

$$(9.17) \quad \mathbf{Q}_n(z) = \mathbf{E}_n^\beta(z) \Phi_\beta(n^{2/3} \psi_\beta(z))$$

so that

$$(9.18) \quad \mathbf{U}_n(z) = \mathbf{E}_n^\beta(z) \Phi_\beta(n^{2/3} \psi_\beta(z)) e^{-n(g(z) - \frac{V(z)}{2} - \frac{l}{2})\sigma_3}$$

where

$$(9.19) \quad \begin{aligned} \mathbf{E}_n^\beta(z) &= \mathbf{M}(z) \mathbf{L}_n^\beta(z)^{-1}, \\ \mathbf{L}_n^\beta(z) &= \frac{1}{2\sqrt{\pi}} \begin{pmatrix} n^{-1/6} \psi_\beta^{-1/4}(z) & 0 \\ 0 & n^{1/6} \psi_\beta^{1/4}(z) \end{pmatrix} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}, \end{aligned}$$

where we take the branch of $\psi_\beta^{1/4}$, which is positive on $(\beta, \beta + \varepsilon)$ and has a cut on $(\beta - \varepsilon, \beta)$. We claim that $\mathbf{E}_n^\beta(z)$ is analytic in $D(\beta, \varepsilon)$; thus $\mathbf{U}_n(z)$ has the jump conditions of j_S . This is clear, as both \mathbf{M} and \mathbf{L}_n^β have the same constant jump, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, on the interval $(\beta - \varepsilon, \beta]$ and are analytic elsewhere. The only other possible singularity for either \mathbf{M} or \mathbf{L}_n^β is the isolated singularity at β , and this is at most a fourth-root singularity and thus removable. It follows that $\mathbf{E}_n^\beta(z) = \mathbf{M}(z) \mathbf{L}_n^\beta(z)^{-1}$ is analytic on $D(\beta, \varepsilon)$; thus \mathbf{U}_n has the prescribed jumps in $D(\beta, \varepsilon)$.

We are left only to prove the matching condition (9.3). Using (9.8), one can check that, for z in each of the sectors of analyticity, $\Phi_\beta(n^{2/3} \psi_\beta(z))$ satisfies the following asymptotics as $n \rightarrow \infty$:

$$(9.20) \quad \begin{aligned} &\Phi_\beta(n^{2/3} \psi_\beta(z)) \\ &= \frac{1}{2\sqrt{\pi}} n^{\frac{1}{6}\sigma_3} \psi_\beta(z)^{-\frac{1}{4}\sigma_3} \left[\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} + \frac{\psi_\beta(z)^{-3/2}}{48n} \begin{pmatrix} -5 & 5i \\ -7 & -7i \end{pmatrix} + O(n^{-2}) \right] \\ &\quad \times e^{-\frac{2}{3}n\psi_\beta(z)^{3/2}\sigma_3}, \end{aligned}$$

where we always take the principal branch of $\psi_\beta(z)^{3/2}$. As such, $\psi_\beta(z)^{3/2}$ is two-valued for $z \in (\beta - \varepsilon, \beta)$, so that

$$(9.21) \quad \left[\frac{2}{3} \psi_\beta(x)^{3/2} \right]_{\pm} = \mp \pi i \int_x^\beta \rho(t) dt.$$

Notice that, by (2.30) and (2.33), for $x \in (\beta - \varepsilon, \beta)$,

$$(9.22) \quad 2g_{\pm}(x) - V(x) = l \pm 2\pi i \int_x^\beta \rho(t) dt.$$

This implies that

$$(9.23) \quad \begin{aligned} [2g_+(\beta) - V(\beta)] - [2g_+(x) - V(x)] &= -2\pi i \int_x^\beta \rho(t) dt, \\ [2g_-(\beta) - V(\beta)] - [2g_-(x) - V(x)] &= 2\pi i \int_x^\beta \rho(t) dt. \end{aligned}$$

Combining these equations with (9.21) gives

$$(9.24) \quad \left[\frac{2}{3} \psi_\beta(x)^{3/2} \right]_\pm = \frac{1}{2} [(2g_\pm(\beta) - V(\beta)) - (2g_\pm(x) - V(x))].$$

This equation can be extended into the upper and lower planes, respectively, giving

$$(9.25) \quad \frac{2}{3} \psi_\beta(z)^{3/2} = \frac{1}{2} [(2g_\pm(\beta) - V(\beta)) - (2g(z) - V(z))] \quad \text{for } \pm \operatorname{Im} z > 0.$$

