GRADUATE/QUALIFIER LINEAR ALGEBRA: COMMENTS AND A PROPOSAL

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1. Overview

From the start of our PhD program and continuing to this time, our PhD students are required to take the Linear Algebra course Math 55400 or pass the qualifying exam based on that course. We inherited this version of Math 55400 from the Mathematics Department at Purdue, West Lafayette. We have graduate students with different interests than those studying math in West Lafayette, and I believe our qualifying exam in linear algebra should be a better fit to our students' interests. This essay will explore the history of this course as a part of our program, discuss what parts of linear algebra are in the course and what parts are not, and finally make a proposal based on my belief that a proof based course in Linear Algebra *IS* an important part of a mathematical sciences PhD education, but that our course as it is, does not serve the students we have. Specifically, I propose a course that also provides the theoretical foundation of applications of linear algebra that our students use, replacing some of the less important, for research in the mathematical sciences in this century, parts of the subject that are included in Math 55400.

2. History and Justification

In December 1969, a single day during the week between Christmas and New Year's, the Presidents of Indiana University and Purdue University got together and, in a document of between 25 and 30 pages, created two new institutions called Indiana University Purdue University Indianapolis (IUPUI) and Indiana Purdue Fort Wayne (IPFW) which were jointly managed by Indiana and Purdue. This blocked the plan of Richard Lugar, then Mayor of Indianapolis, to have the State Legislature create (Senate Bill #1 for the 1970 legislative session) the "University of Indianapolis", which would be a new state-supported research university and would replace the adjunct programs of IU and PU running then in the city. A few years later, Indiana Central College, a liberal arts college on the south side of the city, realized that IUPUI hadn't taken the name Lugar had in mind, and they changed their name to "University of Indianapolis".

The School of Science at IUPUI was created as a Purdue University School, and the programs in the Mathematical Sciences Department would be copies of the Purdue Mathematics Department courses (our 'parent' Department) and Statistics Department courses, and our majors would have the same programs as West Lafayette. Gradually, we in Indianapolis got to do more advanced courses and eventually (mid-1980's?) got to have graduate students whose main PhD advisor was a member of the IUPUI Department. Even so, it was required for there to be at least one member of the WL Department on the Committee, and Indianapolis PhD students took the same qualifying exams as the West Lafayette students, even though they were taking the courses from different faculty members... very tough for Indianapolis students. Eventually, it looks like sometime around 2005 from our Qual Archive, Indianapolis PhD

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students took qualifying exams prepared by the instructors of their courses. Just recently, the IUPUI Mathematical Sciences Department has begun reporting directly to the Purdue University Graduate School, finally(!), with little or no participation of the West Lafayette Math Department.

The qualifying exam system in our Department now is the one set up for the Purdue West Lafayette Mathematics Department in the 80's or maybe earlier: Core Subjects Abstract Algebra, Linear Algebra, Real Analysis and Complex Variables, with four qualifying exams needing to be passed, including two from core subjects, at least one of which is Abstract Algebra or Real Analysis. At that time, and for all of the time I was part of that Department, about a third of the mathematics faculty in West Lafayette did research in algebra and a similar fraction of the PhD students wrote theses in algebra. For this reason, Abstract Algebra and Linear Algebra were taught and had qualifying exams written by faculty whose main research interests were in algebra and they had large numbers of students who were doing so also.

I am not an algebraist, but I think about linear algebra, have done some research in linear algebra, and teach linear algebra a lot. I have now taught (a traditional version, following Hoffman and Kunze) Math 55400 two times, this Spring and Spring 2017.

In our Department, we have very few students interested in writing theses in algebra and very few faculty members who have algebra as their main research interest. From this standpoint, it does not seem necessary to continue to take the West Lafayette perspective of Linear Algebra being a sibling of Abstract Algebra and part of the central core of algebra research. For me, Hoffman and Kunze are writing about 19th century linear algebra and ignoring, in my opinion, most of the 20th century additions to and perspectives on linear algebra.

Most important, I do not believe the course Math 55400 provides the background in linear algebra that many IUPUI Mathematical Sciences PhD students need! I am therefore proposing the creation of a new graduate linear algebra course, maybe Math 55450, that would be an additional "Core Course" and would be an alternative to Math 55400, and *not* allowing students to get credit for taking both courses or using qualifying exams in both versions of Linear Algebra toward the four exams required. In addition, I propose that I should teach this new course in Spring Semester, 2020, with someone else teaching Math 55400, and both courses would have qualifying exams set in August 2020 and January 2021.

