An Empirical Likelihood Approach Of Testing of High Dimensional Symmetries

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Jackknife Empirical likelihood for Multivariate U-statistics

Let \((\mathcal{X}, \mathcal{I})\) be a measurable space and \(P\) be a probability measure on this space. Let \(Z_1, \ldots, Z_n\) be independent copies of a \(\mathcal{X}\)-valued random variable \(Z\) with cumulative distribution function \(F\) under \(P\). Let \(h : \mathbb{R}^m \rightarrow \mathbb{R}^d\) be a known function that is permutation symmetric in its \(m\) arguments. A multivariate or vector U-statistic with kernel \(h\) of order \(m\) is defined as

\[
U_n \equiv U_{nm}(h) = \left( \binom{n}{m} \right)^{-1} \sum_{1 \leq i_1 < \ldots < i_m \leq n} h(Z_{i_1}, \ldots, Z_{i_m}), \quad n \geq 2.
\]
Introduction to U-statistics

- \( h \in L_2(F^m) \), where \( L_2(F^m) = \{ f : \int \|f\|^2 dF^m < \infty \} \)
Introduction to U-statistics

- $h \in L_2(F^m)$, where $L_2(F^m) = \{ f : \int \|f\|^2 dF^m < \infty \}$
- $\theta = E(h) := E(h(Z_1, \ldots, Z_m)) = \int h dF^m$
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- $U_n$ is an unbiased estimate of $\theta$
- Let $U_n^\#$ denote the projection of $U_n$, then $U_n^\#$ is a sum of independent and identically distributed random vectors, as

$$U_n^\# = \sum_{j=1}^n E(U_n|Z_j) - (n - 1)\theta.$$
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  U_n = U_n^\# + \alpha_n, \quad \alpha_n = O_p(n^{-1}).
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Lemma
(The Central Limit Theorem for Multivariate U-statistics)
Suppose the kernel $h$ is square-integrable and the dispersion matrix $\Sigma = \text{Var}(U)$ is positive definite. Then $\sqrt{n}(U_n - \theta)$ and $\sqrt{n}(U^\#_n - \theta)$ are asymptotically equivalent, hence $\sqrt{n}(U_n - \theta)$ is asymptotically normal with mean zero and covariance matrix $m^2 \Sigma$, that is,

$$\sqrt{n}(U_n - \theta) \implies \mathcal{N}(0, m^2 \Sigma).$$
Jackknife pseudo values of U-statistics

Let $U_{n-1}$ denote the U-statistic based on the $n - 1$ observations $Z_1, \ldots, Z_{j-1}, Z_{j+1}, \ldots, Z_n$. 
Jackknife pseudo values of U-statistics

- Let $U_{n-1}^{(-j)}$ denote the U-statistic based on the $n - 1$ observations $Z_1, \ldots, Z_{j-1}, Z_{j+1}, \ldots, Z_n$.
- The Jackknife pseudo values of the U-statistic are defined as
  \[ V_{nj} = nU_n - (n - 1)U_{n-1}^{(-j)}, \quad j = 1, \ldots, n. \]
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V_{nj} = nU_n - (n - 1)U_{n-1}^{(-j)}, \quad j = 1, \ldots, n.
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- $\tilde{V}_{nj} = V_{nj} - \theta$
Some Properties

- $V_{nj}$ is also an unbiased estimator of $\theta$. 

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Some Properties

- $V_{nj}$ is also an unbiased estimator of $\theta$.
- One has

$$\tilde{V}_{nj} = m\tilde{h}_1(Z_j) + O_p(n^{-1/2}), \quad j = 1, \ldots, n.$$ 

where

$$h_c(z_1, \ldots, z_c) = E(h(Z_1, \ldots, Z_m) | Z_1 = z_1, \ldots, Z_c = z_c),$$

$$c = 1, \ldots, m - 1.$$
It shows that each Jackknife value $V_{nj}$ depends asymptotically on $Z_j$, so that $V_{nj}, j = 1, \ldots, n$ are asymptotically independent.
Introduction

