Composition operators on weighted Bergman spaces with admissible Békollé weights

Operators on Spaces of Analytic Functions
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Today’s my talk is organized as follows:

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This study is a joint work with J.K. Sharma.
I. Introduction

$H(\mathbb{D})$: the space of all analytic functions on the unit disk $\mathbb{D}$

d$A$: the normalized area measure on $\mathbb{D}$

$H^p$: Hardy spaces on $\mathbb{D}$ ($0 < p < \infty$)

$A^p_\alpha$: Weighted Bergman spaces on $\mathbb{D}$ with the normalized weighted measure $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$ ($0 < p < \infty$, $\alpha > -1$)

For a given analytic self-map $\phi$ of $\mathbb{D}$, the composition operator $C_\phi : H(\mathbb{D}) \to H(\mathbb{D})$ is defined by $C_\phi f(z) := (f \circ \phi)(z)$ for $z \in \mathbb{D}$, $f \in H(\mathbb{D})$. By the Littlewood subordination principle and the $\mathcal{M}$-invariance of $H^p$ or $A^p_\alpha$, we see that all $C_\phi$ are bounded on $H^p$ and also on $A^p_\alpha$.

However, it is known that there are an analytic self-maps which are not induce compact composition operators on $H^p$ or $A^p_\alpha$. (Example: $\phi(z) = \frac{z + 1}{2}$)
Well-known results

For each analytic self-map $\phi$ of $\mathbb{D}$, we consider the following counting function $N_{\phi,\gamma}$:

$$N_{\phi,\gamma}(z) := \sum_{w \in \phi^{-1}\{z\}} \left( \log \frac{1}{|w|} \right)^\gamma \quad (z \in \mathbb{D} \setminus \{\phi(0)\}, \gamma > 0),$$

where $\phi^{-1}\{z\}$ denotes the zero sequence of $\phi(\cdot) - z$, each point being repeated in the sequence according to its multiplicity. When $\gamma = 1$, $N_{\phi,1}$ is the classical Nevanlinna counting function.

It is known that these counting function play an important role in studies on the compactness of $C_\phi$.

Recall that the essential norm $\|T\|_e$ of a bounded operator $T$ on a Banach space is the distance from $T$ to the closed ideal of compact operators.
Shapiro’s compactness criteria

By using counting functions, J.H. Shapiro gave the following formulas for essential norms of $C_\phi$ on the Hardy space $H^2$ or the weighted Bergman space $A^2_{\alpha}$ ($\alpha > -1$).

\begin{align*}
\text{Theorem (Shapiro ‘87).} & \\
C_\phi \text{ on the Hardy space } H^2 & \Rightarrow \|C_\phi\|_e^2 = \limsup_{|z| \to 1} \frac{N_{\phi,1}(z)}{- \log |z|} \\
C_\phi \text{ on the weighted Bergman space } A^2_{\alpha} & \Rightarrow \|C_\phi\|_e^2 \approx \limsup_{|z| \to 1} \frac{N_{\phi,\alpha+2}(z)}{(- \log |z|)^{\alpha+2}}
\end{align*}

Since the compactness of $C_\phi$ on $H^p$ or $A^p_{\alpha}$ does not depend on $p$, the above results imply

- $C_\phi$ is compact on $H^p$ if and only if $\lim_{|z| \to 1} \frac{N_{\phi,1}(z)}{- \log |z|} = 0$
- $C_\phi$ is compact on $A^p_{\alpha}$ if and only if $\lim_{|z| \to 1} \frac{N_{\phi,\alpha+2}(z)}{(- \log |z|)^{\alpha+2}} = 0$
Generalizations by Smith and Yang

To study cases \( C_\phi : A^p_\alpha \rightarrow A^q_\beta \) \((0 < p, q < \infty, \alpha, \beta \geq -1)\), W. Smith and L. Yang also used the growth of the counting function \( N_{\phi, \gamma} \).

