Continuous Semigroups of Composition Operators on Function Spaces on the Disk

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Session on Function Spaces and Operator Theory

Vanderbilt, 14 April 2018

If φ is an analytic function of $\mathbb{D}$ into itself,

and $\mathcal{H}$ is a Hilbert space of analytic functions on $\mathbb{D}$,

then the composition operator C_φ on $\mathcal{H}$ is the operator

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Usual spaces: f analytic in $\mathbb{D}$, with $f(z) = \sum_{n=0}^{\infty} a_n z^n$

$$\text{Hardy:} \quad H^2(\mathbb{D}) = H^2 = \{f : \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

$$\text{Bergman:} \quad A^2(\mathbb{D}) = A^2 = \{f : \|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 \frac{dA(z)}{\pi} < \infty\}$$

$$\text{weighted Hardy } (\|z^n\| = \beta_n > 0)\text{:} \quad H^2(\beta) = \{f : \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty\}$$

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For many spaces, including the Hardy and Bergman spaces on the disk,

the operators C_φ are bounded for all φ that map the disk into itself.

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Since φ maps $\mathbb{D}$ into itself, the *iterates of* φ, that is

$\varphi_2 = \varphi \circ \varphi, \qquad \varphi_3 = \varphi \circ \varphi_2, \qquad$ etc. $\qquad$ make sense.

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We want to ask "When can this be extended to C_{φ_t} for all $t > 0$?"

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More formally: a *strongly continuous semigroup on the Hilbert space* $\mathcal{H}$ is a map $T : \mathbb{R}_+ \mapsto \mathcal{B}(\mathcal{H})$ such that

- $T(0) = I$
- $T(s + t) = T(s)T(t)$ for $s, t \geq 0$
- For each x in $\mathcal{H}$, $\lim_{t\to 0^+} \|T(t)x - x\| = 0$.

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"When is there a strongly continuous semigroup T for which $T(1) = C_\varphi$?"

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- Bad News: I don't know much!
- Goal: This is an interesting problem;

  I hope some of you will be interested in working on it and do more than I can!

Denjoy-Wolff Theorem (1926).

If φ is an analytic map of $\mathbb{D}$ into itself (not an elliptic automorphism), there is a unique fixed point, a, of φ in $\overline{\mathbb{D}}$ such that $|\varphi'(a)| \leq 1$.

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This distinguished fixed point a is called the *Denjoy-Wolff point of* φ.

The iterates of analytic maps of $\mathbb{D}$ into itself $\mathbb{D}$ have been studied by many mathematicians for about 150 years!!

Some famous names include E. Schroeder, G. Koenigs, P. Fatou, A. Denjoy, J. Wolff, C.L. Siegel, I.N. Baker, Ch. Pommerenke, $\cdots$

The Model for Iteration [C., 1981]

Let φ be an analytic mapping of $\mathbb{D}$ into itself, φ non-constant and not an elliptic automorphism of $\mathbb{D}$, and let a be the Denjoy–Wolff point of φ.

If $\varphi'(a) \neq 0$, then there is a fundamental set V for φ on $\mathbb{D}$,

a domain Ω, either a halfplane or the plane,

an automorphism Φ mapping Ω onto Ω,

and a mapping σ of $\mathbb{D}$ into Ω

such that φ and σ are univalent on V, $\sigma(V)$ is a fundamental set for Φ on Ω, and

$$\Phi \circ \sigma = \sigma \circ \varphi$$

Moreover, Φ is unique up to conjugation by an automorphism of Ω onto Ω, and Φ and σ depend only on φ, not on the particular fundamental set V.

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What's New?? One Proof for All Cases!! Uniqueness of the Model!!

Let φ be an analytic map of $\mathbb{D}$ into itself (not an elliptic automorphism), and let a be the Denjoy-Wolff point of φ.

Analytic self-maps of $\mathbb{D}$ (not elliptic automorphisms) divide into 5 distinct classes and the *Model for Iteration* covers 4 of these cases:

- (Plane/Dilation): $|a| < 1$ and $0 < |\varphi'(a)| < 1$
- (Half-Plane/Dilation): $|a| = 1$ and $0 < \varphi'(a) < 1$
- (Half-Plane/Translation): $|a| = 1$ and $\varphi'(a) = 1$, and $\{\varphi_n(z)\}$ interpolating
- (Plane/Translation): $|a| = 1$ and $\varphi'(a) = 1$, and $\{\varphi_n(z)\}$ not interpolating
- (no LF model): $|a| < 1$ and $\varphi'(a) = 0$