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- If $\pi_j$ is a probability mass placed at $Z_j$, then approximately the same probability mass $\pi_j$ is placed at the Jackknife value $V_{nj}$ for $j = 1, \ldots, n$. 
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- It shows that each Jackknife value $V_{nj}$ depends asymptotically on $Z_j$, so that $V_{nj}, j = 1, \ldots, n$ are asymptotically independent.
- If $\pi_j$ is a probability mass placed at $Z_j$, then approximately the same probability mass $\pi_j$ is placed at the Jackknife value $V_{nj}$ for $j = 1, \ldots, n$.
- The joint likelihood is approximately the product of these $\pi_j$'s.
Jackknife Empirical likelihood with side information

\[ R_n(h, g) = \sup \left\{ \prod_{j=1}^{n} n\pi_j : \pi \in \mathcal{P}_n, \sum_{j=1}^{n} \pi_j \tilde{V}_{nj} = 0, \sum_{j=1}^{n} \pi_j g(Z_j) = 0 \right\}, \]

- \( g \) is a measurable functions from \( \mathcal{X} \) to \( \mathbb{R}^r \) such that \( \int g \, dF = 0 \) and \( \int \|g\|^2 \, dF \) is finite.
- \( r \) is the number of equalities that express the side information, and we shall call them *constraints*.
- We allow \( r \) to depend on the sample size \( n \), \( r = r_n \), and to grow to infinity slowly with \( n \) and study the asymptotic behaviors of the empirical likelihood.
Let $h^{(i)}$ be a measurable functions from $\mathcal{X}^m_i$ to $\mathbb{R}^{d_i}$ which is argument-symmetric and square-integrable for $i = 1, \ldots, r$.

Let $\tilde{V}_{nj}(h^{(i)})$ be the centered jackknife pseduo value of the U-statistic $U_{nmk}(h^{(k)})$ of order $m_i$. 
Let $h^{(i)}$ be a measurable functions from $\mathcal{X}^m_i$ to $\mathbb{R}^{d_i}$ which is argument-symmetric and square-integrable for $i = 1, \ldots, r$. Let $\tilde{V}_{nj}(h^{(i)})$ be the centered jackknife pseudo value of the U-statistic $U_{nm_k}(h^{(k)})$ of order $m_i$. With the U-statistics as side information, we associate the empirical likelihood

$$R_n(h^{(1)}, \ldots, h^{(r)}) = \sup \left\{ \prod_{j=1}^{n} n\pi_j : \pi \in \mathcal{P}_n, \sum_{j=1}^{n} \pi_j \tilde{V}_{nj}(h^{(k)}) = 0, \right.$$  

$$k = 1, \ldots, r \}.$$
Theorem

THEOREM 1 Let \( r_n = r \) for all \( n \). Suppose \( h^{(1)}, \ldots, h^{(r)} \) are argument-symmetric and square-integrable kernels. Assume \( W(h^{(1)}, \ldots, h^{(r)}) \) is positive definite. Then

\[-2 \log \mathcal{R}_n(h^{(1)}, \ldots, h^{(r)}) \overset{d}{\to} \chi^2_{d_1 + \cdots + d_r}.\]
Testing Uniformity

- Suppose $X_1, \ldots, X_n$ is a random sample from uniform distribution on the unit sphere, $\mathcal{U}(S^{d-1})$. 
Testing Uniformity

▶ Suppose $X_1, \ldots, X_n$ is a random sample from uniform distribution on the unit sphere, $\mathcal{U}(S^{d-1})$.

▶ Let $2 \leq m \leq n$. 

Kent, Mardia and Rao (1979) proved that Uniformity $\iff \bar{X}_o^m \perp \perp R_m$, which implies $E(a_k(R_m)\bar{X}_o^m) = 0$, $a_k \in L^2_0(F)$, $k = 1, 2, \ldots$, where $F$ is the distribution function of $R_m$, and $\{a_k\}$ is a basis of $L^2_0(F)$. 
Testing Uniformity

- Suppose $X_1, \ldots, X_n$ is a random sample from uniform distribution on the unit sphere, $\mathcal{U}(S^{d-1})$.

- Let $2 \leq m \leq n$.