Since, by (9.22), $2g_\pm(\beta) - V(\beta) = l$, we get that

$$(9.26) \quad \frac{2}{3} \psi_\beta(z)^{3/2} = -g(z) + \frac{V(z)}{2} + \frac{l}{2}$$

for z throughout $D(\beta, \varepsilon)$. Plugging (9.20) and (9.26) into (9.18), we get

$$(9.27) \quad \begin{aligned} \mathbf{U}_n(z) &= \mathbf{M}(z) \mathbf{L}_n^\beta(z)^{-1} \frac{1}{2\sqrt{\pi}} n^{-\frac{1}{6}\sigma_3} \psi_\beta(z)^{-\frac{1}{4}\sigma_3} \\ &\times \left[\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} + \frac{\psi_\beta(z)^{-3/2}}{48n} \begin{pmatrix} -5 & 5i \\ -7 & -7i \end{pmatrix} + O(n^{-2}) \right] \\ &\times e^{n(g(z) - \frac{V(z)}{2} - \frac{l}{2})\sigma_3} e^{-n(g(z) - \frac{V(z)}{2} - \frac{l}{2})\sigma_3} \\ &= \mathbf{M}(z) \left[I + \frac{\psi_\beta(z)^{-3/2}}{48n} \begin{pmatrix} 1 & 6i \\ 6i & -1 \end{pmatrix} + O(n^{-2}) \right]. \end{aligned}$$

Thus we have that \mathbf{U}_n satisfies conditions (9.1), (9.2), and (9.3).

A similar construction gives the parametrix at α (see [6]). Namely, if we let

$$(9.28) \quad \psi_\alpha(z) = - \left\{ \frac{3\pi}{2} \int_\alpha^z \rho(t) dt \right\}^{2/3},$$

then ψ_α is analytic throughout $D(\alpha, \varepsilon)$, real-valued on the real line, and has negative derivative at α . Also, let

$$(9.29) \quad \Phi_\alpha(z) = \Phi_\beta(z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then we can take

$$(9.30) \quad \mathbf{U}_n(z) = \mathbf{E}_n^\alpha(z) \Phi_\alpha(n^{2/3} \psi_\alpha(z)) e^{-n(g(z) - \frac{V(z)}{2} - \frac{l}{2})\sigma_3}$$

for $z \in D(\alpha, \varepsilon)$, where

$$(9.31) \quad \begin{aligned} \mathbf{E}_n^\alpha(z) &= \mathbf{M}(z)\mathbf{L}_n^\alpha(z)^{-1}, \\ \mathbf{L}_n^\alpha(z) &= \frac{1}{2\sqrt{\pi}} \begin{pmatrix} n^{-1/6}\psi_\alpha^{-1/4}(z) & 0 \\ 0 & n^{1/6}\psi_\alpha^{1/4}(z) \end{pmatrix} \begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix}. \end{aligned}$$

Similar to (9.27), we get, as $n \rightarrow \infty$,

$$(9.32) \quad \mathbf{U}_n(z) = \mathbf{M}(z) \left[I + \frac{\psi_\alpha(z)^{-3/2}}{48n} \begin{pmatrix} 1 & -6i \\ -6i & -1 \end{pmatrix} + O(n^{-2}) \right].$$

10 Parametrix at the Inner Turning Points

We now consider small disks $D(\alpha', \varepsilon)$ and $D(\beta', \varepsilon)$ centered at the inner turning points. Denote $\tilde{D} = D(\alpha', \varepsilon) \cup D(\beta', \varepsilon)$. We will seek a local parametrix $\mathbf{U}_n(z)$ defined on \tilde{D} such that

$$(10.1) \quad \mathbf{U}_n(z) \text{ is analytic on } \tilde{D} \setminus \Sigma_S.$$

$$(10.2) \quad \mathbf{U}_{n+}(z) = \mathbf{U}_{n-}(z)j_S(z) \quad \text{for } z \in \tilde{D} \cap \Sigma_S.$$

$$(10.3) \quad \mathbf{U}_n(z) = \mathbf{M}(z)(I + O(n^{-1})) \quad \text{uniformly for } z \in \partial\tilde{D}.$$

We first construct the parametrix near α' . Let

$$(10.4) \quad \mathbf{U}_n(z) = \tilde{\mathbf{Q}}_n(z) e^{\mp \frac{in\pi z}{2\gamma} \sigma_3} e^{-n(g(z) - \frac{V(z)}{2} - \frac{I}{2})\sigma_3} \quad \text{for } \pm \operatorname{Im} z \geq 0.$$

Then the jumps for $\tilde{\mathbf{Q}}_n$ are

$$(10.5) \quad j_{\tilde{\mathbf{Q}}}(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } z \in (\alpha' - \varepsilon, \alpha'), \\ \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} & \text{for } z \in (\alpha', \alpha' + \varepsilon), \\ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} & \text{for } z \in (\alpha', \alpha' + i\varepsilon), \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{for } z \in (\alpha', \alpha' - i\varepsilon), \end{cases}$$

where orientation is taken from left to right on horizontal contours, and down to up on vertical contours according to Figure 7.1 (for a calculation of the jumps see

appendix C in [6]). We now take

$$(10.6) \quad \Phi_{\alpha'}(z) = \begin{cases} \begin{pmatrix} y_2(z) & -y_0(z) \\ y_2'(z) & -y_0'(z) \end{pmatrix} & \text{for } \arg z \in (0, \frac{\pi}{2}), \\ \begin{pmatrix} y_2(z) & y_1(z) \\ y_2'(z) & y_1'(z) \end{pmatrix} & \text{for } \arg z \in (\frac{\pi}{2}, \pi), \\ \begin{pmatrix} y_1(z) & -y_2(z) \\ y_1'(z) & -y_2'(z) \end{pmatrix} & \text{for } \arg z \in (-\pi, -\frac{\pi}{2}), \\ \begin{pmatrix} y_1(z) & y_0(z) \\ y_1'(z) & y_0'(z) \end{pmatrix} & \text{for } \arg z \in (-\frac{\pi}{2}, 0). \end{cases}$$

Then $\Phi_{\alpha'}(z)$ solves a RHP similar to that of $\tilde{\mathbf{Q}}_n$, but for jumps emanating from the origin rather than from α' .

