Unitary Equivalence of One-parameter Groups of Toeplitz and Composition Operators

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Joint work with Eva Gallardo Gutiérrez, U. Zaragoza, Spain
The Hardy Hilbert space on the unit disk, \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) is:

\[
H^2 = \left\{ f \text{ analytic in } \mathbb{D} : f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ with } \|f\|^2 = \sum |a_n|^2 < \infty \right\}
\]

where for \( f \) and \( g \) in \( H^2 \), we have \( \langle f, g \rangle = \sum a_n b_n \)

and we will consider two types of operators on \( H^2 \):

For \( \psi \) an analytic map of \( \mathbb{D} \) into the complex plane,

the analytic Toeplitz operator \( T_{\psi} \) is

\[
(T_{\psi}f)(z) = \psi(z)f(z) \quad \text{for } f \text{ in } H^2
\]

and, for \( \varphi \) an analytic map of \( \mathbb{D} \) into itself,

the composition operator \( C_{\varphi} \) is

\[
(C_{\varphi}f)(z) = f(\varphi(z)) \quad \text{for } f \text{ in } H^2
\]
For example, if $\varphi$ is defined by

$$\varphi(z) = \frac{3z + 1}{z + 3}$$

then $\varphi$ is an automorphism of the disk $\mathbb{D}$ with $\varphi(\pm 1) = \pm 1$ and $\varphi'(1) = \frac{1}{2}$.

The spectrum of $C_\varphi$ is the annulus

$$\sigma(C_\varphi) = \{\lambda : \frac{1}{\sqrt{2}} \leq |\lambda| \leq \sqrt{2}\}$$

and each $\lambda$ with

$$\frac{1}{\sqrt{2}} < |\lambda| < \sqrt{2}$$

is an eigenvalue of infinite multiplicity for $C_\varphi$. 
Similarly, if $\psi$ is defined by
\[
\psi(z) = \left( \frac{1 - z}{1 + z} \right)^{(i \log 2)/\pi}
\]
then $\psi$ is the covering map of the disk $\mathbb{D}$ onto the annulus
\[
\psi(\mathbb{D}) = \{ \lambda : \frac{1}{\sqrt{2}} < |\lambda| < \sqrt{2} \}
\]

The spectrum of the Toeplitz operator $T_\psi = T_{\psi}^*$ is the annulus
\[
\psi(\mathbb{D}) = \{ \lambda : \frac{1}{\sqrt{2}} \leq |\lambda| \leq \sqrt{2} \}
\]
and each $\lambda$ with
\[
\frac{1}{\sqrt{2}} < |\lambda| < \sqrt{2}
\]
is an eigenvalue of infinite multiplicity for $T_{\psi}^*$. 
Both of these operators are part of one-parameter groups of operators:

$$\varphi_t(z) = \frac{(1 + e^{-t})z + (1 - e^{-t})}{(1 - e^{-t})z + (1 + e^{-t})}$$

with $C_{\varphi_s}C_{\varphi_t} = C_{\varphi_s \circ \varphi_t} = C_{\varphi_{s+t}}$ for $-\infty < s, t < \infty$

and

$$\psi_t(z) = \left(\frac{1 - z}{1 + z}\right)^{it/\pi}$$

with $T^*_{\psi_s}T^*_{\psi_t} = T^*_{\psi_s \psi_t} = T^*_{\psi_{s+t}}$ for $-\infty < s, t < \infty$

and each $\lambda$ with

$$e^{-t/2} < |\lambda| < e^{t/2}$$

is an eigenvalue of infinite multiplicity for each of $C_{\varphi_t}$ and $T^*_{\psi_t}$. 
IDEA!!

If there were a connection (e.g. similarity or unitary equivalence) between these operators,
then the eigenvectors for each of these operators should correspond to the eigenvectors for the same eigenvalue for the other operator!
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BUT there are too many eigenvectors!
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BUT there are too many eigenvectors!