- $\bar{X}_m = m^{-1} \sum_{i=1}^{m} X_i$ (the sample mean vector);
- $\bar{X}_m^o = \bar{X}_m / \|\bar{X}_m\|$ (the direction of sample mean);
- $R_m = \|\bar{X}_m\|$ (length of the sample resultant).

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$$E(a_k(R_m)\bar{X}_m^o) = 0, \quad a_k \in L_{2,0}(F), \; k = 1, 2, \ldots,$$

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Example

For convenience, let \( m = 2 \) and \( d = 3 \).
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- $X \sim \mathcal{U}(S^2)$, $X = (x, y, z)^\top$, and the density is $f(x, y, z) = \frac{1}{4\pi}$, $(x, y, z) \in S^2$.
- Let $X_1, \ldots, X_n$ be i.i.d copies of $X$.

\[
R(X_i, X_j) = \|X_i + X_j\|, \quad S(X_i, X_j) = \frac{X_i + X_j}{\|X_i + X_j\|}, \quad i, j = 1, \ldots, n.
\]
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\]

- It follows that

\[
E(a_k(R(\mathbf{X}_i, \mathbf{X}_j))S(\mathbf{X}_i, \mathbf{X}_j)) = 0, \quad a_k \in L_{2,0}(F), \quad k = 1, 2, \ldots,
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It follows that

$$E\left( a_k(R(\mathbf{X}_i, \mathbf{X}_j))S(\mathbf{X}_i, \mathbf{X}_j) \right) = 0, \quad a_k \in L_{2,0}(F), \quad k = 1, 2, \ldots,$$

Assume that $F$ is continuous. Then a basis of $L_{2,0}(F)$ is

$$\{ \varphi_k \circ F \}, \text{ where } \{ \varphi_k \} = \{ \varphi_k : k = 1, 2, \ldots \} \text{ is a basis of } L_{2,0}(\mathcal{U}) \text{ with } \mathcal{U} \text{ the uniform distribution over } [0, 1].$$
Example

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\begin{align*}
R(\mathbf{X}_i, \mathbf{X}_j) &= \|\mathbf{X}_i + \mathbf{X}_j\|, \\
S(\mathbf{X}_i, \mathbf{X}_j) &= \frac{\mathbf{X}_i + \mathbf{X}_j}{\|\mathbf{X}_i + \mathbf{X}_j\|}, \quad i, j = 1, \ldots, n.
\end{align*}
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- It follows that
\[
E(a_k(R(\mathbf{X}_i, \mathbf{X}_j))S(\mathbf{X}_i, \mathbf{X}_j)) = 0, \quad a_k \in L_{2,0}(F), \quad k = 1, 2, \ldots,
\]
- Assume that $F$ is continuous. Then a basis of $L_{2,0}(F)$ is $\{\varphi_k \circ F\}$, where $\{\varphi_k\} = \{\varphi_k : k = 1, 2, \ldots\}$ is a basis of $L_{2,0}(\mathcal{U})$ with $\mathcal{U}$ the uniform distribution over $[0, 1]$.
- We usually use the trigonometric basis $\varphi_k(t) = \sqrt{2} \cos k\pi t$. 
In this case, an estimator of the above expected value is the vector U-statistics

\[ U_n = \binom{n}{m}^{-1} \sum_{1 \leq i < j \leq n} a_k(R(X_i, X_j))S(X_i, X_j) \]
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\[ U_n = \left( \begin{array}{c} \ n \\ m \end{array} \right)^{-1} \sum_{1 \leq i < j \leq n} a_k(R(X_i, X_j))S(X_i, X_j) \]

We can show that this is a minimum variance unbiased estimator of \( E(a_k(R(X_i, X_j))S(X_i, X_j)) \).
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Jackknifing this vector U-statistics by the Jackknife pseudo values

\[ V_{nj} = nU_n - (n - 1)U_{n-1}^{(-j)} \]

where \( U_{n-1}^{(-j)} \) denote the vector U-statistic based on the \( n - 1 \) observations \( X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n \).
State the null hypothesis $H_0 : X_1, \ldots, X_n \sim \mathcal{U}(S^2)$
Example

