Composition Operators on
Spaces of Analytic Functions

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Some History
Functional analysis began a little more than 100 years ago

Questions had to do with interpreting differential operators as linear transformations on vector spaces of functions

Sets of functions needed structure connected to the convergence implicit in the limit processes of the operators

Concrete functional analysis developed with results on spaces of integrable functions, with special classes of differential operators, and sometimes used better behaved inverses of differential operators
The abstraction of these ideas led to:

- Banach and Hilbert spaces

- Bounded operators, unbounded closed operators, compact operators

- Spectral theory as a generalization of Jordan form and diagonalizability

- Multiplication operators as an extension of diagonalization of matrices

Concrete examples and development of theory interact:

- Shift operators as an examples of asymmetric behavior possible in operators on infinite dimensional spaces

Studying *composition operators* can be seen as extension of this process
The classical Banach spaces are spaces of functions on a set $X$: if $\varphi$ is a map of $X$ onto itself, we can imagine a composition operator with symbol $\varphi$,

$$C_{\varphi}f = f \circ \varphi$$

for $f$ in the Banach space.

This operator is formally linear:

$$(a.f + b.g) \circ \varphi = a.f \circ \varphi + b.g \circ \varphi$$

But other properties, like “Is $f \circ \varphi$ in the space?” clearly depend on the map $\varphi$ and the Banach space of functions.
Some Examples
Several classical operators are composition operators. For example, we may regard $\ell^p(\mathbb{N})$ as the space of functions of $\mathbb{N}$ into $\mathbb{C}$ that are $p^{th}$ power integrable with respect to counting measure by thinking $x$ in $\ell^p$ as the function $x(k) = x_k$. If $\varphi : \mathbb{N} \to \mathbb{N}$ is given by $\varphi(k) = k + 1$, then

$$(C_\varphi x)(k) = x(\varphi(k)) = x(k + 1) = x_{k+1},$$

that is,

$$C_\varphi : (x_1, x_2, x_3, x_4, \cdots) \mapsto (x_2, x_3, x_4, x_5, \cdots)$$

so $C_\varphi$ is the “backward shift”.

In fact, backward shifts of all multiplicities can be represented as composition operators.
Moreover, composition operators often come up in studying other operators. For example, if we think of the operator of multiplication by $z^2$,

$$(M_{z^2} f)(z) = z^2 f(z)$$

it is easy to see that $M_{z^2}$ commutes with multiplication by any bounded function. Also, $C_{-z}$ commutes with $M_{z^2}$:

$$(M_{z^2} C_{-z} f)(z) = M_{z^2} f(-z) = z^2 f(-z)$$

and

$$(C_{-z} M_{z^2} f)(z) = C_{-z} (z^2 f(z)) = (-z)^2 f(-z) = z^2 f(-z)$$

In fact, in some contexts, the set of operators that commute with $M_{z^2}$ is the algebra generated by the multiplication operators and the composition operator $C_{-z}$. 
Also, Forelli showed that all isometries of $H^p(\mathbb{D})$, $1 < p < \infty$, $p \neq 2$, are weighted composition operators.

In these lectures, we will not consider absolutely arbitrary composition operators; a more interesting theory can be developed by restricting our attention to more specific cases.
Our Context
Definition

Banach space of functions on set $X$ is called a \textit{functional Banach space} if

1. vector operations are the pointwise operations
2. $f(x) = g(x)$ for all $x$ in $X$ implies $f = g$ in the space
3. $f(x) = f(y)$ for all $f$ in the space implies $x = y$ in $X$
4. $x \mapsto f(x)$ is a bounded linear functional for each $x$ in $X$

We denote the linear functional in 4. by $K_x$, that is, for all $f$ and $x$,

$$K_x(f) = f(x)$$

and if the space is a Hilbert space, $K_x$ is the function in the space with

$$\langle f, K_x \rangle = f(x)$$
Examples

(1) $\ell^p(\mathbb{N})$ is a functional Banach space, as above

(2) $C([0, 1])$ is a functional Banach space

(3) $L^p([0, 1])$ is not a functional Banach space because

$$f \mapsto f(1/2)$$

is not a bounded linear functional on $L^p([0, 1])$

Exercise 1: Prove the assertion in (3) above.

