Composition Operators on Hilbert Spaces of Analytic Functions

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Otoño 2012

At this time, there are several active areas, in the world and in España, in functional analysis and the theory connected with Hilbert space operators and these include both study of very abstract questions and the study of more concrete problems. It is well known and easy to prove that all separable, complex, infinite dimensional Hilbert spaces are isomorphic. Since this is the case, it is reasonable to ask why questions are set in different concrete Hilbert spaces like $\ell_2$, $L^2([0,1])$, or the Hardy Hilbert space $H^2$. One answer is that different contexts suggest different tools and different approaches to problems and it may be that one context might provide more easily discoverable solutions to an abstract problem than other contexts do. This course will provide an introduction to the study of Hilbert space problems in a context closely connected to the study of analytic functions and their links to algebra and geometry, an area in which there is active interest in many parts of the world.

The study of composition operators lies at the interface of analytic function theory and operator theory and uses tools from both areas to solve problems that involve the connections between them. As a part of operator theory, research on composition with a fixed function acting on a space of analytic functions is of fairly recent origin, dating back to work of E. Nordgren in the mid 1960’s. The first explicit reference to composition operators in the Mathematics Subject Classification Index appeared in 1990. This was a recognition that over the intervening years the literature had grown enough to deserve notice and it continues to grow today. There are themes that have developed so that it is possible to see important groups of papers as exploring the same theme. This course will try to synthesize some of the contexts suggest different tools and different approaches to problems and it may be that one context might provide more easily discoverable solutions to an abstract problem than other contexts do. This course will provide an introduction to the study of Hilbert space problems in a context closely connected to the study of analytic functions and their links to algebra and geometry, an area in which there is active interest in many parts of the world.

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One of the attractive features of this subject is that the prerequisites are minimal. The lectures will presume a basic background in analysis, both functional analysis and complex analysis. Rather than seeking the utmost generality, the theory will be developed in a context that is comfortable and illustrates the nature of the general results.

The approach is based on a philosophy that mathematics develops best from a base of well chosen examples and that its theorems describe and generalize what is true about the characteristic objects in a subject. Thus the lectures are about the concrete operator theory that arises when we study the operation of composition of analytic functions in the
context of the classical spaces. In particular, we study the relationship between properties of the operator $C_\varphi$ and properties of the function $\varphi$: the goal is to see the boundedness, spectrum, adjoint, etc., of $C_\varphi$ as resulting from the particular geometric and analytic features of the symbol $\varphi$. The theory of multiplication operators, arising from the spectral theorem for normal operators, has developed and branched into the study of Toeplitz operators, subnormal operators and so on. I believe composition operators can similarly inform the development of operator theory because they are very diverse and occur naturally in a variety of problems. Composition operators have arisen in the study of commutants of multiplication operators and more general operators and play a role in the theory of dynamical systems. De Branges' original proof of the Bieberbach conjecture depended on composition operators (he called them substitution operators) on a space of analytic functions.

There will be opportunities for questions and some exercises will be posed.


**Other References:**


Basic outline for the course

1. **Introduction & related function theory.** Functional Banach spaces, description of the classical Hardy, Bergman, Dirichlet spaces, and their generalizations. Littlewood subordination, Schwarz–Pick, Julia–Carathéodory, and Denjoy–Wolff theorems, the Denjoy–Wolff point, $a$, the model for iteration.

2. **Basic results.** Boundedness, norms, compactness. Properties of linear fractional maps and their composition operators, adjoints.

3. **Adjoints.** $C_\varphi^*$ for $\varphi$ rational and $C_\varphi$ defined on $H^2$.

4. **Spectra.** $C_\varphi$ invertible or Fredholm, spectra of composition operators with inner symbol, spectra for symbol with $|a| = 1$ and $\varphi'(a) \leq 1$. Spectra for symbol with $|a| < 1$ and $|\varphi'(a)| < 1$, $C_\varphi$ compact and non-compact composition operators, weighted composition operators.

5. **Connections to Invariant Subpaces & other Problems.** Discussion of future research directions and unsolved problems.