TRY infinitesimal generators!
The infinitesimal generator of a (semi)group $A_t$ of operators

is an operator $G$ such that for each $f$ (in the domain of $G$)

$$Gf = \left. \frac{d}{dt} \right|_{t=0} A_t f$$

and analogous to the ideas from solution of first order linear elementary
differential equations, we imagine that

$$A_t \, \overset{\text{“=”}}{=} \, e^{tG}$$
The infinitesimal generator of the group of composition operators is

\[
\left( \frac{d}{dt} \bigg|_{t=0} \mathcal{C}_{\varphi_t} f \right)(z) = \frac{d}{dt} \bigg|_{t=0} f(\varphi_t(z))
\]

\[
= f'(\varphi_t(z)) \frac{2e^{-t}(1 - z^2)}{\left[ (1 - e^{-t})z + (1 + e^{-t}) \right]^2} \bigg|_{t=0}
\]

\[
= f'(z) \frac{1 - z^2}{2}
\]

so \( G \) is the differential operator

\[
(Gf)(z) = \frac{1}{2}(1 - z^2)f'(z)
\]

A similar calculation gives the infinitesimal generator, \( H \), of the group \( T_{\psi_t}^* \).

Eigenvectors for the same eigenvalues of \( G \) and \( H \) should also be connected!
GOOD NEWS!!

The corresponding eigenspaces are 1-dimensional!

Let’s try to match them up!
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The corresponding eigenspaces are 1-dimensional!

Let’s try to match them up!

For $-1/2 < \text{Re}\,\lambda < 1/2$, the eigenvectors of $G$ and $H$ are multiples of

$$w_\lambda = \left(\frac{1-z}{1+z}\right)^{-\lambda} \quad \text{and} \quad v_\lambda = \left(1 - \frac{-i\sin\left(\frac{\lambda\pi}{2}\right)}{\cos\left(\frac{\lambda\pi}{2}\right)} z\right)^{-1}$$

and, for each $G$ and $H$,

the eigenvectors corresponding to $-1/2 < \lambda < 1/2$ have dense span in $H^2$
If $G$ and $H$ are to correspond to each other,

for $-1/2 < \lambda, \mu < 1/2$,

the relationship between $w_\lambda$ and $w_\mu$ should be analogous to

the relationship between $v_\lambda$ and $v_\mu$. 
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**BAD NEWS!!**

Nasty computations give:

$$2\langle v_\lambda, v_\mu \rangle = \langle w_\lambda, w_\mu \rangle + 1$$

Not a good correspondence!
If \( G \) and \( H \) are to correspond to each other,

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**BAD NEWS!!**

Nasty computations give:

\[
2\langle v_\lambda, v_\mu \rangle = \langle w_\lambda, w_\mu \rangle + 1
\]

Not a good correspondence!

AND it can’t be fixed by multiplying the vectors by a constant!
If $G$ and $H$ are to correspond to each other,
the relationship between $w_\lambda$ and $w_\mu$ should be analogous to
the relationship between $v_\lambda$ and $v_\mu$.

When does a unitary operator take a pair of one dimensional subspaces onto
another (specific) pair??
If $G$ and $H$ are to correspond to each other,
the relationship between $w_\lambda$ and $w_\mu$ should be analogous to
the relationship between $v_\lambda$ and $v_\mu$.

When does a unitary operator take a pair of one dimensional subspaces onto
another (specific) pair??

**Lemma.**

Let $w_\lambda$, $w_\mu$, $v_\lambda$, and $v_\mu$ be non-zero vectors in $\mathcal{H}$, and let $M_\lambda = \text{span}\{w_\lambda\}$, $M_\mu = \text{span}\{w_\mu\}$, $N_\lambda = \text{span}\{v_\lambda\}$, and $N_\mu = \text{span}\{v_\mu\}$. There is a unitary operator $U$ on $\mathcal{H}$ such that $UM_\lambda = N_\lambda$ and $UM_\mu = N_\mu$ if and only if

$$\frac{|\langle w_\lambda, w_\mu \rangle|}{\|w_\lambda\|\|w_\mu\|} = \frac{|\langle v_\lambda, v_\mu \rangle|}{\|v_\lambda\|\|v_\mu\|}$$
Lemma.

There is a unitary operator $U$ on $\mathcal{H}$ such that $U M_\lambda = N_\lambda$ and $U M_\mu = N_\mu$ if and only if

$$\frac{|\langle w_\lambda, w_\mu \rangle|}{\|w_\lambda\| \|w_\mu\|} = \frac{|\langle v_\lambda, v_\mu \rangle|}{\|v_\lambda\| \|v_\mu\|}$$

Theorem.

There is no unitary operator $U$ on $H^2$ such that $U^* C_{\varphi_t} U = T_{\psi_t}^*$ for every real number $t$.

Proof:

Such a unitary $U$ would take eigenspaces $M_\lambda$ and $M_\mu$ onto the eigenspaces $N_\lambda$ and $N_\mu$, but for $\lambda = 0$ and $\mu = 1/4$, the angles don’t match.
If $G$ and $H$ are to correspond to each other,

for $-1/2 < \lambda, \mu < 1/2$,

the relationship between $w_\lambda$ and $w_\mu$ should be analogous to

the relationship between $v_\lambda$ and $v_\mu$.