**Theorem (Smith ’96).** For \( 0 < p \leq q < \infty \) and \( \alpha, \beta \geq -1 \),
\[
C_\phi : A^p_\alpha \rightarrow A^q_\beta \text{ is bounded} \iff N_{\phi, \beta+2}(z) = O((\log |z|)^{\alpha+2})
\]
\[
C_\phi : A^p_\alpha \rightarrow A^q_\beta \text{ is compact} \iff N_{\phi, \beta+2}(z) = o((\log |z|)^{\alpha+2})
\]

Remark: When \( \alpha = -1 \) (or \( \beta = -1 \)), \( A^p_{-1} \) is considered to be the Hardy space \( H^p \).

**Theorem (Smith-Yang ’98).** For \( 0 < q < p < \infty \) and \( \alpha, \beta > -1 \), the followings are equivalent:
\[
C_\phi : A^p_\alpha \rightarrow A^q_\beta \text{ is bounded} \iff C_\phi : A^p_\alpha \rightarrow A^q_\beta \text{ is compact}
\]
\[
\iff \text{The function } \mathbb{D} \ni z \mapsto \frac{N_{\phi, \beta+2}(z)}{(1-|z|^2)^{\alpha+2}} \text{ belongs to the space } L^{p-q}_p(dA_\alpha)
\]
Our purpose

Let \( \sigma(r) \) be a non-negative continuous function on \([0, 1)\). We extend it by \( \sigma(z) = \sigma(|z|) \) for \( z \in \mathbb{D} \) and call such \( \sigma(z) \) a weight function.

For a given weight function \( \sigma(z) \) and \( 0 < p < \infty \), we consider the following weighted Bergman space \( A^p(\sigma dA) \):

\[
A^p(\sigma dA) := \left\{ f \in H(\mathbb{D}) : \|f\|_\sigma^p = \int_\mathbb{D} |f(z)|^p \sigma(z) dA(z) < \infty \right\}.
\]

Shapiro’s or Smith and Yang’s results suggest a problem:

**Find the condition on \( \phi \) which characterize the boundedness and the compactness of \( C_\phi \) from \( A^p(\sigma_1 dA) \) into \( A^q(\sigma_2 dA) \).**
II. Preliminaries

- Generalized Nevanlinna counting function

Let $\phi$ be an analytic self-map of $\mathbb{D}$ and $\sigma$ a weight function on $\mathbb{D}$. We define the function $N_{\phi,\sigma}$ as follows:

$$N_{\phi,\sigma}(z) := \sum_{w \in \phi^{-1}\{z\}} \sigma(w) \quad (z \in \mathbb{D} \setminus \{\phi(0)\}).$$

As in the classical Nevanlinna counting function $N_{\phi,\gamma}$, we understand that $N_{\phi,\sigma}(z) = 0$ if $z \notin \phi(\mathbb{D})$ and $\phi^{-1}\{z\}$ denotes the zero sequence of $\phi(\cdot) - z$, each point being repeated in the sequence according to its multiplicity. Conventionally, we consider that $N_{\phi,\sigma}(z) = 0$ if $z = \phi(0)$.

When $\sigma(z) = -\log |z|$, this $N_{\phi,\sigma}$ coincides with $N_{\phi,1}$. So we call $N_{\phi,\sigma}$ a generalized Nevanlinna counting function associated to $\phi$ and $\sigma$. This $N_{\phi,\sigma}$ can be found in the paper of K. Kellay and P. Lefèvre (JMAA, vol.386 (2012)).
• Békollé weight condition

For $1 < p < \infty$ and $\alpha > -1$, the class $B_p(\alpha)$ consists of all weight functions $\sigma$ with the property: there is a constant $C > 0$ such that

$$\left( \int_{S(a)} \sigma \, dA_\alpha \right) \cdot \left( \int_{S(a)} \sigma^{-\frac{p'}{p}} \, dA_\alpha \right)^{-\frac{p'}{p'}} \leq C \{ A_\alpha(S(a)) \}^p,$$

where $S(a) = \{ \varphi_a(z) : \Re(z\bar{a}) \leq 0 \} \ (a \in \mathbb{D})$ and $\frac{1}{p} + \frac{1}{p'} = 1$. We call this condition the Békollé weight condition.

