Recent Progress in Understanding the Structure of Composition Operators on Spaces of Analytic Functions

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If \( \varphi \) is an analytic function of \( \mathbb{D} \) into itself,

and \( \mathcal{H} \) is a Hilbert space of analytic functions on \( \mathbb{D} \),

then the composition operator \( C_\varphi \) on \( \mathcal{H} \) is the operator

\[
C_\varphi f = f \circ \varphi \quad \text{for} \quad f \in \mathcal{H}
\]

Usual spaces: \( f \) analytic in \( \mathbb{D} \), with \( f(z) = \sum_{n=0}^{\infty} a_n z^n \)

Hardy: \( H^2(\mathbb{D}) = H^2 = \{ f : \| f \|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty \} \)

Bergman: \( A^2(\mathbb{D}) = A^2 = \{ f : \| f \|^2 = \int_{\mathbb{D}} |f(z)|^2 \frac{dA(z)}{\pi} < \infty \} \)

weighted Bergman (\( \alpha > 0 \)): \( A^2_\alpha = \{ f : \| f \|^2 = \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha \frac{dA(z)}{\pi} < \infty \} \)

weighted Hardy (\( \| z^n \| = \beta_n > 0 \)): \( H^2(\beta) = \{ f : \| f \|^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty \} \)
We will also be interested in a generalization of composition operators:
Let $\varphi$ be an analytic map of $\mathbb{D}$ into itself and let $\psi$ be analytic on $\mathbb{D}$.

The \textit{weighted composition operator} $W_{\psi,\varphi}$ is the operator on $\mathcal{H}$ given by

$$(W_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z)) \quad \text{for } z \text{ in } \mathbb{D}$$

Naturally, we are most interested in $\varphi$ and $\psi$ for which $W_{\psi,\varphi}$ is bounded!

For most of the spaces that we consider, $W_{\psi,\varphi}$ is bounded for all $\varphi$ mapping the disk into itself and all $\psi$ analytic and bounded on the disk.

On the other hand, for some $\varphi$, there are unbounded functions $\psi$ in $\mathcal{H}$ for which $W_{\psi,\varphi}$ is bounded, or even compact!

For example, on $H^2$, if $\|\varphi\|_\infty < 1$ and $\psi$ is any function in $H^2$, then $W_{\psi,\varphi}$ is compact on $H^2$!
It is somewhat standard that calling $\mathcal{H}$ a

\begin{align*}
\textit{Hilbert space of analytic functions on} \ D
\end{align*}

means that for each $w$ in $\mathbb{D}$, the linear functional $f \mapsto f(w)$ is continuous.

So the Riesz representation theorem implies there is a vector $K_w$ in $\mathcal{H}$ so that

$\langle f, K_w \rangle = f(w)$ for all $f$ in $\mathcal{H}$.

For example,

for $H^2$, we have $K_w(z) = (1 - \overline{w} z)^{-1}$ and

for $A^2$, we have $K_w(z) = (1 - \overline{w} z)^{-2}$
In this talk, we will mostly consider spaces $H^2(\beta_\kappa)$ for $\kappa \geq 1$ which are the weighted Hardy spaces with

$$K_w(z) = (1 - \overline{w}z)^{-\kappa}$$

The spaces $H^2(\beta_\kappa)$ include the usual Hardy and Bergman spaces and all the weighted Bergman spaces ($\alpha = \kappa + 2$).

On all of these spaces, for any $\varphi$ analytic map of $\mathbb{D}$ into itself, the composition operator $C_\varphi$ is a bounded operator and for all $w$ in $\mathbb{D}$

$$C_\varphi^* K_w = K_{\varphi(w)}$$
The study of composition and weighted composition operators is about 50 years old today!!

In 1964, Frank Forelli showed that all the isometries of the $H^p$ spaces of the disk are weighted composition operators.

In 1969, Howard Schwartz, apparently inspired by Peter Rosenthal, completed his thesis defining composition operators on $H^2$ and describing some of their properties.
In 2010, the Mathematics Subject Classification list was expanded to include 47B33 for “Composition operators”

For the past five years or so, MathSciNet has reviewed about 100 papers a year that included this as a primary classification!
Many basic properties of these operators have been described over the past 50 years, including criteria for boundedness and compactness on a wide variety of spaces.