Notice that, by (2.18), for $t \in [\alpha, \alpha']$, as $t \rightarrow \alpha'$,

$$(10.7) \quad \rho(t) = \frac{1}{2\gamma} - C(\alpha' - t)^{1/2} + O((\alpha' - t)^{3/2}), \quad C > 0.$$

It follows that, as $z \rightarrow \alpha'$ for $z \in [\alpha, \alpha']$,

$$(10.8) \quad \int_z^{\alpha'} \left(\frac{1}{2\gamma} - \rho(t) \right) dt = C_0(\alpha' - z)^{3/2} + O((\alpha' - z)^{5/2}), \quad C_0 = \frac{2}{3}C.$$

Thus,

$$(10.9) \quad \psi_{\alpha'}(z) = - \left\{ \frac{3\pi}{2} \int_z^{\alpha'} \left(\frac{1}{2\gamma} - \rho(t) \right) dt \right\}^{2/3}$$

is analytic at α' , and so extends to a conformal map from $D(\alpha', \varepsilon)$ onto a convex neighborhood of the origin. Furthermore,

$$(10.10) \quad \psi_{\alpha'}(\alpha') = 0, \quad \psi'_{\alpha'}(\alpha') > 0;$$

consequently, $\psi_{\alpha'}$ is real negative on $(\alpha' - \varepsilon, \alpha')$ and real positive on $(\alpha', \alpha' + \varepsilon)$. Again, we can slightly deform the vertical pieces of the contour Σ_S close to α' so that

$$(10.11) \quad \psi_{\alpha'}\{D(\alpha', \varepsilon) \cap \Sigma_S\} = (-\varepsilon, \varepsilon) \cup (-i\varepsilon, i\varepsilon).$$

We now take

$$(10.12) \quad \tilde{\mathbf{Q}}_n(z) = \mathbf{E}_n^{\alpha'}(z) \Phi_{\alpha'}(n^{2/3} \psi_{\alpha'}(z))$$

where

$$(10.13) \quad \begin{aligned} \mathbf{E}_n^{\alpha'}(z) &= \mathbf{M}(z) e^{\pm \frac{i\Omega n}{2} \sigma_3} \tilde{\mathbf{L}}_n(z)^{-1} \quad \text{for } \pm \operatorname{Im} z \geq 0, \\ \mathbf{L}_n^{\alpha'}(z) &= \frac{1}{2\sqrt{\pi}} \begin{pmatrix} n^{-1/6} \psi_{\alpha'}^{-1/4}(z) & 0 \\ 0 & n^{1/6} \psi_{\alpha'}^{1/4}(z) \end{pmatrix} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}, \end{aligned}$$

and we take the branch of $\psi_{\alpha'}^{1/4}$ that is positive on $(\alpha', \alpha' + \varepsilon)$ and has a cut on $(\alpha' - \varepsilon, \alpha')$. \mathbf{U}_n then becomes

$$(10.14) \quad \begin{aligned} \mathbf{U}_n(z) &= \mathbf{M}(z) e^{\pm \frac{i\Omega n}{2} \sigma_3} \mathbf{L}_n^{\alpha'}(z)^{-1} \Phi_{\alpha'}(n^{2/3} \psi_{\alpha'}(z)) \\ &\quad \times e^{\mp \frac{in\pi z}{2\gamma} \sigma_3} e^{-n(g(z) - \frac{V(z)}{2} - \frac{1}{2}) \sigma_3} \quad \text{for } \pm \operatorname{Im} z \geq 0. \end{aligned}$$

The function $\Phi_{\alpha'}(n^{2/3} \psi_{\alpha'}(z))$ has the jumps j_S , and we claim that the prefactor $\mathbf{E}_n^{\alpha'}$ is analytic in $D(\alpha', \varepsilon)$, and thus does not change these jumps. This can be seen, as

$$(10.15) \quad \mathbf{M}_+(z) e^{\frac{i\Omega n}{2} \sigma_3} = \mathbf{M}_-(z) e^{-\frac{i\Omega n}{2} \sigma_3} e^{\frac{i\Omega n}{2} \sigma_3} j_M e^{\frac{i\Omega n}{2} \sigma_3};$$

thus the jump for the function $\mathbf{M}(z) e^{\pm \frac{i\Omega n}{2} \sigma_3}$ is