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Specifically, for $\sigma(z) = (1+z)(1-z)^{-1}$ and $\Phi(z) = z+1$, we have

$$\sigma \circ \varphi(z) = \frac{1 + \frac{z+1}{-z+3}}{1 - \frac{z+1}{-z+3}} = \frac{2}{-z+1} = \frac{1+z}{1-z} + 1 = \sigma(z) + 1 = \Phi \circ \sigma(z)$$

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In fact, φ is part of the continuous semi-group:

$$\varphi_t(z) = \frac{(2-t)z + t}{-tz + (t+2)}$$

and

$$\sigma \circ \varphi_t(z) = \frac{1 + \frac{(2-t)z+t}{-tz+(t+2)}}{1 - \frac{(2-t)z+t}{-tz+(t+2)}} = \frac{-tz + (t+2) + ((2-t)z+t)}{-tz + (t+2) - ((2-t)z+t)} = \sigma(z) + t = \Phi_t \circ \sigma(z)$$

This example makes everything look easy!

That is an illusion, but there some things that carry over to more complicated cases.

Moreover, the conclusion can be partially recovered for all maps φ with $\varphi'(a) \neq 0$!

For simplicity, the result will be stated in the plane-translation case, although analogous results hold for other cases.

Let φ be an analytic mapping on the disk with Denjoy–Wolff point, a,

with $|a| = 1$ and $\varphi'(1) = 1$.

In addition, suppose V is a fundamental set for φ on $\mathbb{D}$ and σ is a map of $\mathbb{D}$ into $\Omega = \mathbb{C}$ such that φ and σ are univalent on V, $\sigma(V)$ is a fundamental set for $\Phi(z) = z + 1$ on $\mathbb{C}$, and

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For z in $\mathbb{D}$, let $\nu(z) = \min\{n : \varphi_n(z) \in V\}$

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For z in $\mathbb{D}$, let $\nu(z) = \min\{n : \varphi_n(z) \in V\}$

Define τ by

$$\tau(z) = \inf\{t : \nu(z) \leq t \text{ and } \sigma(z) + t_1 \in \sigma(V) \text{ for all } t_1 > t\}$$

(Note: $\tau(z) < \infty$ for every z in $\mathbb{D}$ and $\tau(z) = 0$ for some z in $\mathbb{D}$)

Partially Defined Semigroups [C., 1981]

Let φ be an analytic mapping of $\mathbb{D}$ into itself that falls into the Plane/Translation case of the Model Theorem.

There is a function $h(z,t)$ complex analytic in the first argument for z in $\mathbb{D}$ and real analytic in the second argument for $t > \tau(z)$ such that:

- $h(z,n) = \varphi_n(z)$ for $n > \tau(z)$

and • $h(h(z,t_1),t_2) = h(z,t_1+t_2)$ for $t_1 > \tau(z)$ and $t_2 > 0$.

Moreover, there is a function g meromorphic in $\mathbb{D}$ and holomorphic in V that agrees with the infinitesimal generator of $h(z,t)$ on the set $\{z : \tau(z) = 0\}$.

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In essence, this result is defining the tail of a continuous semigroup $\varphi_t(z)$ for each z in $\mathbb{D}$ and $t > \tau(z)$.

Corollary: Let φ be an analytic function mapping $\mathbb{D}$ into itself such that

- φ is real valued on the interval $(-1, 1)$
- $\varphi'(x) > 0$ for $-1 < x < 1$

and • the Denjoy-Wolff point of φ is 1

then there is a function $h(x, t)$ real analytic in each variable for $|x| < 1$ and $t \geq 0$ such that $h(x, 1) = \varphi(x)$ and

$$h(h(z, t_1), t_2) = h(z, t_1 + t_2)$$

for t_1 and t_2 non-negative real numbers.

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for t_1 and t_2 non-negative real numbers.

In other words, $\varphi_t(x)$ is a continuous semigroup on $(-1, 1)$.

An Example:

Let $\varphi(z) = \frac{1}{4}(1 + z)^2$

Clearly, φ is an analytic map of $\mathbb{D}$ into itself and $\varphi(1) = \varphi'(1) = 1$,

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Moreover, φ acting on $\mathbb{D}$ is univalent and satisfies the hypotheses of the Corollary above, so $\varphi_t(x)$ is a real-analytic semigroup on $(-1, 1]$.

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In addition, for each z in $\mathbb{D}$, the angle between the ray from 1 to z and the real axis is greater than angle between the ray from 1 to $\varphi(z)$ and the real axis, so that, in some sense, φ is mapping points in $\mathbb{D}$ toward the real axis.

Let $\varphi(z) = \frac{1}{4}(1+z)^2$

Conjecture:

Composition operator C_φ is part of a continuous semigroup of operators on H^2.

THANK YOU!

A version of these slides is posted on my website:

`http://www.math.iupui.edu/~ccowen/`