But also, there are many properties that are still not completely understood!

A prominent and important example includes finding a useful description of the adjoints of composition operators. In the late 1980’s, formulas for adjoints of composition operators whose symbols are linear fractional maps were found.

Not until about ten years ago, was work by Gallardo-Gutíerrez and C.; Hammond, Moorhouse, and Robbins; Martín and Vukotic; Bourdon and Shapiro; and Elliott completed on describing the adjoints for composition operators on the Hardy spaces of the disk and upper half plane whose symbols are rational functions.
Not until about ten years ago, was work by Gallardo-Gutíerrez and C.; Hammond, Moorhouse, and Robbins; Martín and Vukotic; Bourdon and Shapiro; and Elliott completed on describing the adjoints for composition operators on the Hardy spaces of the disk and upper half plane whose symbols are rational functions.

Just two years ago(!), Goshabulaghi and Vaezi extended this work by finding the descriptions for the adjoints of these operators on the Bergman and Dirichlet spaces.

The narrowness of these results challenges us to find adjoints for composition operators for more general symbols!
Spectra of composition operators give other examples of questions about composition operators that have been partially, but not completely answered.

Every non-automorphism $\varphi$ mapping the disk into itself has a distinguished fixed point, $a$, the Denjoy-Wolff point, in the closed disk for which $|\varphi'(a)| \leq 1$.

It has been known since the 1970’s that the properties of composition operators vary greatly depending whether $|a| = 1$ or $|a| < 1$ and on the value of $\varphi'(a)$.

For $|a| < 1$, and for $|a| = 1$ and $\varphi'(a) < 1$, maps for which $\varphi$ has a dilation model, we have had a fairly good understanding of the spectra of $C_\varphi$ for some time.
However, if $|a| = 1$ and $\varphi'(a) = 1$, the maps $\varphi$ separate into two types and spectra of $C_\varphi$ for these maps are not known except for special cases. About 20 years ago, I conjectured that for $\varphi$ having a plane translation model, the spectrum $C_\varphi$ should be a union of spirals, each passing through 1 and spiraling down to 0.

Last year, however, Paul Bourdon gave an example of a map $\varphi$ of this sort for which the spectrum of $C_\varphi$ was a ‘lollipop’: the disk of radius $1/3$ centered at the origin together with the line segment $[0, 1]$!

Bourdon’s result shows us that we still have much to learn about spectra of composition operators!
Other areas where recent progress has been made:

- understanding which composition operators have special properties like being complex symmetric or hyponormal
- understanding composition operators whose symbols act on higher-dimensional domains like the ball or the polydisk

A few things in several variables have been known for some time, like the unboundedness of some composition operators with ‘nice’ symbols on the ball in $\mathbb{C}^2$ described by Wogen and Cima and Wogen 25 years ago. But even after recent results in several variables, we are very far from understanding the structure of composition operators on higher dimensional domains, and these questions deserve further study.
At this point, I’d like to change directions. I’d like to describe several recent ideas connecting composition operators and the theory of invariant subspaces.
For $A$, a bounded operator on $\mathcal{H}$, a (closed) subspace $M$ is called a (non-trivial) invariant subspace of $A$ if $M \neq 0$, $M \neq \mathcal{H}$, and

$$v \in M \implies Av \in M$$

In finite dimensional spaces, every operator has invariant subspaces and understanding the structure of the invariant subspaces has been critical in understanding the structure of these operators.

We want the same for operators on infinite dimensional spaces!

**Invariant Subspace Problem:**

Does every bounded operator have a (non-trivial) invariant subspace?

No! in general, for Banach spaces! (Enflo, C. J. Read and others 1984–)

Still open for Hilbert spaces!
BUT,

for a Hilbert space operator whose lattice of invariant subspaces is known,

we feel we have a basic understanding of the structure of the operator!