$$(10.16) \quad e^{\frac{i\Omega n}{2} \sigma_3} j_M e^{\frac{i\Omega n}{2} \sigma_3} = \begin{cases} e^{\frac{i\Omega n}{2} \sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{\frac{i\Omega n}{2} \sigma_3} & \text{for } z \in (\alpha' - \varepsilon, \alpha'), \\ e^{\frac{i\Omega n}{2} \sigma_3} e^{-i\Omega n \sigma_3} e^{\frac{i\Omega n}{2} \sigma_3} & \text{for } z \in (\alpha', \alpha' + \varepsilon), \end{cases}$$

or, equivalently,

$$(10.17) \quad e^{\frac{i\Omega n}{2} \sigma_3} j_M e^{\frac{i\Omega n}{2} \sigma_3} = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } z \in (\alpha' - \varepsilon, \alpha'), \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for } z \in (\alpha', \alpha' + \varepsilon), \end{cases}$$

which is exactly the same as the jump conditions for $\mathbf{L}_n^{\alpha'}$. Thus

$$\mathbf{E}_n^{\alpha'}(z) = \mathbf{M}(z) e^{\pm \frac{i\Omega n}{2} \sigma_3} \mathbf{L}_n^{\alpha'}(z)^{-1}$$

has no jumps in $D(\alpha', \varepsilon)$. The only other possible singularity for $\mathbf{E}_n^{\alpha'}$ is at α' , and this singularity is at most a fourth-root singularity and thus removable. Thus, $\mathbf{E}_n^{\alpha'}$ is analytic in $D(\alpha', \varepsilon)$, and $\tilde{\mathbf{Q}}_n$ has the prescribed jumps.

We are left to check that \mathbf{U}_n satisfies the matching condition (10.3). The large- n asymptotics of $\Phi_{\alpha'}(n^{2/3} \psi_{\alpha'}(z))$ are given in the different regions of analyticity as follows:

$$(10.18) \quad \begin{aligned} &\Phi_{\alpha'}(n^{2/3} \psi_{\alpha'}(z)) \\ &= \frac{1}{2\sqrt{\pi}} n^{-\frac{1}{6} \sigma_3} \psi_{\alpha'}(z)^{-\frac{1}{4} \sigma_3} \\ &\quad \times \left[\mp \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix} \pm \frac{\psi_{\alpha'}(z)^{-3/2}}{48n} \begin{pmatrix} -5i & 5 \\ 7i & 7 \end{pmatrix} + O(n^{-2}) \right] \\ &\quad \times e^{\frac{2}{3} n \psi_{\alpha'}(z)^{3/2} \sigma_3} \quad \text{for } \pm \operatorname{Im} z > 0, \end{aligned}$$

where we always take the principal branch of $\psi_{\alpha'}(z)^{3/2}$. As such, $\psi_{\alpha'}(z)^{3/2}$ is two-valued for $x \in (\alpha' - \varepsilon, \alpha')$, so that

$$(10.19) \quad \begin{aligned} \left[\frac{2}{3} \psi_{\alpha'}(x)^{3/2} \right]_{\pm} &= \mp \pi i \int_x^{\alpha'} \left(\frac{1}{2\gamma} - \rho(t) \right) dt \\ &= \mp \frac{\pi i}{2\gamma} (\alpha' - x) \pm \pi i \int_x^{\alpha'} \rho(t) dt. \end{aligned}$$

From (2.30) and (2.33), we have that

$$(10.20) \quad \begin{aligned} 2g_+(x) - V(x) &= l + 2\pi i \int_x^{\beta} \rho(t) dt, \\ 2g_-(x) - V(x) &= l - 2\pi i \int_x^{\beta} \rho(t) dt, \end{aligned}$$

for $x \in (\alpha' - \varepsilon, \alpha')$. These equations imply that

$$(10.21) \quad (2g_{\pm}(x) - V(x)) - (2g_{\pm}(\alpha') - V(\alpha')) = \pm 2\pi i \int_x^{\alpha'} \rho(t) dt.$$

We can therefore write (10.19) as

$$(10.22) \quad \begin{aligned} \left[\frac{2}{3} \psi_{\alpha'}(x)^{3/2} \right]_{\pm} &= \mp \frac{\pi i}{2\gamma} (\alpha' - x) \\ &\quad + \frac{1}{2} [(2g_{\pm}(x) - V(x)) - (2g_{\pm}(\alpha') - V(\alpha'))]. \end{aligned}$$

We can extend these equations into the upper and lower half-plane, respectively, obtaining

$$(10.23) \quad \begin{aligned} \frac{2}{3} \psi_{\alpha'}(z)^{3/2} &= \\ &\quad \mp \frac{\pi i}{2\gamma} (\alpha' - z) + \frac{1}{2} [(2g(z) - V(z)) - (2g_{\pm}(\alpha') - V(\alpha'))] \end{aligned}$$

for $\pm \operatorname{Im} z > 0$.