- State the null hypothesis $H_0: X_1, \ldots, X_n \sim \mathcal{U}(S^2)$

- The null hypothesis implies $E(a_k(R(X_i, X_j))S(X_i, X_j)) = 0$. This suggests the jackknife empirical likelihood

$$ R_n = \sup \left\{ \prod_{i=1}^{n} n\pi_i : \pi \in \mathcal{P}_n, \sum_{j=1}^{n} \pi_j v_{nj}(a_k) = 0, \; k = 1, \ldots, r \right\}. $$
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\]

- We will show that

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-2 \log \mathcal{R}_n \implies \chi^2_{r \times d}
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▶ We will show that

$$-2 \log R_n \implies \chi^2_{r \times d}$$

▶ In this case, we accept $H_0$ if $-2 \log R_n \leq \chi^2_{3 \times r}(1 - \alpha)$, where $\alpha$ is the level of significance. $\chi^2_r(1 - \alpha)$ is the $(1 - \alpha) \times 100\%$ percentile of $\chi^2$ distribution with degrees of freedom $r$. 

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Testing Symmetries

- Spherical Symmetry
- Rotational Symmetry
Testing Spherical Symmetry

Suppose a random vector $X \in \mathcal{R}^d$ has a distribution spherically symmetric about $\theta$, i.e.,

$$
X - \theta \overset{d}{=} \Gamma(X - \theta),
$$

for every orthogonal $d \times d$ matrix $\Gamma$. 

$V = \|X - \theta\|$, $U = (X - \theta)/\|X - \theta\|$. We have $U \sim U(S_{d-1})$. 

Spherical symmetry $\iff V \perp \perp U \iff E[a(V)b(U)] = 0$, $a \in L^2(FV)$, $b \in L^2(FU)$. 

The choices for $a, b$ are uncountably many.
Testing Spherical Symmetry

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- $V = \|X - \theta\|$, $U = (X - \theta)/\|X - \theta\|$. We have $U \sim \mathcal{U}(S^{d-1})$.

- Spherical symmetry

$$\iff V \perp U \iff E[a(V)b(U)] = 0, \quad a \in L_{2,0}(F_V), \ b \in L_{2,0}(F_U).$$

The choices for $a, b$ are uncountably many.
However $a, b$ can be reduced to countably many equations.
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Let $\{a_j\}$ denote a basis of $L_{2,0}(F_V)$ and $\{b_k\}$ denote a basis of $L_{2,0}(F_U)$.
Take $a_j = \varphi_j \circ F_V$ and $b_k = \varphi_k \circ F_U$. $\varphi_k(t) = \sqrt{2} \cos k\pi t$. 
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Take $a_j = \varphi_j \circ F_V$ and $b_k = \varphi_k \circ F_U$. $\varphi_k(t) = \sqrt{2} \cos k\pi t$.

Using the first few basis functions, we can construct empirical likelihood ratio:

$$R_{ss}^n = \sup \left\{ \prod_{i=1}^{n} n\pi_i : \pi \in \mathcal{P}_n, \sum_{i=1}^{n} \pi_i a_j(V_i)b_k(U_i) = 0, \quad j = 1, \ldots, J, \quad k = 1, \ldots, K \right\}$$

where $(V_i, U_i), i = 1, \ldots, n$ is a random sample of $(V, U)$.
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Let $\{a_j\}$ denote a basis of $L_{2,0}(F_V)$ and $\{b_k\}$ denote a basis of $L_{2,0}(F_U)$.

Take $a_j = \varphi_j \circ F_V$ and $b_k = \varphi_k \circ F_U$. $\varphi_k(t) = \sqrt{2} \cos k \pi t$.

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where $(V_i, U_i), i = 1, \ldots, n$ is a random sample of $(V, U)$.

By Owen’s theorem, $-2 \log R_{ss}^n \Rightarrow \chi^2_{JK}$.
Consider a vector function $U \rightarrow f(U)$ for some known function $f : \mathcal{R}^d \rightarrow \mathcal{R}^e$. (For example, $f(U) = U$.)
Consider a vector function $\mathbf{U} \rightarrow f(\mathbf{U})$ for some known function $f : \mathcal{R}^d \rightarrow \mathcal{R}^e$. (For example, $f(\mathbf{U}) = \mathbf{U}$.)