Goal today:

Outline four sets of ideas about invariant subspaces of composition operators

and thereby persuade you that now is a good time to think about this topic!
BUT,

for a Hilbert space operator whose lattice of invariant subspaces is known,

we feel we have a basic understanding of the structure of the operator!

Beurling’s Theorem (1949):

Let $S$ be the operator of multiplication by $z$ on $H^2(\mathbb{D})$. A closed subspace $M$ of $H^2(\mathbb{D})$ is invariant for $S$ if and only if there is an inner function $\psi$ such that $M = \psi H^2(\mathbb{D})$. 
First Example: A complete lattice!

Theorem (Montes-Rodríguez, Ponce-Escudero, & Shkarin, 2010)

For $\text{Re} \, a > 0$, let

$$
\varphi_a(z) = \frac{(2 - a)z + a}{-az + 2 + a}
$$

A closed subspace $M$ of $H^2(\mathbb{D})$ is invariant for $C_{\varphi_a}$ if and only if there is a closed set $F$ of $[0, \infty)$ such that

$$
M = \text{closed span}\{e^{t\frac{z+1}{z-1}} : t \in F\}
$$

The relevance of the functions $e^{t\frac{z+1}{z-1}}$ is that they are eigenvectors for $C_{\varphi_a}$:

$$
C_{\varphi_a} \left( e^{t\frac{z+1}{z-1}} \right) = e^{-at} e^{t\frac{z+1}{z-1}}
$$

In other words, each of the invariant subspaces for $C_{\varphi_a}$ is the closed span of a collection of eigenvectors.
Theorem (Montes-Rodríguez, Ponce-Escudero, & Shkarin, 2010)

For $\text{Re } a > 0$, let

$$\varphi_a(z) = \frac{(2 - a)z + a}{-az + 2 + a}$$

A closed subspace $M$ of $H^2(\mathbb{D})$ is invariant for $C_{\varphi_a}$ if and only if there is a closed set $F$ of $[0, \infty)$ such that

$$M = \text{closed span}\{e^{t\frac{z+1}{z-1}} : t \in F\}$$

Corollary

If $\text{Re } a > 0$ and $\text{Re } b > 0$,

then the lattices of invariant subspaces for $C_{\varphi_a}$ and for $C_{\varphi_b}$ are the same.

Corollary

If $\text{Re } a > 0$, then $C_{\varphi_a}$ has no (non-trivial) reducing subspaces.
Their proof is based on two quite different ideas.

First, suppose $\mathcal{A}$ is a Banach algebra. If $\tau$ is in $\mathcal{A}$, we say $\tau$ is a cyclic element if the algebra generated by $\tau$ is dense in $\mathcal{A}$.

Let $M_\tau$ be the operator on $\mathcal{A}$ of multiplication by $\tau$, that is,

$$M_\tau \omega = \tau \omega \text{ for } \omega \text{ in } \mathcal{A}.$$ 

**Proposition**

If $\tau$ is a cyclic element in the Banach algebra $\mathcal{A}$,

then the invariant subspaces of $M_\tau$ are the closed ideals of $\mathcal{A}$. 

Let $W^{1,2}[0, \infty)$ be the Sobolev space with inner product

$$\langle f, g \rangle_{1,2} = \frac{1}{2} \int_0^\infty f(t)\overline{g(t)} + f'(t)\overline{g'(t)} \, dt$$

where $f$ and $g$ are functions in $L^2[0, \infty)$ that are absolutely continuous on each bounded subinterval of $[0, \infty)$ and whose derivatives $f'$ and $g'$ are in $L^2[0, \infty)$.

They give a unitary equivalence between $H^2(\mathbb{D})$ and the Sobolev space and they show that $W^{1,2}[0, \infty)$ is a Banach algebra.

Finally, they show that the unitary equivalence of these spaces carries the adjoints of the composition operators to multiplication by cyclic elements of the Banach algebra to which they can apply the Proposition.
Second Example:

Invariant subspaces with application to function theory

If \( \varphi \) is an analytic map of \( \mathbb{D} \) into itself and \( \psi \) is a bounded analytic function on \( \mathbb{D} \), the weighted composition operator \( W_{\psi,\varphi} \) is bounded operator on \( H^2(\beta_\kappa) \), although \( W_{\psi,\varphi} \) might be a bounded operator even if \( \psi \) is not bounded.