3. **Proposed Course Description**

My vision for this course is *NOT* as a numerical linear algebra course or a computational linear algebra course, but rather as a course in abstract mathematics focussed on the linear algebra foundations of both classical topics and also 20th century topics like singular value decomposition or properties of matrices with all entries being non-negative real numbers, and having more emphasis on inner product spaces than is typical in Math 55400.

For example, clearly, eigenvectors and eigenvalues, invariant subspaces, characteristic polynomials, minimal polynomials, and non-diagonalizable matrices for linear transformations are very important in all linear algebra courses. On the other hand, the focus of the Jordan Canonical Form theorem, a central goal for Hoffman and Kunze's book, is the creation of very special basis for the vector space that is both very tedious to develop and, in my opinion, unimportant in understanding the structure of a linear transformation in terms of its invariant subspaces, eigenvectors, and eigenvalues.

In addition, while linear algebra includes vector spaces over arbitrary fields, and there are important applications of linear algebra to vector spaces over finite fields, the vast majority of important applications and interest are to $\mathbb{R}^n$ and $\mathbb{C}^n$, vector spaces over the real and complex fields. The proposed course will introduce vector spaces over finite fields, give some examples, and some problems including linear independence and bases for such vector spaces, but the majority of the material will cover $\mathbb{R}^n$ and $\mathbb{C}^n$ and inner products will be introduced relatively early and used when convenient after that point.

While applications will not be part of the course, I will mention applications of the material, as I do in *all* courses I teach, at the time we are developing the structures and proving the theorems that form the foundation for the applications that our students might see in their later work. For example, in the linear algebra courses we teach, in Math 35100 and higher, we talk about different bases for a given vector space, rather than having just one, favorite, basis. It might not be obvious to students why we bother: one important reason is that in using linear algebra to study a problem, some bases might be more helpful than others in understanding what is happening in the problem. In the same way, this course will include theoretical topics that make the problems our students might see later in their work easier to understand. Another example, appearing in both Math 55400 and in the proposed course, the definition of positive definite matrices and theorems about some of their properties are included. As part of that discussion, I would point out that, to quote Professor Sarkar, "the variance-covariance matrix of a random vector is positive definite with probability one."

4. **Sources**

[RB] R. Bellman, *Introduction to Matrix Analysis*, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1995.

[CC] C. C. Cowen, *Linear Algebra for Engineering and Science*, West Pickle Press, Indianapolis, IN, 1996. (text)

[PH] P. R. Halmos, *Finite Dimensional Vector Spaces, (2^{nd} edition)*, Van Nostrand Co., Princeton, NJ, 1958.

[HK] K. Hoffman and R. Kunze, *Linear Algebra (2^{nd} edition)*, 1971 (paperback reprint, 2015).

[HM] H. Minc, *Nonnegative Matrices*, John Wiley & Sons, Inc., 1988.

5. **Approximate Course Outline**

Topics	*Sources*
Fields, matrices, partitioned matrices, systems of linear equations, determinants	CC, PH, HK
Vector spaces, subspaces, bases, dimension, sums and intersections of subspaces	CC, PH, HK
Linear transformations, linear functionals, matrices for linear transformations, rank–nullity theorem, coordinates, change of coordinates	CC, PH, HK
Polynomials, algebras, ideals	HK
Inner products, orthogonality, Gram–Schmidt, Householder, QR-factorization, norms, adjoints, Fundamental Theorem of Linear Algebra	CC
Midterm Test	
Projections, orthogonal projections, inconsistent systems and least squares 'solutions'	CC
Eigenvalues, eigenvectors, spectral mapping thm., diagonalizability, Cayley-Hamilton, characteristic and minimal polynomials, block Jordan form	CC
Hermitian, unitary, normal, and positive definite matrices and transformations Schur's triangularization theorem, spectral theorem for normal operators	CC, PH
Singular Value Decomposition	CC
Matrices with positive entries, Perron-Frobenius Theorem, Schur/Hadamard product	RB, HM
Final Exam	