Using (10.20) at $x = \alpha'$, we can write

$$(10.24) \quad \begin{aligned} \frac{2}{3} \psi_{\alpha'}(z)^{3/2} &= \mp \frac{\pi i}{2\gamma} (\alpha' - z) + g(z) - \frac{V(z)}{2} - \frac{l}{2} \\ &\quad \mp \pi i \int_{\alpha'}^{\beta} \rho(t) dt \quad \text{for } \pm \operatorname{Im} z > 0 \end{aligned}$$

or, equivalently,

$$(10.25) \quad \frac{2}{3} \psi_{\alpha'}(z)^{3/2} = g(z) - \frac{V(z)}{2} - \frac{l}{2} \pm \frac{\pi i z}{2\gamma} \mp \frac{i(\Omega_n - \pi)}{2n} \quad \text{for } \pm \operatorname{Im} z > 0.$$

Plugging (10.18) and (10.24) into (10.14) gives

$$\begin{aligned}
\mathbf{U}_n(z) &= \mathbf{M}(z) e^{\pm \frac{i\Omega n}{2} \sigma_3} \mathbf{L}_n^{\alpha'}(z)^{-1} \frac{1}{2\sqrt{\pi}} n^{-\frac{1}{6} \sigma_3} \psi_{\alpha'}(z)^{-\frac{1}{4} \sigma_3} \\
&\quad \times \left[\mp \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix} \pm \frac{\psi_{\alpha'}(z)^{-3/2}}{48n} \begin{pmatrix} -5i & 5 \\ 7i & 7 \end{pmatrix} + O(n^{-2}) \right] \\
&\quad \times e^{n(g(z) - \frac{1}{2} - \frac{V(z)}{2}) \sigma_3} e^{\mp \frac{i\Omega n}{2} \sigma_3} e^{\pm \frac{i\pi}{2} \sigma_3} e^{\pm \frac{i n \pi z}{2\gamma} \sigma_3} e^{\mp \frac{i n \pi \bar{z}}{2\gamma} \sigma_3} \\
&\quad \times e^{-n(g(z) - \frac{V(z)}{2} - \frac{1}{2}) \sigma_3} \\
(10.26) \quad &= \mathbf{M}(z) e^{\pm \frac{i\Omega n}{2} \sigma_3} \mathbf{L}_n^{\alpha'}(z)^{-1} \frac{1}{2\sqrt{\pi}} n^{-\frac{1}{6} \sigma_3} \psi_{\alpha'}(z)^{-\frac{1}{4} \sigma_3} \\
&\quad \times \left[\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} + \frac{\psi_{\alpha'}(z)^{-3/2}}{48n} \begin{pmatrix} 5 & -5i \\ -7 & -7i \end{pmatrix} + O(n^{-2}) \right] e^{\mp \frac{i\Omega n}{2} \sigma_3} \\
&= \mathbf{M}(z) \left[I + \frac{\psi_{\alpha'}(z)^{-3/2}}{48n} e^{\pm \frac{i\Omega n}{2} \sigma_3} \begin{pmatrix} -1 & -6i \\ -6i & 1 \end{pmatrix} e^{\mp \frac{i\Omega n}{2} \sigma_3} + O(n^{-2}) \right] \\
&= \mathbf{M}(z) \left[I + \frac{\psi_{\alpha'}(z)^{-3/2}}{48n} \begin{pmatrix} -1 & -6i e^{\pm i\Omega n} \\ -6i e^{\mp i\Omega n} & 1 \end{pmatrix} + O(n^{-2}) \right] \\
&\quad \text{for } \pm \operatorname{Im} z > 0.
\end{aligned}$$

We can make a similar construction near β' (see [6]). Let

$$(10.27) \quad \psi_{\beta'}(z) = - \left\{ \frac{3\pi}{2} \int_{\beta'}^z \left(\frac{1}{2\gamma} - \rho(t) \right) dt \right\}^{2/3}.$$

This function is analytic in $D(\beta', \varepsilon)$ and has negative derivative at β' ; thus $\operatorname{Im} z$ and $\operatorname{Im} \psi_{\beta'}(z)$ have opposite signs for $z \in D(\beta', \varepsilon)$. Also, let

$$(10.28) \quad \Phi_{\beta'}(z) = \Phi_{\alpha'}(z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then we can take for $z \in D(\beta', \varepsilon)$,

$$(10.29) \quad \begin{aligned} \mathbf{U}_n(z) &= \mathbf{M}(z) e^{\pm \frac{i\Omega n}{2} \sigma_3} \mathbf{L}_n^{\beta'}(z)^{-1} \Phi_{\beta'}(n^{2/3} \psi_{\beta'}(z)) \\ &\quad \times e^{\mp \frac{i n \pi z}{2\gamma} \sigma_3} e^{-n(g(z) - \frac{V(z)}{2} - \frac{1}{2}) \sigma_3} \quad \text{for } \pm \operatorname{Im} z > 0, \end{aligned}$$

where

$$(10.30) \quad \mathbf{L}_n^{\beta'}(z) = \frac{1}{2\sqrt{\pi}} \begin{pmatrix} n^{-1/6} \psi_{\beta'}^{-1/4}(z) & 0 \\ 0 & n^{1/6} \psi_{\beta'}^{1/4}(z) \end{pmatrix} \begin{pmatrix} -1 & i \\ -1 & -i \end{pmatrix}.$$