It is natural to ask “When is \( W_{\psi,\varphi} \) Hermitian?”
Theorem. (Ko & C. for $H^2(\mathbb{D})$ and Gunatillake, Ko, and C. for $H^2(\beta_\kappa)$)

For $\kappa \geq 1$, $\varphi$ a map of the disk into itself, and $\psi$ in $H^2(\beta_\kappa)$,

$W_{\psi, \varphi}$ is a bounded Hermitian weighted composition operator if and only if

$$
\psi(z) = c(1 - \overline{a_0}z)^{-\kappa} \quad \text{and} \quad \varphi(z) = a_0 + \frac{a_1 z}{1 - \overline{a_0}z}
$$

where $c = \psi(0)$ and $a_1 = \varphi'(0)$ are real numbers

and $a_1$ and $a_0 = \varphi(0)$ are such that $\varphi$ maps the unit disk into itself.
The parametrization above can be reformulated to see a semigroup:

**Theorem.**
For \( \kappa \geq 1 \) and \( 0 \leq t < \infty \), let \( A_t = W_{\psi_t, \varphi_t} \) where

\[
\psi_t(z) = (1 + t - tz)^{-\kappa} \quad \text{and} \quad \varphi_t(z) = (t + (1 - t)z)/(1 + t - tz)
\]

The \( A_t \) form a strongly continuous semigroup of Hermitian weighted composition operators on \( H^2(\beta_\kappa) \). If \( \Delta \) is the infinitesimal generator of this semigroup, \( \mathcal{D}_A = \{ f \in H^2(\beta_\kappa) : (z - 1)^2 f' \in H^2(\beta_\kappa) \} \) is the domain of \( \Delta \) and \( (\Delta f)(z) = (z - 1)^2 f'(z) + \kappa(z - 1) f(z) \) for \( f \) in \( \mathcal{D}_A \).

**Corollary.**
For \( \kappa \geq 1 \) and for \( t > 0 \), the operator \( A_t \) on \( H^2(\beta_\kappa) \) has no eigenvalues.

**Proof:** There are no non-zero functions in \( H^2(\beta_\kappa) \) that satisfy

\[
(z - 1)^2 f' + \kappa(z - 1) f = \lambda f
\]
Theorem.

For \( \kappa \geq 1 \) and \( 0 \leq t < \infty \), let \( A_t = W_{\psi_t, \varphi_t} \) where

\[
\psi_t(z) = (1 + t - tz)^{-\kappa} \quad \text{and} \quad \varphi_t(z) = \frac{(t + (1-t)z)}{(1 + t - tz)}
\]

For each \( t > 0 \), the operator \( A_t \) is a cyclic Hermitian weighted composition operator on \( H^2(\beta_\kappa) \). Indeed, the vector \( 1 \) is a cyclic vector for \( A_t \).

If \( \mu \) is the absolutely continuous probability measure given by

\[
d\mu = \frac{(\log(1/x))^{\kappa-1}}{\Gamma(\kappa)} dx
\]

the operator \( U \) given by \( U(\psi_t) = x^t \) for \( 0 \leq t < \infty \) is a unitary map of \( H^2(\beta_\kappa) \) onto \( L^2([0,1], \mu) \) and satisfies \( UA_t = M_{x^t}U \).

In particular, for each \( t > 0 \), these operators satisfy \( \|A_t\| = 1 \) and have spectrum \( \sigma(A_t) = [0, 1] \).
We define subspaces $H_c$ of $H^2(\beta_\kappa) = A^2_{\kappa-2}$ as follows:

Let $H_0 = H^2(\beta_\kappa)$. For $c < 0$, define the subspace $H_c$ by

$$H_c = \text{closure}\{e^{c1+z}f : f \in H^2(\beta_\kappa)\}$$

For $0 \leq t$ and $c \leq 0$, the subspace $H_c$ is invariant for $A_t$.