**Theorem (Békollé '81).** Let $1 < p < \infty$ and $\alpha > -1$. For a weight function $\sigma$, the followings are equivalent:

$P_\alpha$ is a bounded projection from $L^p(\sigma dA)$ onto $A^p(\sigma dA)$

$$\iff \frac{\sigma(z)}{(1 - |z|^2)^\alpha} \text{ belongs to the class } B_p(\alpha)$$

Remark: $P_\alpha$ is the usual Bergman projection for $A^p_\alpha$. 

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By using this Békollé’s result, D.H. Luecking gave the dual relation of $A^p(\sigma dA)$.

**Theorem (Luecking ’85).** Let $1 < p < \infty$ and $\alpha > -1$. Suppose that 
\[
\frac{\sigma(z)}{(1 - |z|^2)^\alpha}
\] 
belongs to the class $B_p(\alpha)$. Then,

\[
(A^p(\sigma dA))^* = A^{p'}(\sigma^{\frac{p'}{p}} dA_{\alpha p'})
\] 
under the integral pairing

\[
\langle f, g \rangle = \int_{\mathbb{D}} f(z)\overline{g(z)} dA_\alpha(z).
\]
• Constantin’s result on $C_{\phi} : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$

O. Constnatin proved the Carleson’s embedding theorem for $A^p(\sigma dA)$ under the Békolé weight condition. By applying this embedding theorem to the pull-back measure $(\sigma dA) \circ \phi^{-1}$, Constantin got the following characterizations for $C_{\phi}$.

**Theorem (Constantin ’10).** Suppose that $\frac{\sigma_1(z)}{(1 - |z|^2)^\alpha} \in B_{p_0}(\alpha)$ for some $p_0 > 1$ and $\alpha > -1$. Let $D_{\lambda,r}$ denote the disk $\{z \in \mathbb{D} : |z - \lambda| < r(1 - |\lambda|)\}$.

- For $0 < p \leq q < \infty$, $C_{\phi} : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$ is compact
  \[ \Leftrightarrow \int_{\phi^{-1}(D_{\lambda,r})} \sigma_2(z) dA(z) = o \left( \left\{ \int_{D_{\lambda,r}} \sigma_1(z) dA(z) \right\}^{q/p} \right) \quad (|\lambda| \to 1). \]

- For $0 < q < p < \infty$, $C_{\phi} : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$ is bounded
  \[ \Leftrightarrow C_{\phi} : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA) \text{ is compact} \]
  \[ \Leftrightarrow \mathbb{D} \ni \lambda \mapsto \frac{\int_{\phi^{-1}(D_{\lambda,r})} \sigma_2(z) dA(z)}{\int_{D_{\lambda,r}} \sigma_1(z) dA(z)} \text{ belongs to } L^{p-q}(\sigma_1 dA). \]

Remark: The choice of $\{p, q\}$ is independent of $p_0$.  

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• Admissible Békollé weight

Our arguments in the proof are based on the above Luecking’s and Constantin’s results and require some growth conditions on \( \sigma \), so we will consider the following conditions:

\[
(W_1) \quad \frac{\sigma(z)}{(1 - |z|^2)^{\alpha}} \in B_{p_0}(\alpha) \text{ for some } p_0 > 1 \text{ and } \alpha > -1.
\]

\[
(W_2) \quad \sigma(r) \text{ is non-increasing on } [0, 1).
\]

\[
(W_3) \quad \frac{\sigma(r)}{(1 - r^2)^{\delta+1}} \text{ is non-decreasing for some } \delta > 0.
\]

To formulate our results, we need to introduce another weight function. For each \( \sigma \), we put

\[
\omega_{\sigma}(z) := \int_{|z|}^{1} (t - |z|) \sigma(t) \, dt.
\]

Then we see that \( \omega_{\sigma}(z) \) is non-increasing, convex and \( \omega_{\sigma}(z) \to 0 \) as \( |z| \to 1 \). If \( \sigma \) satisfies the above \((W_2)\) and \((W_3)\), then it holds that

\[
\omega_{\sigma}(z) \approx (1 - |z|^2)^2 \sigma(z) \approx \int_{E(z,r)} \sigma(w) \, dA(w) \quad \text{for every } z \in \mathbb{D} \text{ and } r \in [0, 1).
\]
III. Results

In our results, we always assume that $\sigma_1$ is an admissible Békollé weight and $\sigma_2$ is a weight function. By using the growth of the generalized Nevanlinna counting function $N_{\phi, \omega_\sigma_2}$ associated to $\phi$ and $\omega_\sigma_2$, we had a different characterization from Constantin’s one.