Similar to (10.26), we obtain

$$(10.31) \quad \begin{aligned} \mathbf{U}_n(z) &= \mathbf{M}(z) \left[I + \frac{\psi_{\beta'}(z)^{-3/2}}{48n} \begin{pmatrix} -1 & 6i e^{\pm i\Omega n} \\ 6i e^{\mp i\Omega n} & 1 \end{pmatrix} + O(n^{-2}) \right] \\ &\quad \text{for } \pm \operatorname{Im} z > 0. \end{aligned}$$

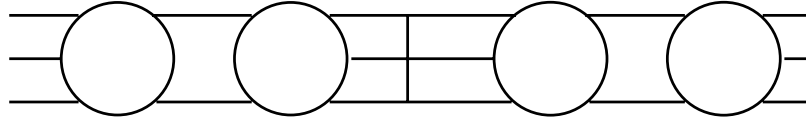


FIGURE 11.1. The contour Σ_X .

11 The Third and Final Transformation of the RHP

We now consider the contour Σ_X , consisting of the circles $\partial D(\alpha, \varepsilon)$, $\partial D(\alpha', \varepsilon)$, $\partial D(\beta', \varepsilon)$, and $\partial D(\beta, \varepsilon)$, all oriented counterclockwise, together with the parts of $\Sigma_S \setminus ([\alpha, \alpha'] \cup [\beta', \beta])$ that lie outside of the disks $D(\alpha, \varepsilon)$, $D(\alpha', \varepsilon)$, $D(\beta', \varepsilon)$, and $D(\beta, \varepsilon)$; see Figure 11.1.

We let

$$(11.1) \quad \mathbf{X}_n(z) = \begin{cases} \mathbf{S}_n(z)\mathbf{M}(z)^{-1} & \text{for } z \text{ outside disks } D(\alpha, \varepsilon), D(\alpha', \varepsilon), D(\beta', \varepsilon), D(\beta, \varepsilon), \\ \mathbf{S}_n(z)\mathbf{U}_n(z)^{-1} & \text{for } z \text{ inside disks } D(\alpha, \varepsilon), D(\alpha', \varepsilon), D(\beta', \varepsilon), D(\beta, \varepsilon). \end{cases}$$

Then $\mathbf{X}_n(z)$ solves the following RHP:

- (1) $\mathbf{X}_n(z)$ is analytic on $\mathbb{C} \setminus \Sigma_X$.
- (2) $\mathbf{X}_n(z)$ has the jump properties

$$(11.2) \quad \mathbf{X}_{n+}(x) = \mathbf{X}_{n-}(z)j_X(z)$$

where

$$(11.3) \quad j_X(z) = \begin{cases} \mathbf{M}(z)\mathbf{U}_n(z)^{-1} & \text{for } z \text{ on the circles,} \\ \mathbf{M}(z)j_S\mathbf{M}(z)^{-1} & \text{otherwise.} \end{cases}$$

- (3) As $z \rightarrow \infty$,

$$(11.4) \quad \mathbf{X}_n(z) \sim \mathbf{I} + \frac{\mathbf{X}_1}{z} + \frac{\mathbf{X}_2}{z^2} + \dots$$

Additionally, we have that $j_X(z)$ is uniformly close to the identity in the following sense:

$$(11.5) \quad j_X(z) = \begin{cases} I + O(n^{-1}) & \text{uniformly on the circles,} \\ I + O(e^{-C(z)n}) & \text{on the rest of } \Sigma_X, \end{cases}$$

where $C(z)$ is a positive, continuous function satisfying (6.7). If we set

$$(11.6) \quad j_X^0(z) = j_X(z) - I,$$

then (11.5) becomes

$$(11.7) \quad j_X^0(z) = \begin{cases} O(n^{-1}) & \text{uniformly on the circles,} \\ O(e^{-C(z)n}) & \text{on the rest of } \Sigma_X. \end{cases}$$

The solution to the RHP for \mathbf{X}_n is based on the following lemma:

LEMMA 11.1 *Suppose $v(z)$ is a function on Σ_X solving the equation*

$$(11.8) \quad v(z) = I - \frac{1}{2\pi i} \int_{\Sigma_X} \frac{v(u)j_X^0(u)}{z_- - u} du \quad \text{for } z \in \Sigma_X$$

where z_- means the value of the integral on the minus side of Σ_X . Then

$$(11.9) \quad \mathbf{X}_n(z) = I - \frac{1}{2\pi i} \int_{\Sigma_X} \frac{v(u)j_X^0(u)}{z - u} du \quad \text{for } z \in \mathbb{C} \setminus \Sigma_X$$

solves the RHP for \mathbf{X}_n .