For $0 \leq \delta \leq 1$ define the subspace $L_\delta$ of $L^2([0,1],\mu)$ by

$$L_\delta = \{f \in L^2([0,1],\mu) : f(x) = 0 \text{ for } \delta < x \leq 1\}$$

These are the spectral subspaces of the multiplication operators $M_{x^t}$.

Our computation requires the use of the \textit{incomplete Gamma function}

$$\Gamma(a, w) = \int_w^\infty t^{a-1}e^{-t} dt$$

where $a$ is a complex parameter and $w$ is a real parameter. An alternate definition in which both $a$ and $w$ are complex parameters is

$$\Gamma(a, w) = e^{-w}w^a\int_0^\infty e^{-wu}(1+u)^{a-1} du$$
Suppose $N$ is a subspace of $H^2(\beta_\kappa)$ that is invariant for the operator of multiplication by $z$.

If there is $f$ in $N$ with $f(0) \neq 0$ and $G$ is a function of $N$ so that

$$\|G\| = 1 \quad \text{and} \quad G(0) = \sup \{ \Re f(0) : f \in N \quad \text{and} \quad \|f\| = 1 \}$$

then we say $G$ solves the extremal problem for the invariant subspace $N$.

Of course, the usual Bergman space is the case $\kappa = 2$, and these extremal functions were developed to understand the shift invariant subspaces in that case.
Our computation requires the use of the *incomplete Gamma function*

\[ \Gamma(a, w) = \int_w^{\infty} t^{a-1}e^{-t} \, dt \]

where \( a \) is a complex parameter and \( w \) is a real parameter. An alternate definition in which both \( a \) and \( w \) are complex parameters is

\[ \Gamma(a, w) = e^{-w}w^a \int_0^{\infty} e^{-wu}(1 + u)^{a-1} \, du \]

**Theorem.**

For \( c < 0 \), if \( H_c \) is the invariant subspace for \( M_z \) in \( H^2(\beta_\kappa) \) defined by

\[ H_c = \text{closure}\{e^{c \frac{1+z}{1-z}} f : f \in H^2(\beta_\kappa)\} \]

then the extremal function for \( H_c \) is

\[ G_c(z) = \frac{\Gamma(\kappa, -2c/(1 - z))}{\Gamma(\kappa)\Gamma(\kappa, -2c)} \]
Theorem.

For $0 < r < 1$, let $P_r$ be the orthogonal projection onto the subspace $H_{(\log r)/2}$ in $H^2(\beta \kappa)$. If $u$ is any point of the open disk, then for $K_u(z) = (1 - \overline{u}z)^{-\kappa}$

$$(P_r K_u)(z) = \frac{1}{\Gamma(\kappa)(1 - \overline{u}z)^\kappa} \Gamma \left( \kappa, -\frac{(\log r)(1 - \overline{u}z)}{(1 - \overline{u})(1 - z)} \right)$$

This gives the kernel functions for the invariant subspaces $H_c$ in $H^2(\beta \kappa)$, including for the usual Bergman space ($\kappa = 2$).

This result generalizes the formula for the usual Bergman space computed in a different way by W. Yang in his thesis.
Third Example:

Common invariant subspaces for $C_\varphi$ and $S$, multiplication by $z$
(Wahl & C., 2011)

Always assume that $\varphi$ is non-constant and not an elliptic automorphism.

Without loss of generality, if $a$, the Denjoy-Wolff point of $\varphi$, is in $\mathbb{D}$, we can assume $a = 0$ and if $|a| = 1$, we can assume $a = 1$.

For simplicity, we will assume that the Hilbert space is $H^2(\mathbb{D})$, although many of the results hold for $H^2(\beta_\kappa)$. When weighted composition operators, $W_{\psi,\varphi}$, are discussed, we will assume that $\psi$ is in $H^\infty(\mathbb{D})$. 
Third Example:

Common invariant subspaces for $C_\varphi$ and $S$, multiplication by $z$
(Wahl & C., 2011)

Theorem.