Main tools are the following equivalent norm for $\|f\|_\sigma$:

$$
\|f\|_\sigma^p \approx |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^p \sigma(z) \, dA(z)
$$

$$
\approx |f(0)|^p + \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \left\{ \int_1^1 \left( \log \frac{r}{|z|} \right) \sigma(r) r \, dr \right\} \, dA(z)
$$

and the change of variable formula:

$$
\|f \circ \phi\|_\sigma^p \approx |f(\phi(0))|^p + \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \left\{ \int_0^1 N_{\phi}(r, z) \sigma(r) r \, dr \right\} \, dA(z).
$$

Here $N_{\phi}(r, z)$ denotes the partial counting function for $\phi$. 
Theorem A. (Boundedness for the case $0 < p \leq q < \infty$)

\[ C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA) \text{ is bounded } \iff \mathcal{N}_{\phi,\omega\sigma_2}(z) = O(\omega_1(z)^{\frac{q}{p}}). \]

Outline of the proof: The direction ($\Rightarrow$) can be verified by the test function

\[ f_z(w) := \frac{(1 - |z|^2)^{\alpha+2-2/p}}{\sigma_1(z)^{1/p}(1 - \overline{zw})^{\alpha+2}}. \]

Consider the pseudohyperbolic disk \( E(z, \frac{1-|z|}{2}) = \{w : |\varphi_z(w)| < \frac{1-|z|}{2}\} \). Then the relation \( |f_z(w)|^{q-2}|f'_z(w)|^2 \approx \frac{1}{\omega_1(z)^{q/p}(1-|z|^2)^2} \) for \( w \in E(z, \frac{1-|z|}{2}) \) gives that

\[ \frac{1}{\omega_1(z)^{q/p}} \cdot \frac{1}{(1 - |z|^2)^2} \int_{E(z, \frac{1-|z|}{2})} \mathcal{N}_{\phi,\omega\sigma_2}(w) dA(w) \lesssim \|C_\phi f_z\|_{\sigma_2}^{q} \lesssim 1. \]

Combining this with the sub-mean value property:

\[ \mathcal{N}_{\phi,\omega_2}(z) \lesssim \frac{1}{(1 - |z|^2)^2} \int_{E(z, \frac{1-|z|}{2})} \mathcal{N}_{\phi,\omega_2}(w) dA(w) \quad \text{for } |z| \text{ sufficiently close to } 1, \]

we have \( \mathcal{N}_{\phi,\omega_2}(z) = O(\omega_1(z)^{\frac{q}{p}}) \) as \( |z| \to 1 \).

Another direction ($\Leftarrow$) is straightforward. Q.E.D.
Theorem B. (Essential norm)

Let \( p_0 \leq p \leq q < \infty \) and \( q \geq 2 \). Suppose that \( C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA) \) is bounded. Then

\[
\|C_\phi\|_e^q \approx \limsup_{|z| \to 1} \frac{N_{\phi, \omega_2}(z)}{\omega_1(z)^{\frac{q}{p}}}.
\]

Outline of the proof: The lower estimate is verified by the same argument in Theorem A (\( \Rightarrow \)), so we only state the proof of the upper estimate.