We will consider functional Banach spaces whose functions are analytic on the underlying set $X$;

this what we mean by “Banach space of analytic functions”
Examples (cont’d) Some Hilbert spaces of analytic functions:

★(4) Hardy Hilbert space: \( X = \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \)

\[
H^2(\mathbb{D}) = \{ f \text{ analytic in } \mathbb{D} : f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ with } \| f \|_{H^2}^2 = \sum |a_n|^2 < \infty \}
\]

where for \( f \) and \( g \) in \( H^2(\mathbb{D}) \), we have \( \langle f, g \rangle = \sum a_n \overline{b_n} \)

★(5) Bergman Hilbert space: \( X = \mathbb{D} \)

\[
A^2(\mathbb{D}) = \{ f \text{ analytic in } \mathbb{D} : \| f \|_{A^2}^2 = \int_{\mathbb{D}} |f(\zeta)|^2 \frac{dA(\zeta)}{\pi} < \infty \}
\]

where for \( f \) and \( g \) in \( A^2(\mathbb{D}) \), we have \( \langle f, g \rangle = \int f(\zeta) \overline{g(\zeta)} \frac{dA(\zeta)}{\pi} \)

Exercise 2: Prove that the Bergman space is complete.

(6) Dirichlet space: \( X = \mathbb{D} \)

\[
\mathcal{D} = \{ f \text{ analytic in } \mathbb{D} : \| f \|_{\mathcal{D}}^2 = \| f \|_{H^2}^2 + \int_{\mathbb{D}} |f'(\zeta)|^2 \frac{dA(\zeta)}{\pi} < \infty \}
\]

(7) generalizations where \( X = B_N \), the ball, or \( X = \mathbb{D}^N \), the polydisk.
If $\mathcal{H}$ is a Hilbert space of complex-valued analytic functions on the domain $\Omega$ in $\mathbb{C}$ or $\mathbb{C}^N$ and $\varphi$ is an analytic map of $\Omega$ into itself, the \textit{composition operator} $C_\varphi$ on $\mathcal{H}$ is the operator given by

$$ (C_\varphi f)(z) = f(\varphi(z)) \quad \text{for } f \text{ in } \mathcal{H} $$

At least formally, this defines $C_\varphi$ as a linear transformation.

In this context, the study of composition operators was initiated about 40 years ago by Nordgren, Schwartz, Rosenthal, Caughran, Kamowitz, and others.
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**Goal:**

relate the properties of $\varphi$ as a function with properties of $C_\varphi$ as an operator.
Kernel Functions
Backtrack: Show $H^2$ is a functional Hilbert space.

For a point $\alpha$ in the disk $\mathbb{D}$, the kernel function $K_\alpha$ is the function in $H^2(\mathbb{D})$ such that for all $f$ in $H^2(\mathbb{D})$, we have

$$\langle f, K_\alpha \rangle = f(\alpha)$$

$f$ and $K_\alpha$ are in $H^2$, so $f(z) = \sum a_n z^n$ and $K_\alpha(z) = \sum b_n z^n$

for some coefficients. Thus, for each $f$ in $H^2$,

$$\sum a_n \alpha^n = f(\alpha) = \langle f, K_\alpha \rangle = \sum a_n \overline{b_n}$$

The only way this can be true is for $b_n = \overline{a^n} = \overline{\alpha^n}$ and

$$K_\alpha(z) = \sum \overline{\alpha^n} z^n = \frac{1}{1 - \overline{\alpha}z}$$
Today, we’ll mostly consider the Hardy Hilbert space, $H^2$

$$H^2 = \{ f = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n|^2 < \infty \}$$

which is also described as

$$H^2 = \{ f \text{ analytic in } \mathbb{D} : \sup_{0<r<1} \int |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty \}$$

and for $f$ in $H^2$

$$\|f\|^2 = \sup_{0<r<1} \int |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} = \int |f(e^{i\theta})|^2 \frac{d\theta}{2\pi} = \sum |a_n|^2$$

