When Is the Inner Factor of $f - f(a)$ an Interpolating Blaschke Product for All $a$?

Carl Cowen

IUPUI

Eva Gallardo Gutiérrez
Universidad Complutense de Madrid

and

Pamela Gorkin
Bucknell University

UM Dearborn, Analysis Day, 29 April 2017
When Is the Inner Factor of $f - f(a)$ an Interpolating Blaschke Product for All $a$?

Carl Cowen

Eva Gallardo Gutiérrez

and

Pamela Gorkin

Thanks to: Simons Foundation Collaboration Grant 358080

Plan Nacional I+D MTM2013-42105-P

and

Simons Foundation Collaboration Grant 243653.
Dedicated to the Memory of Donald E. Sarason

January 26, 1933 – April 8, 2017
Some terminology:

$H^\infty(\mathbb{D}) = H^\infty$ is Banach space of bounded analytic functions on unit disk $\mathbb{D}$

Buerling: Every function in $H^\infty$ has an inner-outer factorization $f = Ig$

where $g$ is outer – polynomials in $z$ times $g$ are dense in $H^p$ for $p < \infty$

and $I$ is inner – $\left| \lim_{r \to 1^-} I(rie^{i\theta}) \right| = 1$ almost all $\theta$

The inner function can be factored as $I = BS$ with both $B$ and $S$ inner and

for $|\lambda| = 1$ and integer $N \geq 0$, the Blaschke product $B$ with zeros $\{z_k\}$ is

$$B(z) = \lambda z^N \prod_k \frac{|z_k|}{z_k} \left( \frac{z_k - z}{1 - \bar{z}_k z} \right)$$

and, for $\mu$ a singular measure on $\partial \mathbb{D}$, the singular inner factor $S$ is

$$S_\mu(z) = \exp \left( - \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \right)$$
In particular, if \( f = Ig = BSg \) is the inner-outer factorization of \( f \),

the Blaschke factor, \( B \), tells us where, in the disk, the value of \( f \) is 0,

and the singular inner factor \( S \), tells us where,

near the unit circle, the value of \( f \) is nearly 0.

In this talk, we’ll be interested in the different factorizations, for \( a \) in \( \mathbb{D} \):

\[
f - f(a) = I_ag_a = B_aS_ag_a
\]

and, in this factorization, the Blaschke factor tells us

where the value of \( f \) is the same as its value at \( a \).
It turns out that the singular factor, and repeated zeros in the Blaschke factor, are rarely there!

Rudin’s Theorem (1967):

For \( f \) in \( H^\infty \), the inner factor of \( f - f(a) \) is a Blaschke product with distinct zeros, except for \( a \) in a set of capacity zero in \( \mathbb{D} \).
We know that a function analytic on $\mathbb{D}$ can take the value 0

only on a countable set with no accumulation points in $\mathbb{D}$.

For an $H^\infty$ function $f$, the set where $f$ takes the value 0

are the zeros of the Blaschke product $B$ in the factorization $f = BSg$

The infinite product

$$B(z) = \lambda z^N \prod_{k} \frac{|z_k|}{z_k} \left( \frac{z_k - z}{1 - \overline{z_k}z} \right)$$

converges if and only if $\sum_k (1 - |z_k|) < \infty$ ($\{z_k\}$ is a Blaschke sequence)

so $H^\infty$ functions vanish on a smaller set than general analytic functions!
If \( \{z_k\} \) is any sequence in \( \mathbb{D} \), the map \( E \) from \( H^\infty \) into \( \ell^\infty \)
given by \( E : f \mapsto (f(z_k))_{k=1}^\infty \) is clearly continuous.