The proof of this lemma is immediate from the jump property of the Cauchy transform. By assumption

$$(11.10) \quad \mathbf{X}_{n-}(z) = v(z),$$

and the additive jump of the Cauchy transform gives

$$(11.11) \quad \mathbf{X}_{n+}(z) - \mathbf{X}_{n-}(z) = v(z)j_X^0(z) = \mathbf{X}_{n-}(z)j_X^0(z);$$

thus $\mathbf{X}_{n+}(z) = \mathbf{X}_{n-}(z)j_X(z)$. Asymptotics at infinity are given by (11.9).

The solution to equation (11.8) is given by a series of perturbation theory. Namely, the solution is

$$(11.12) \quad v(z) = I + \sum_{k=1}^{\infty} v_k(z)$$

where

$$(11.13) \quad v_k(z) = -\frac{1}{2\pi i} \int_{\Sigma_X} \frac{v_{k-1}(u)j_X^0(u)}{z - u} du, \quad v_0(z) = I.$$

This function clearly solves (11.8) provided the series converges, which it does for sufficiently large n . Indeed, by (11.5),

$$(11.14) \quad |v_k(z)| \leq \left(\frac{C}{n}\right)^k \frac{1}{1 + |z|} \quad \text{for some constant } C > 0;$$

thus the series (11.12) is dominated by a convergent geometric series and thus converges absolutely. This in turn gives

$$(11.15) \quad \mathbf{X}_n(z) = I + \sum_{k=1}^{\infty} \mathbf{X}_{n,k}(z)$$

where

$$(11.16) \quad \mathbf{X}_{n,k}(z) = -\frac{1}{2\pi i} \int_{\Sigma_X} \frac{v_{k-1}(u)j_X^0(u)}{z - u} du.$$

We will need to compute

$$(11.17) \quad \mathbf{X}_{n,1}(z) = -\frac{1}{2\pi i} \int_{\Sigma_X} \frac{j_X^0(u)}{z-u} du.$$

12 Evaluation of \mathbf{X}_1

We are interested in the matrix \mathbf{X}_1 , which gives the $\frac{1}{z}$ -term of $\mathbf{X}_n(z)$ at infinity; see (11.4). By (11.9),

$$(12.1) \quad \mathbf{X}_1 = -\frac{1}{2\pi i} \int_{\Sigma_X} v(u) j_X^0(u) du;$$

hence by (11.12) and (11.14),

$$(12.2) \quad \mathbf{X}_1 = -\frac{1}{2\pi i} \int_{\Sigma_X} j_X^0(u) du + O(n^{-2}).$$

We would like to evaluate the integral

$$(12.3) \quad -\frac{1}{2\pi i} \int_{\Sigma_X} j_X^0(u) du,$$

with an error of the order of n^{-2} . By (11.7), it is enough to evaluate this integral over the circles $\partial D(\alpha, \varepsilon)$, $\partial D(\alpha', \varepsilon)$, $\partial D(\beta', \varepsilon)$, and $\partial D(\beta, \varepsilon)$. It can be shown (see [6]) that the matrix-valued function $j_X^0(z)$ is analytic in the punctured disks; hence

$$(12.4) \quad \mathbf{X}_1 = -\left(\operatorname{Res}_{z=\alpha} + \operatorname{Res}_{z=\alpha'} + \operatorname{Res}_{z=\beta'} + \operatorname{Res}_{z=\beta}\right) j_X^0(z) + O(n^{-2}).$$

In particular, we are interested in the [12] entry of this matrix. Calculation of these residues gives that

$$(12.5) \quad [\mathbf{X}_1]_{12} = \frac{c(n)}{n} + O(n^{-2}),$$

where $c(n)$ is an explicit quasi-periodic function of n ; see [6].

13 Large- n Asymptotic Formula for h_n

We evaluate the large- n asymptotic behavior of h_{nn} and then we use formula (1.28). By (4.7), $h_{nn} = [\mathbf{P}_1]_{12}$, and by (5.16),

$$(13.1) \quad [\mathbf{P}_1]_{12} = [\mathbf{R}_1]_{12} \left(-\frac{n\pi i}{\gamma}\right);$$

hence

$$(13.2) \quad h_{nn} = [\mathbf{R}_1]_{12} \left(-\frac{n\pi i}{\gamma}\right).$$

Furthermore, from (6.4) we obtain that

$$(13.3) \quad h_{nn} = e^{nl} [\mathbf{T}_1]_{12} \left(-\frac{n\pi i}{\gamma} \right),$$

and from (7.2) that

$$(13.4) \quad h_{nn} = e^{nl} [\mathbf{S}_1]_{12} \left(-\frac{n\pi i}{\gamma} \right).$$

It follows from (11.1) that

$$(13.5) \quad \mathbf{S}_1 = \mathbf{M}_1 + \mathbf{X}_1.$$

By (8.33),

$$(13.6) \quad [\mathbf{M}_1]_{12} = \frac{iA\vartheta_4((n+1)\omega)}{\vartheta_4(n\omega)}, \quad \omega = \frac{\pi(1+\zeta)}{2}, \quad A = \frac{\pi\vartheta_1'(0)}{2\vartheta_1(\omega)}.$$