For $f$ in $H^2$, $\lim_{r \to 1^-} f(re^{i\theta}) = f^*(e^{i\theta})$ exists for almost all $0 \leq \theta \leq 2\pi$ and $f^*$ is in $L^2(\partial \mathbb{D})$ with $\|f\|^2 = \|f^*\|^2 = \sum_{n=0}^{\infty} |a_n|^2$

(so we will immediately start systematically confusing $f^*$ and $f$ by saying $f$ is an analytic function on the (open) disk and $f$ is an $L^2$ function on the circle with no non-zero negative Fourier coefficients).
Notice that the set \( \{z^n\}_{n=0}^\infty \) is an orthonormal basis for \( H^2 \), so we get an obvious isomorphism of \( \ell^2 \) and \( H^2 \) by \((a_0, a_1, a_2, \cdots) \leftrightarrow \sum_{n=0}^\infty a_n z^n \) and this isomorphism relates the right shift on \( \ell^2 \) to multiplication by \( z \) in \( H^2 \).

**Exercise 3:** Similarly, we want another way to think about \( A^2(\mathbb{D}) \).

(a) Show that the set \( \{z^n\}_{n=0}^\infty \) is an *orthogonal* basis for \( A^2(\mathbb{D}) \).

(b) Find the norm of \( z^n \) in \( A^2(\mathbb{D}) \) for each non-negative integer \( n \).

(c) Find a condition (*) on the coefficients \( a_n \) so that if \( f \) is an analytic function on the disk with \( f(z) = \sum_{n=0}^\infty a_n z^n \), then \( f \) is in \( A^2(\mathbb{D}) \) if and only if (*).

(d) Use the ideas of (a)–(c) to show that for \( \alpha \) in the disk, the function \( K_\alpha \) in \( A^2(\mathbb{D}) \) so that \( \langle f, K_\alpha \rangle = f(\alpha) \) for every \( f \) in \( A^2(\mathbb{D}) \) is

\[
K_\alpha(z) = \frac{1}{(1 - \bar{\alpha}z)^2}
\]
Theorems from Complex Analysis
Theorem: (Littlewood Subordination Theorem)

Let \( \varphi \) be an analytic map of the unit disk into itself such that \( \varphi(0) = 0 \).

If \( G \) is a subharmonic function in \( \mathbb{D} \), then for \( 0 < r < 1 \)

\[
\int_0^{2\pi} G(\varphi(re^{i\theta})) \, d\theta \leq \int_0^{2\pi} G(re^{i\theta}) \, d\theta
\]

For \( H^2 \), the Littlewood subordination theorem plus some easy calculations for changes of variables induced by automorphisms of the disk yields the following estimate of the norm for composition operators on \( H^2 \):

\[
\left( \frac{1}{1 - |\varphi(0)|^2} \right)^{\frac{1}{2}} \leq \| C_\varphi \| \leq \left( \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right)^{\frac{1}{2}}
\]

and a similar estimate for the norm of \( C_\varphi \) on \( A^2 \).
On $H^2$ and on $A^2$, the operator $C_\varphi$ is bounded for all functions $\varphi$ that are analytic and map $\mathbb{D}$ into itself.

Not always true:

If the function $z$ is in $\mathcal{H}$, and $C_\varphi$ is bounded on $\mathcal{H}$, then $C_\varphi z = \varphi$ is in $\mathcal{H}$. For some maps $\varphi$ of the disk into itself, $\varphi$ is not a vector in the Dirichlet space, so $C_\varphi$ is not bounded for such $\varphi$.

This is the sort of result we seek, connecting the properties of the operator $C_\varphi$ with the analytic and geometric properties of $\varphi$. 