For the Taylor expansion \( f(z) = \sum a_k z^k \), we consider the operator \( R_n f(z) = \sum_{k=n}^\infty a_k z^k \). Then \( \{R_n\}_{n \geq 1} \) is uniformly bounded on \( A^p(\sigma_1 dA) \) if \( p > 1 \). The assumption \( p \geq p_0 \) implies that \( \frac{\sigma_1(z)}{(1 - |z|^2)^\alpha} \in B_p(\alpha) \) by Hölder’s inequality, and so \( P_\alpha \) is a bounded projection from \( L^p(\sigma_1 dA) \) onto \( A^p(\sigma_1 dA) \). Fix \( t \in (0, 1) \). These facts and \( q \geq 2 \) show that

\[
|R_n f(z)|^{q-2} \lesssim \frac{\|f\|_{\sigma_1}^{q-2}}{(\int_D \sigma_1 dA)^{(q-2)/p}} \left( \sum_{k=n}^\infty \frac{\Gamma(k + \alpha + 2)}{k! \Gamma(\alpha + 2)} t^k \right)^{q-2},
\]

for \( |z| \leq t \). Letting \( n \to \infty \), this implies that

\[
\int_{|z| \leq t} |R_n f(z)|^{q-2} |R_n f'(z)| \left\{ \int_0^1 N_\phi(r, z) \sigma_2(r) r \, dr \right\} \, dA(z) \to 0.
\]
On the other hand, inequalities:

$$\int_0^1 N_\phi(r, z)\sigma_2(r) r \, dr \leq \frac{1}{t} N_{\phi, \omega \sigma_2}(z) \quad \text{and} \quad |R_n f(z)|^{q-2} \lesssim \frac{\|R_n\|^{q-p} |R_n f(z)|^{p-2}}{\omega \sigma_1(z)^{(q-p)/p}}$$

for $|z| \in (t, 1)$ give that

$$\int_{|z|>t} |R_n f(z)|^{q-2} |R_n f'(z)|^2 \left\{ \int_0^1 N_\phi(r, z)\sigma_2(r) r \, dr \right\} dA(z)$$

$$\lesssim \sup_{n \geq 1} \|R_n\|^q \cdot \sup_{|z|>t} \frac{N_{\phi, \omega \sigma_2}(z)}{\omega \sigma_1(z)^{q/p}}.$$  

Thus we have that

$$\|C_\phi\|^q \leq \liminf_{n \to \infty} \|C_\phi R_n\|^q \lesssim \sup_{|z|>t} \frac{N_{\phi, \omega \sigma_2}(z)}{\omega \sigma_1(z)^{q/p}},$$

for any $t \in (\frac{1}{2}, 1)$. Letting $t \to 1$ in this inequality, we obtain the desired upper estimate.

Q.E.D.
**Corollary.** (Compactness for the case \(0 < p \leq q < \infty\))

\[ C'_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA) \text{ is compact } \iff \mathcal{N}_{\phi, \omega_\sigma_2}(z) = o(\omega_{\sigma_1}(z)^{\frac{q}{p}}). \]

**Outline of the proof:** The case \(p \geq p_0\) and \(q \geq 2\) is an immediate consequence of Theorem B. It is enough to prove that the case \(p < p_0\) and \(q < 2\) because the rest of cases are verified by the same argument.

Now we choose a positive integer \(m\) such that \(mp \geq p_0\) and \(mq \geq 2\). Then Theorem B implies that \(C'_\phi : A^{mp}(\sigma_1 dA) \to A^{mq}(\sigma_2 dA)\) is compact if and only if

\[ \mathcal{N}_{\phi, \omega_\sigma_2}(z) = o(\omega_{\sigma_1}(z)^{\frac{mq}{mp}}) = o(\omega_{\sigma_1}(z)^{\frac{q}{p}}) \quad (|z| \to 1). \]

On the other hand, Constantin's compactness criterion for the case \(0 < p \leq q < \infty\) gives the following equivalence:

\[ C'_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA) \text{ is compact } \iff C'_\phi : A^{mp}(\sigma_1 dA) \to A^{mq}(\sigma_2 dA) \text{ is compact}. \]

Thus we obtain the desired result. Q.E.D.
Theorem C. (The case $0 < q < p < \infty$)

\[ C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA) \text{ is bounded} \iff \frac{N_{\phi, \omega \sigma_2}(z)}{\omega \sigma_1(z)} \in L^{\frac{p}{p-q}}(\sigma_1 dA). \]

Moreover, if $C_\phi$ is bounded, then $C_\phi$ is also compact.