If $\varphi$ is an analytic map of $\mathbb{D}$ into itself, $\psi$ is in $H^\infty$, and $M$ is an invariant subspace for $C_\varphi$ and for $S$, then $M$ is an invariant subspace for $W_{\psi,\varphi}$.

Conversely, if $\psi^{-1}$ is in $H^\infty$ and $M$ is an invariant subspace for $W_{\psi,\varphi}$ and for $S$, then $M$ is invariant for $C_\varphi$. 
The simplest singular inner function shift invariant subspaces are those associated with atomic measures:

**Theorem.**

If $\varphi$ is an analytic map of the unit disk into itself with $\varphi(1) = 1$ and $\varphi'(1) \leq 1$, then $e^{\frac{\alpha z+1}{z-1}}H^2$ is an invariant subspace for $C_\varphi$ when $\alpha > 0$.

Conversely, if $\varphi$ is an analytic map of the disk into itself and $e^{\frac{\alpha z+1}{z-1}}H^2$ is an invariant subspace for $C_\varphi$ for some $\alpha > 0$, then $\varphi(1) = 1$ and $\varphi'(1) \leq 1$.

**Conclusion:**

Suppose $\varphi$ maps the disk into itself and $\alpha > 0$. Then the subspace $e^{\frac{\alpha z+1}{z-1}}H^2$ is invariant for $C_\varphi$ if and only if 1 is the Denjoy-Wolff point of $\varphi$. 
For $|\lambda| = 1$ and $z_j$ for $j = 1, 2, \cdots$, points in $\mathbb{D}$ satisfying $\sum_j (1 - |z_j|) < \infty$, the function

$$B(z) = \lambda \prod_j \frac{|z_j| z_j - z}{z_j 1 - \bar{z}_j z}$$

is a Blaschke product. The zero set, $\{z_j\}$, for $B$ is denoted $Z(B)$.

**Lemma.**

Let $C_\varphi$ be a composition operator on $H^2$. Then $BH^2$ is invariant for $C_\varphi$ if and only if $z_j$ in $Z(B)$ implies $\varphi_n(z_j)$ is in $Z(B)$ for all non-negative integers $j$ and $n$ and if $w$ is in $Z(B)$, then multiplicity $\varphi(w) \geq$ multiplicity $w$. 
Theorem.

Suppose $C_\varphi$ is a composition operator on $H^2$ with $\varphi(a) = a$ for $a$ in $\mathbb{D}$.

If $BH^2$ is a non-trivial invariant subspace for $C_\varphi$, then

(i) $a$ is in $Z(B)$

and (ii) for every $z_j$ in $Z(B)$, there is an integer $n_j$ so that $\varphi(z_{n_j}) = a$.

Corollary.

Let $\varphi$ be a univalent analytic function mapping the disk into itself with $\varphi(a) = a$ for some $a$ in $\mathbb{D}$.

Then the subspaces $\left( \frac{z - a}{1 - \overline{a}z} \right)^k H^2$ are the only non-trivial Blaschke-product induced subspaces invariant for both $C_\varphi$ and $S$. 
Theorem.

Let \( \varphi \) be an analytic function mapping the disk into itself with Denjoy-Wolff point on the unit circle.

If \( B \) is a Blaschke product and \( BH^2 \) is invariant for \( C_\varphi \), then for each \( w \) in \( Z(B) \), the set \( \{ \varphi_n(w) : n \in \mathbb{N} \} \) is an infinite set in \( Z(B) \).

In particular, there are no finite Blaschke products \( B \) so that \( BH^2 \) is a (non-trivial) invariant subspace for \( C_\varphi \).
Theorem.

Let $\varphi$ be an analytic function mapping the disk into itself with Denjoy-Wolff point 1 on the unit circle.

If $B$ is a Blaschke product and $BH^2$ is invariant for $C_\varphi$, then for each $w$ in $Z(B)$, the set $\{\varphi_n(w) : n \in \mathbb{N}\}$ is an infinite set in $Z(B)$.