**Definition:** We say \( \{z_k\} \) is an interpolating sequence if the map given by

\[
E : f \mapsto (f(z_k))_{k=1}^\infty \quad \text{maps} \quad H^\infty \quad \text{onto} \quad \ell^\infty.
\]

**Definition:** We say \( B \) is an interpolating Blaschke product

if \( B \) is a Blaschke product

\[
B(z) = \lambda \prod_k \frac{|z_k|}{z_k} \left( \frac{z_k - z}{1 - \overline{z_k}z} \right)
\]

and the zero set \( \{z_k\} \) is an interpolating sequence.
**Definition:** We say \( \{z_k\} \) is an interpolating sequence if the map given by

\[
E : f \mapsto (f(z_k))_{k=1}^\infty \quad \text{maps } H^\infty \text{ onto } \ell^\infty.
\]

Interpolating sequences have come up in many different ways in problems related to analytic functions on the disk and spaces of such analytic functions:

- Identifying the algebras between \( H^\infty(\mathbb{D}) \) and \( L^\infty(\partial \mathbb{D}) \)
- Understanding the topology of \( H^\infty \) in the corona
- Constructions of Toeplitz or composition operators with special properties
- Creating tools to study Toeplitz operators and composition operators
- Characterizing commutants of analytic Toeplitz operators
- Building universal operators to study the invariant subspace problem
In 1958, Carleson proved:

**Theorem**

The set of points \( \{z_j\}_{j \geq 1} \) in the disk is an interpolating sequence if and only if there exists \( \delta > 0 \) such that

\[
\inf_{k} \prod_{j \geq 1, j \neq k} \left| \frac{z_j - z_k}{1 - \overline{z}_j z_k} \right| \geq \delta
\]

Because \( \rho(z, w) = \left| \frac{z - w}{1 - \overline{z}w} \right| \) is the ‘pseudo-hyperbolic distance’ between \( z \) and \( w \), Carleson’s result says the points in an interpolating sequence are, in a very strong way, uniformly far apart.
In functional Hilbert spaces,

functional \( f \mapsto f(\alpha) \) for \( \alpha \) in \( \mathbb{D} \) is continuous, so there is vector \( K_\alpha \),
called the *kernel function for \( \alpha \)*, so that \( f(\alpha) = \langle f, K_\alpha \rangle \).

For \( H^2 \), the kernel functions are

\[
K_\alpha(z) = \frac{1}{1 - \overline{\alpha}z}
\]

One tool, in studying operators on Hilbert spaces, is choosing a good basis.

In 1961, H. S. Shapiro and A. L. Shields showed that

if \( \{\alpha_j\}_{j>0} \) is an interpolating sequence, then the set

\[
\frac{K_{\alpha_j}}{\|K_{\alpha_j}\|} \quad j > 0
\]

is (Banach) equivalent to an orthonormal basis of the subspace \( H^2 \ominus BH^2 \)

where \( B \) is the Blaschke product with zero set \( \{\alpha_j\}_{j>0} \).
Using the equivalence of \( \left\{ \frac{K_{\alpha_j}}{\|K_{\alpha_j}\|} \right\}_{j>0} \) to an orthonormal basis, the following characterization of certain commutants is obtained:

**Theorem** [C., 1978]

Let \( \Omega \) be bounded domain in \( \mathbb{C} \), let \( \varphi \) be the covering map of \( \mathbb{D} \) onto \( \Omega \), and let \( G \) be the group of linear fractional maps on \( \mathbb{D} \) satisfying \( \varphi \circ I = \varphi \).

If \( S \) is a bounded operator with \( ST_\varphi = T_\varphi S \), then for \( I \) in \( G \), there is an analytic function \( C_I \) defined on \( \mathbb{D} \) so that for each \( \alpha \) in \( \mathbb{D} \) and each \( g \) in \( H^2 \),

\[
(Sg)(\alpha) = \sum_{I \in G} C_I(\alpha)g(I(\alpha))
\]

where the series converges absolutely and uniformly on compact subsets of \( \mathbb{D} \).