Combining this with (12.5) gives

$$(13.7) \quad h_{nn} = e^{nl} \left[\frac{iA\vartheta_4((n+1)\omega)}{\vartheta_4(n\omega)} + \frac{c(n)}{n} + O(n^{-2}) \right] \left(-\frac{n\pi i}{\gamma} \right).$$

By (2.36),

$$(13.8) \quad e^{l/2} = \frac{\pi\vartheta_1'(0)}{2e\vartheta_1(\omega)} = \frac{A}{e};$$

hence

$$(13.9) \quad \begin{aligned} h_{nn} &= \left(\frac{A}{e} \right)^{2n} \left[iA \frac{\vartheta_4((n+1)\omega)}{\vartheta_4(n\omega)} + \frac{c(n)}{n} + O(n^{-2}) \right] \left(-\frac{n\pi i}{\gamma} \right) \\ &= \frac{n\pi A^{2n+1} \vartheta_4((n+1)\omega)}{\gamma e^{2n} \vartheta_4(n\omega)} \left(1 + \frac{c_1(n)}{n} + O(n^{-2}) \right) \end{aligned}$$

where

$$(13.10) \quad c_1(n) = \frac{c(n)\vartheta_4(n\omega)}{iA\vartheta_4((n+1)\omega)}.$$

From (1.28) and the Stirling formula we obtain that

$$(13.11) \quad \frac{h_n}{(n!)^2} = \frac{n^{2n} h_{nn}}{(n!)^2 (2\gamma)^{2n}} = \left(\frac{e}{2\gamma} \right)^{2n} \frac{h_{nn}}{2\pi n} \left(1 - \frac{1}{6n} + O(n^{-2}) \right);$$

hence by (13.9),

$$(13.12) \quad \begin{aligned} \frac{h_n}{(n!)^2} &= \left(\frac{e}{2\gamma} \right)^{2n} \frac{1}{2\pi n} \frac{n\pi A^{2n+1} \vartheta_4((n+1)\omega)}{\gamma e^{2n} \vartheta_4(n\omega)} \\ &\quad \times \left(1 + \frac{c_1(n)}{n} - \frac{1}{6n} + O(n^{-2}) \right) \\ &= G^{2n+1} \frac{\vartheta_4((n+1)\omega)}{\vartheta_4(n\omega)} \left(1 + \frac{c_2(n)}{n} + O(n^{-2}) \right), \end{aligned}$$

where

$$(13.13) \quad G = \frac{A}{2\gamma} = \frac{\pi \vartheta_1'(0)}{4\gamma \vartheta_1(\omega)}, \quad c_2(n) = c_1(n) - \frac{1}{6}.$$

Observe that $c_1(n)$ has the form

$$(13.14) \quad c_1(n) = f(n\omega, \omega),$$

where $f(x, \omega)$ is a real analytic function, periodic with respect to both x and ω , of periods π and 2π , respectively. Remarkably, it can be shown using classical theta function identities (see, e.g., [29]), that $f(x, \omega)$ does not depend on either x or ω . In fact (see [6]),

$$(13.15) \quad c_1(n) \equiv \frac{1}{6};$$

thus

$$(13.16) \quad c_2(n) \equiv 0.$$

We can summarize these results in the following proposition:

PROPOSITION 13.1 *As $n \rightarrow \infty$,*

$$(13.17) \quad \frac{h_n}{(n!)^2} = G^{2n+1} \frac{\vartheta_4((n+1)\omega)}{\vartheta_4(n\omega)} (1 + O(n^{-2})),$$

where

$$(13.18) \quad G = \frac{\pi \vartheta_1'(0)}{4\gamma \vartheta_1(\omega)}.$$

14 Large- n Asymptotics of Z_n

By substituting (13.17) into (1.23) we obtain that

$$(14.1) \quad \begin{aligned} \frac{\tau_n}{\prod_{k=0}^{n-1} (k!)^2} &= 2^{n^2} \prod_{k=0}^{n-1} \frac{h_k}{(k!)^2} \\ &= 2^{n^2} h_0 \prod_{k=1}^{n-1} \left[G^{2k+1} \frac{\vartheta_4((k+1)\omega)}{\vartheta_4(k\omega)} (1 + O(k^{-2})) \right] \\ &= C \vartheta_4(n\omega) (2G)^{n^2} (1 + O(n^{-1})), \end{aligned}$$

where $C > 0$ does not depend on n . Thus, by (1.14),

$$(14.2) \quad Z_n = \frac{[\sinh(\gamma - t) \sinh(\gamma + t)]^{n^2} \tau_n}{(\prod_{k=0}^{n-1} k!)^2} = C \vartheta_4(n\omega) F^{n^2} (1 + O(n^{-1})),$$

where

$$(14.3) \quad F = 2G \sinh(\gamma - t) \sinh(\gamma + t) = \frac{\pi \sinh(\gamma - t) \sinh(\gamma + t) \vartheta_1'(0)}{2\gamma \vartheta_1(\omega)}.$$

Theorem 1.1 is proved.

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