Outline of the proof: To prove the direction ($\iff$), it is enough to prove that

\[ \int_\mathbb{D} |f(z)|^{q-2} |f'(z)|^2 \left\{ \int_0^1 N_{\phi}(r, z) \sigma_2(r) r \, dr \right\} dA(z) < \infty \quad (f \in A^p(\sigma_1 dA)). \]

By an application of the sub-mean value property of $N_{\phi, \omega \sigma_2}$, we have that

\[ \int_\mathbb{D} |f(z)|^{q-2} |f'(z)|^2 \left\{ \int_0^1 N_{\phi}(r, z) \sigma_2(r) r \, dr \right\} dA(z) \leq \int_\mathbb{D} |f(z)|^q H(z) dA(z) \quad \left( \text{where } H(z) := \int_{E(z, \frac{1}{2})} \frac{N_{\phi, \omega \sigma_2}(w)}{(1 - |w|^2)^4} dA(w) \right) \]

\[ \leq \left\{ \int_\mathbb{D} |f(z)|^p \sigma_1(z) dA(z) \right\}^{\frac{q}{p}} \cdot \left\{ \int_\mathbb{D} H(z) \sigma_1^\frac{p}{p-q} dA(z) \right\}^{\frac{p-q}{p}} \quad \text{(by Hölder’s inequality)} \]

\[ \leq \|f\|_{\sigma_1}^q \cdot \| \mathcal{M} \left[ \frac{N_{\phi, \omega \sigma_2}}{\omega \sigma_1} \right] \|_{L^{\frac{p}{p-q}}} \quad \left( \text{where } \mathcal{M} \left[ \frac{N_{\phi, \omega \sigma_2}}{\omega \sigma_1} \right] \text{ is the Hardy-Littlewood maximal function} \right). \]
For another direction ($\Rightarrow$), we consider the following function:

\[
f_t(z) := \sum_{j=1}^{\infty} \frac{c_j}{z_j} \frac{r_j(t)}{\sigma_1(z_j)^1/p} \frac{(1 - |z_j|^2)^{\alpha+2}}{(1 - z_j z)^{\alpha+2}},
\]

where \( \{c_j\} \in l^p \), \( \{r_j(t)\} \) is the Rademacher functions on \([0, 1)\) and \( \{z_j\} \subset \mathbb{D} \) with \(\inf_{j \neq k} |\varphi_{z_j}(z_k)| > 0\). As in the argument of Smith-Yang (PAMS, vol. 126 (1998)), it follows from Khinchine's inequality that

\[
\left\{ \begin{array}{l}
\int_{E(z_j, \frac{1}{2})} \mathcal{N}_{\phi, \omega \sigma_2}(z) \, dA(z) \\
\frac{\sigma_1(z_j)^{q/p} (1 - |z_j|^2)^{2+2q/p}}{\varphi_{z_j}(z_k)}
\end{array} \right\}_{j \geq 1} \in \left( l^\frac{p}{q} \right)^* = l^{\frac{p}{p-q}}.
\]

By a sub-mean value property, we obtain that

\[
\int_{\mathbb{D} \setminus \frac{1}{4}\mathbb{D}} \left( \frac{\mathcal{N}_{\phi, \omega \sigma_2}(z)}{\omega_{\sigma_1}(z)} \right)^{\frac{p}{p-q}} \sigma_1(z) \, dA(z) \lesssim \int_{\mathbb{D} \setminus \frac{1}{4}\mathbb{D}} \left( \frac{\int_{E(z_j, \frac{1}{2})} \mathcal{N}_{\phi, \omega \sigma_2}(w) \, dA(w)}{\omega_{\sigma_1}(z) (1 - |z|^2)^2} \right)^{\frac{p}{p-q}} \sigma_1(z) \, dA(z)
\]

\[
\lesssim \sum_{j=1}^{\infty} \left( \frac{\int_{E(z_j, \frac{1}{2})} \mathcal{N}_{\phi, \omega \sigma_2}(z) \, dA(z)}{\sigma_1(z_j)^{q/p} (1 - |z_j|^2)^{2+2q/p}} \right)^{\frac{p}{p-q}} < \infty.
\]

The integrability on \(\frac{1}{4}\mathbb{D}\) is clear. 

Q.E.D.
Thank you for your attention.