Conversely, if bounded operator \( S \) has representation (1), then \( ST_\varphi = T_\varphi S \).
An easy example of an interpolating sequence is \( \{z_n\} \) where \( z_n = 1 - 2^{-n} \)

**Theorem:** If \( \varphi \) maps the unit disk into itself, \( \varphi(a) = a \) for \( |a| = 1 \) and \( \varphi'(a) < 1 \), then for any point \( z_0 \) in \( \mathbb{D} \), the sequence \( \{\varphi_n(z_0)\} \) is an interpolating sequence.

For \( \varphi(z) = (1 + z)/2 \), which maps \( \mathbb{D} \) into itself, \( \varphi(1) = 1 \), and \( \varphi'(1) = 1/2 < 1 \).

Since \( \varphi_n(0) = 1 - 2^{-n} \), this sequence is an interpolating sequence.
For another easy example, consider the singular inner function

\[ S_1(z) = \exp \left( \frac{1 + z}{1 - z} \right) \]

which is associated with the singular measure \( \mu = \delta_1 \), point mass at \{1\}.

Clearly, 0 is not in the range of \( S_1 \), but every other point of \( \mathbb{D} \) is in the range and

\[ S_1(\partial \mathbb{D} \setminus \{1\}) = \partial \mathbb{D} \]

Indeed, \( S_1 \) is a covering map of \( \mathbb{D} \) onto \( \mathbb{D} \setminus \{0\} \)

For \( p \neq 0 \) in \( \mathbb{D} \), the set

\[ S_1^{-1}(\{p\}) = \{ z : S_1(z) = p \} \]

is an interpolating sequence in \( \mathbb{D} \).
In thinking about ‘covering maps’, we follow W. A. Veech’s book: 

_A Second Course in Complex Analysis_, 1967.

**Definition:** Let \( \Omega \) and \( \Omega_1 \) be regions. A triple \((f, \Omega_1, \Omega)\) is a covering if

a) \( f : \Omega_1 \to \Omega \) is analytic and,

b) if \( z_0 \in \Omega_1 \) and \( w_0 \in \Omega \) satisfy \( f(z_0) = w_0 \) and if \( \gamma \) is an arc from \( w_0 \) in \( \Omega \), then \( f^{-1} \) can be continued along \( \gamma \) with values in \( \Omega_1 \) and initial value \( z_0 \).

We say \( \Omega_1 \) is a covering surface of \( \Omega \) and \( f \) is a covering map.

If \( f \) is a covering map, then \( f'(z) \neq 0 \) for \( z \in \Omega_1 \), and \( f(\Omega_1) = \Omega \).

For \( S_1 \) above, the triple \( (S_1, \mathbb{D}, \mathbb{D} \setminus \{0\}) \) is a covering.
If \( f \) is a covering map and \( \Omega \) is simply connected, then \( f \) is also one-to-one.

We say that a region \( \Omega_1 \) is a universal covering of a region \( \Omega \) if \( \Omega_1 \) is simply connected and there is a covering, \((f, \Omega_1, \Omega)\).

The name “universal covering” stems from the “universal property”:

**Theorem** Let \((f_1, \Omega_1, \Omega)\) be a covering with \(\Omega_1\) simply connected.

If \((f_2, \Omega_2, \Omega)\) is a covering,

then there is a covering \((c, \Omega_1, \Omega_2)\) such that \(f_1 = f_2 \circ c\).

**Example:** The triple \((z^2, \mathbb{D} \setminus \{0\}, \mathbb{D} \setminus \{0\})\) is a covering.

The triple \((S_1, \mathbb{D}, \mathbb{D} \setminus \{0\})\) is a universal covering.
**Theorem**  If $\Omega$ is analytically equivalent to a bounded region and $z_0$ is a prescribed point in $\Omega$, then there exists a unique covering $f : \mathbb{D} \to \Omega$ such that $f(0) = z_0$ and $f'(0) > 0$.