Examples:

1. Let $\varphi(z) = (z + 1)/2$. Then $\varphi(\mathbb{D}) \subset \mathbb{D}$ with $\varphi(1) = 1$ and $\varphi'(1) = 1/2$.

   If $B_0$ is any finite Blaschke product, there is a Blaschke product $B$ so that $B_0$ divides $B$ and $BH^2$ is an invariant subspace for $C_\varphi$.

2. Let $\varphi(z) = 1/(2 - z)$. Then $\varphi(\mathbb{D}) \subset \mathbb{D}$ with $\varphi(1) = \varphi'(1) = 1$.

   There is NO Blaschke product $B$ so that $BH^2$ is invariant for $C_\varphi$. 
There is space between our results: If $J$ is a singular inner function whose singular measure has no atom, then our work says nothing about possible non-trivial spaces of the form $JH^2$ that are invariant for $C_\varphi$!
Fourth Example:

Universal operators commuting with a compact operator
(Gallardo Gutíerrez & C., 2015)

Rota’s Universal Operators:

Let $\mathcal{H}$ be a Hilbert space and let $U$ be a bounded operator on $\mathcal{H}$.

We say $U$ is universal for $\mathcal{H}$ if for each bounded operator $A$ on $\mathcal{H}$,
there is an invariant subspace $M$ for $U$ and a non-zero number $\lambda$
such that $\lambda A$ is similar to $U|_M$

Rota proved in 1960 that there are universal operators!
Rota’s Universal Operators:

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Theorem (Caradus (1969))

If $\mathcal{H}$ is separable Hilbert space and $U$ is bounded operator on $\mathcal{H}$ such that:

- The null space of $U$ is infinite dimensional.
- The range of $U$ is $\mathcal{H}$.

then $U$ is universal for $\mathcal{H}$.
Some well known Universal Operators (in sense of Rota):

Best Known: adjoint of a unilateral shift of infinite multiplicity:

If $S$ is analytic Toeplitz operator whose symbol is an inner function that is not a finite Blaschke product, then $S^*$ is a universal operator.

Also well known: (Nordgren, Rosenthal, Wintrobe ('84,'87)):

If $\varphi$ is an automorphism of $\mathbb{D}$ with fixed points $\pm 1$ and Denjoy-Wolff point 1, that is, $\varphi(z) = \frac{z + s}{1 + sz}$ for $0 < s < 1$,

then a translate of the composition operator $C_\varphi$ is a universal operator.

(Gallardo Gutiérrez and C. showed (2011) that this translate, restricted to a co-dimension one invariant subspace on which it is universal, is unitarily equivalent to the adjoint of an analytic Toeplitz operator $T_\psi$.)
Some well known Universal Operators (in sense of Rota):

Best Known: adjoint of a unilateral shift of infinite multiplicity:

If $S$ is analytic Toeplitz operator whose symbol is an inner function that is \textit{not} a finite Blaschke product, then $S^*$ is a universal operator.

Also well known: (Nordgren, Rosenthal, Wintrobe ('84,'87)):

If $\varphi$ is an automorphism of $\mathbb{D}$ with fixed points $\pm 1$ and Denjoy-Wolff point $1$,

that is, $\varphi(z) = \frac{z + s}{1 + sz}$ for $0 < s < 1$,

then a translate of the composition operator $C_\varphi$ is a universal operator.

In C.’s thesis ('76): The analytic Toeplitz operators $S$ and $T_\psi$ (hence $C_\varphi$) \textit{DO NOT} commute with non-trivial compact operators.

Also proved: \textit{IF} an analytic Toeplitz operator commutes with a non-trivial compact, then the compact operator is quasi-nilpotent.
Main Theorem of 2013 paper (Gallardo Gutiérrez and C.):

There are bounded analytic functions $\varphi$ and $\psi$ on the unit disk

and an analytic map $J$ of the unit disk into itself

so that the Toeplitz operator $T_\varphi^*$ is a universal operator in the sense of Rota

and the weighted composition operator $W_{\psi,J}^*$

is an injective, compact operator with dense range

that commutes with the universal operator $T_\varphi^*$. 
Using this theorem, because $T_z^*$, the adjoint of the operator of multiplication by $z$, commutes with $T_\varphi^*$, the universal operator, which commutes with $W_{\psi,J}^*$, a non-zero compact operator, the following is a trivial, but quite surprising, application of Lomonosov’s theorem:

**Theorem. (!!)**

There is a backward shift invariant subspace,

$$L = (\eta H^2)\perp$$ for some inner function $\eta$,

that is invariant for every operator in $\{T_\varphi^*\}'$, the commutant of $T_\varphi^*$.  