**IDEA!!**

Could the relationship

$$2\langle v_\lambda, v_\mu \rangle = \langle w_\lambda, w_\mu \rangle + 1$$

come from invariant subspaces??
Lemma.

For $D$ bounded operator on Hilbert space $\mathcal{H}$ and $M$ an invariant subspace, then $M \perp$ is an invariant subspace for $D^*$.

$$D = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad D^* = \begin{pmatrix} A^* & 0 \\ B^* & C^* \end{pmatrix}$$
Lemma.

For $D$ bounded operator on Hilbert space $\mathcal{H}$ and $M$ an invariant subspace, then $M^\perp$ is an invariant subspace for $D^*$. Furthermore, if $r$ is an eigenvector for $D$ with eigenvalue $\lambda$ and $r = p + q$ where $p$ is in $M$ and $q$ is in $M^\perp$, then either $q = 0$ or $q$ is an eigenvector for the eigenvalue $\lambda$ for the compression of $D$ to $M^\perp$, which is the adjoint of the restriction of $D^*$ to its invariant subspace $M^\perp$.

\[
\begin{pmatrix} \lambda p \\ \lambda q \end{pmatrix} = \lambda \begin{pmatrix} p \\ q \end{pmatrix} = \lambda r = Dr = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} Ap + Bq \\ Cq \end{pmatrix}
\]

so $\lambda q = Cq$

For each $t$, the subspace $zH^2$ is an invariant subspace for $T_{\psi_t}$ and for $C_{\varphi_t}^*$.

Letting $x_\lambda = w_\lambda - 1$ and $u_\lambda = v_\lambda - 1$, each in $zH^2$, work above means that $x_\lambda$ and $u_\lambda$ are each eigenvectors of the compressions of $C_{\varphi_t}$ and $T_{\psi_t}^*$ to $zH^2$ and they are eigenvectors of the compressions of $G$ and $H$ to $zH^2$ that correspond to the same eigenvalues.

$zH^2$ is an invariant subspace for $T_{\psi_t}$ and for $C_{\varphi_t}^*$ for each $t$.

Letting $x_\lambda = w_\lambda - 1$ and $u_\lambda = v_\lambda - 1$, each in $zH^2$, this means that

- $x_\lambda$ and $u_\lambda$ are each eigenvectors of the compressions of $C_{\varphi_t}$ and $T_{\psi_t}^*$ to $zH^2$
- and they are eigenvectors of the compressions of $G$ and $H$ to $zH^2$
- that correspond to the same eigenvalues.

**A MIRACLE:**

$$2\langle u_\lambda, u_\mu \rangle = \langle x_\lambda, x_\mu \rangle$$
Theorem.

(1) The operator $U$ defined by

$$U(x_\lambda) = \sqrt{2}u_\lambda$$

can be extended to a unitary operator of $zH^2$ onto itself.
Theorem.

(1) The operator \( U \) defined by

\[ U(x_\lambda) = \sqrt{2} u_\lambda \]

can be extended to a unitary operator of \( zH^2 \) onto itself.

(2) For each real number \( t \),

\[ U \left( C_{\varphi_t}^* \right|_{zH^2} = T_{\psi_t} \right|_{zH^2} U \]

so the operators \( C_{\varphi_t}^* \left|_{zH^2} \right. \) and \( T_{\psi_t} \left|_{zH^2} \right. \) are unitarily equivalent.
Theorem.

(1) The operator $U$ defined by

$$U(x_\lambda) = \sqrt{2}u_\lambda$$

can be extended to a unitary operator of $zH^2$ onto itself.

(2) For each real number $t$,

$$U \, C^{\ast}_{\varphi_t}|_{zH^2} = T_{\psi_t}|_{zH^2} \, U$$

so the operators $C^{\ast}_{\varphi_t}|_{zH^2}$ and $T_{\psi_t}|_{zH^2}$ are unitarily equivalent.

That is, there is a unitary operator on $zH^2$ that shows the restrictions of $C^{\ast}_{\varphi_t}$ and $T_{\psi_t}$ to $zH^2$ are unitarily equivalent for each $t$. 
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Slides posted on webpage:

www.math.iupui.edu/~ccowen/Richmond1011.pdf

Paper posted on webpage:

www.math.iupui.edu/~ccowen/Downloads.html