We also have the following:

**Proposition**  If $\Omega$ is analytically equivalent to a bounded region, not simply connected, and $f : \mathbb{D} \to \Omega$ is a covering map, then the fiber
$$\{z \in \mathbb{D} : f(z) = f(a)\}$$
is an infinite sequence for all $a$ in $\mathbb{D}$.

**Theorem**[C., 1978]  Let $c$ be a covering map from $\mathbb{D}$ onto bounded domain $\Omega$. If $\Omega$ is not simply connected, then for every $a \in \mathbb{D}$, the inner factor of $c - c(a)$ is an interpolating Blaschke product.
Theorem[C., Gallardo-Gutiérrez, Gorkin, 2016]

Let $f$ be non-constant analytic mapping of $\mathbb{D}$ onto bounded domain, $f(\mathbb{D})$.

Then $f'(z) \neq 0$ for every $z$ in $\mathbb{D}$

if and only if

there is an analytic, universal covering, $c$, mapping the simply connected domain, $\mathbb{U}$, not equal to $\mathbb{C}$, onto the domain $f(\mathbb{D})$

and a univalent, analytic map, $\sigma$ of $\mathbb{D}$ into $\mathbb{U}$ so that $f = c \circ \sigma$. 
**Example:** Let \( f(z) = (1 + .5z)^8 \) and consider \( f(D) \).

The set \( f(D) \) is contained in the annulus \( .0039 < 2^{-8} < |w| < 1.5^8 < 26 \) and is *not* simply connected.

The derivative \( f'(z) = 4(1 + .5z)^7 \neq 0 \) for \( z \) in the unit disk.

We can take \( U \subset \{ u : -8 \log(2) < \text{Re}(u) < 8 \log(1.5) \} \) and \( c(u) = \exp(u) \)

and \( \sigma(D) \subset U \), but \( |\text{Im}(\sigma(z))| < 1.5\pi \) (!!) so that \( f(z) = c(\sigma(z)) \).

Indeed, \( \sigma(z) = 8\log(1 + .5z) = \log((1 + .5z)^8) \)

so that \( f(z) = \exp(\log((1 + .5z)^8)) = (1 + .5z)^8 \) !
Figure 1: The set $f(D)$ for $f(z) = (1 + .5 \cdot z)^8$
Sketch of proof: (Assume $f'(z) \neq 0$ for $z$ in $\mathbb{D}$)

First, choose 0 as a base point in $\mathbb{D}$ and $w_0 = f(0)$ as a base point in $f(\mathbb{D})$. 
Sketch of proof: (Assume $f'(z) \neq 0$ for $z$ in $\mathbb{D}$)

First, choose $0$ as a base point in $\mathbb{D}$ and $w_0 = f(0)$ as a base point in $f(\mathbb{D})$.

Since $f(\mathbb{D})$ is a bounded domain, there is a universal cover on any domain $U$ that is conformally equivalent to the unit disk. Choose $u_0$ as a base point in $U$; there is a unique universal cover, $c$, of $U$ onto $f(\mathbb{D})$ for which $c(u_0) = w_0$ and $c'(u_0) > 0$. 
Sketch of proof: (Assume $f'(z) \neq 0$ for $z$ in $\mathbb{D}$)

First, choose 0 as a base point in $\mathbb{D}$ and $w_0 = f(0)$ as a base point in $f(\mathbb{D})$.

Since $f(\mathbb{D})$ is a bounded domain, there is a universal cover on any domain $U$ that is conformally equivalent to the unit disk. Choose $u_0$ as a base point in $U$; there is a unique universal cover, $c$, of $U$ onto $f(\mathbb{D})$ for which $c(u_0) = w_0$ and $c'(u_0) > 0$.