Main Theorem of 2013 paper (Gallardo Gutiérrez and C.):

There are bounded analytic functions \( \varphi \) and \( \psi \) on the unit disk and an analytic map \( J \) of the unit disk into itself so that the Toeplitz operator \( T_\varphi^* \) is a universal operator in the sense of Rota and the weighted composition operator \( W_{\psi,J}^* \) is an injective, compact operator with dense range that commutes with the universal operator \( T_\varphi^* \).

Let \( K_\varphi \) be the set of compact operators:

\[
K_\varphi = \{ G \in B(H^2) : G \text{ is compact, and } T_\varphi^* G = G T_\varphi^* \}
\]

Theorem  The set \( K_\varphi \) is a closed ideal in \( \{ T_\varphi^* \}' \) and, in particular, \( \eta \) in \( H^\infty \) and \( G \) in \( K_\varphi \) implies \( T_\eta^* G \) and \( GT_\eta^* \) are also in \( K_\varphi \).

Moreover, every operator in \( K_\varphi \) is quasi-nilpotent.
A strategy for using $T^{*}_\varphi$ to solve the Invariant Subspace Problem is to also consider operators that commute with it, like $T^{*}_\eta$ for $\eta$ in $H^\infty$.

**Theorem.**

Let $M$ be an infinite dimensional, proper invariant subspace for $T^{*}_\varphi$.

If $W$ is an operator on $H^2$ that commutes with $T^{*}_\varphi$, then

- either kernel $(W) \cap M$ is a proper subspace of $M$ that is invariant for $T^{*}_\varphi$,
- or kernel $(W) \cap M = (0)$, or $M \subset$ kernel $(W)$.

**Corollary.**

Let $M$ be an infinite dimensional, proper invariant subspace for $T^{*}_\varphi$.

If $M$ contains a vector $w$ that is cyclic for the backward shift and a vector $v \neq 0$ that is non-cyclic for the backward shift, then kernel $(T^{*}_\eta) \cap M$ is a non-trivial invariant subspace for $T^{*}_\varphi$, where $\eta$ is an inner function with $T^{*}_\eta v = 0$. 
This suggests the question

*Does every closed, infinite dimensional subspace of $H^2$ include a non-zero, non-cyclic vector for the backward shift?*

but Prof. N. Nikolski pointed out that the answer to this question is “No!”.

On the other hand, we are not interested in arbitrary subspaces of $H^2$ so we specialize our query to address the issue at hand:

**Question 1:** *Does every closed, infinite dimensional, invariant subspace for $T^*_\varphi$ include a non-zero vector that is not cyclic for the backward shift?*
The another issue in the Corollary above is that there may be an inner function \( \eta \) for which \( M \subset \text{kernel}(T_\eta^*) \) which would mean every vector in \( M \) is non-cyclic for the backward shift and there would be no \( w \) for the hypothesis.

This possibility leads to the following question:

**Question 2:** Let \( M \) be an infinite dimensional, proper invariant subspace for \( T_\varphi^* \). Suppose \( \eta \) is an inner function for which \( M \subset \text{kernel}(T_\eta^*) \).

Is there always an inner function \( \zeta \) dividing \( \eta \) so that

\[
(0) \neq M \cap \text{kernel}(T_\zeta^*) \neq M?
\]

If the answers to both Question 1 and Question 2 are ‘Yes’, then every bounded operator on a Hilbert space of dimension at least 2 has a non-trivial invariant subspace!
THANK YOU!

This version of these slides, including references, is posted on my website:

http://www.math.iupui.edu/~ccowen/
References


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