Since $f'(z) \neq 0$, the function $f$ is locally univalent at every point of $\mathbb{D}$. For any path, $\gamma$, from 0 to $z_1$ in $\mathbb{D}$, $f(\gamma)$ is a path from $w_0$ to $w_1$ in $f(\mathbb{D})$. Because $f$ is locally univalent, there is a unique lifting of this path in $U$ from $u_0$ to $u_1$. Now, define $\sigma(z_1) = u_1$. Because $\mathbb{D}$ and $U$ are simply connected, all is well defined and $\sigma$ is univalent.
Sketch of proof: (Assume $f'(z) \neq 0$ for $z$ in $\mathbb{D}$)

First, choose 0 as a base point in $\mathbb{D}$ and $w_0 = f(0)$ as a base point in $f(\mathbb{D})$.

Since $f(\mathbb{D})$ is a bounded domain, there is a universal cover on any domain $U$ that is conformally equivalent to the unit disk. Choose $u_0$ as a base point in $U$; there is a unique universal cover, $c$, of $U$ onto $f(\mathbb{D})$ for which $c(u_0) = w_0$ and $c'(u_0) > 0$.

Since $f'(z) \neq 0$, the function $f$ is locally univalent at every point of $\mathbb{D}$. For any path, $\gamma$, from 0 to $z_1$ in $\mathbb{D}$, $f(\gamma)$ is a path from $w_0$ to $w_1$ in $f(\mathbb{D})$. Because $f$ is locally univalent, there is a unique lifting of this path in $U$ from $u_0$ to $u_1$. Now, define $\sigma(z_1) = u_1$. Because $\mathbb{D}$ and $U$ are simply connected, all is well defined and $\sigma$ is univalent.

(Conversely, suppose $f = c \circ \sigma$) $f'(z) = c'(\sigma(z))\sigma'(z)$; since $c'$ and $\sigma'$ are never 0, $f'$ is never 0.
Corollary [C., Gallardo-Gutiérrez, Gorkin, 2016]

Let $f$ be non-constant analytic mapping of $D$ onto bounded domain, $f(D)$. Then $f'(z) \neq 0$ for every $z$ in $D$

if and only if

for each $a$ in $D$, the zero sequence of $f - f(a)$ is a single point,

or for each $a$ in $D$, the zero sequence is an interpolating sequence.
Sketch of proof: (Assume $f'(z) \neq 0$ for $z$ in $\mathbb{D}$)

If $f(\mathbb{D})$ is simply connected, then the universal covering map is univalent.
Sketch of proof: (Assume $f'(z) \neq 0$ for $z$ in $\mathbb{D}$)

If $f(\mathbb{D})$ is simply connected, then the universal covering map is univalent.

If $f(\mathbb{D})$ is not simply connected, there is a universal covering map so that $f = c \circ \sigma$ and the zeros of $c - c(u_0)$ are all interpolating.
Sketch of proof: (Assume $f'(z) \neq 0$ for $z$ in $\mathbb{D}$)

If $f(\mathbb{D})$ is simply connected, then the universal covering map is univalent.

If $f(\mathbb{D})$ is not simply connected, there is a universal covering map so that $f = c \circ \sigma$ and the zeros of $c - c(u_0)$ are all interpolating.

Conversely, suppose $f'(z_0) = 0$ for some $z_0$ in $\mathbb{D}$. In this case, for $g(z) = f(z) - f(z_0)$, we have $g(z_0) = g'(z_0) = 0$, so the zero set of $g$, that is, the zero sequence of $f - f(z_0)$, is not an interpolating sequence.
**Theorem**[C., Gallardo-Gutiérrez, Gorkin, 2016]

Let $f$ be an $H^\infty$ function.

Then $f$ is an exactly $n$-to-1 map of $\mathbb{D}$ onto a bounded domain, where $n > 1$, if and only if

$f(\mathbb{D})$ is simply connected and there is a Riemann map $c$ of $\mathbb{D}$ onto $f(\mathbb{D})$ such that $f = c \circ B$, where $B$ is a finite Blaschke product of order $n$. 
THANK YOU!