The empirical likelihood takes the form

$$R_{n}^{ssh} = \sup \left\{ \prod_{i=1}^{n} n \pi_i : \pi \in \mathcal{P}_n, \sum_{i=1}^{n} \pi_i a_j(V_i) f(\mathbf{U}_i) = 0, \quad j = 1, \ldots, J \right\}$$
Consider a vector function $\mathbf{U} \rightarrow f(\mathbf{U})$ for some known function $f : \mathcal{R}^d \rightarrow \mathcal{R}^e$. (For example, $f(\mathbf{U}) = \mathbf{U}$.)

The empirical likelihood takes the form

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$$

In this case, under certain conditions we have

$$
-2 \log \mathcal{R}_{n}^{ssh} \Rightarrow \chi^2_{Je}
$$
Simulations with Jackknife pseudo values

- Suppose $X_1, \ldots, X_n$ is a random sample from a spherically symmetrical distribution.
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- $V_i = \|X_i - \theta\|$, $U_i = (X_i - \theta)/\|X_i - \theta\|$, $U_i \sim \mathcal{U}(S^{d-1}), i = 1, \ldots, n$. 
Simulations with Jackknife pseudo values

- Suppose $X_1, \ldots, X_n$ is a random sample from a spherically symmetrical distribution.
- $V_i = \|X_i - \theta\|$, $U_i = (X_i - \theta) / \|X_i - \theta\|$, $U_i \sim \mathcal{U}(S^{d-1})$, $i = 1, \ldots, n$.
- Let $m = 2$, for any $p, q = 1, \ldots, n$, let $R = U_p + U_q$, $R^0 = R / \|R\|$.
Simulations with Jackknife pseudo values

- Suppose \( X_1, \ldots, X_n \) is a random sample from a spherically symmetrical distribution.

- \( V_i = \|X_i - \theta\|, U_i = (X_i - \theta)/\|X_i - \theta\|, \)
  \( U_i \sim \mathcal{U}(S^{d-1}), i = 1, \ldots, n. \)

- Let \( m = 2 \), for any \( p, q = 1, \ldots, n \), let \( \mathbf{R} = U_p + U_q, \)
  \( R^0 = \mathbf{R}/\|\mathbf{R}\|. \)

- We have the fact that \( U_i \sim \mathcal{U}(S^{d-1}) \iff \|\mathbf{R}\| \perp \mathbf{R}^0. \)
Simulations

Testing Symmetries

Let $b_k = \varphi_k \circ G$, $\varphi_k(t) = \sqrt{2} \cos k\pi t$, $k = 1, \ldots, K$. 

$b_K = (b_1, \ldots, b_K) \top$.

$G(u) = \binom{n}{2}^{-1} \sum_{1 \leq p < q \leq n} 1[\|U_p + U_q\| \leq u]$. 

$\varphi_k(t)$ is argument-symmetric.
Let \( b_k = \varphi_k \circ G, \varphi_k(t) = \sqrt{2} \cos k\pi t, k = 1, \ldots, K. \)
\[
b_K = (b_1, \ldots, b_K)^\top.
\]
\[
G(u) = \left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right)^{-1} \sum_{1 \leq p < q \leq n} 1[\|U_p + U_q\| \leq u].
\]
Let the kernel function
\[
h(U_p, U_q) = b_K(\|U_p + U_q\|) \otimes (\|U_p + U_q\|/\|U_p + U_q\|),
\]
which is argument-symmetric.
Let $b_k = \varphi_k \circ G$, $\varphi_k(t) = \sqrt{2} \cos k\pi t$, $k = 1, \ldots, K$.

$\mathbf{b}_K = (b_1, \ldots, b_K)^\top$.

$G(u) = \binom{n}{2}^{-1} \sum_{1 \leq p < q \leq n} 1[\|\mathbf{U}_p + \mathbf{U}_q\| \leq u]$.

Let the kernel function

$h(\mathbf{U}_p, \mathbf{U}_q) = b_K(\|\mathbf{U}_p + \mathbf{U}_q\|) \otimes ((\mathbf{U}_p + \mathbf{U}_q)/\|\mathbf{U}_p + \mathbf{U}_q\|)$, which is argument-symmetric.

The U-statistics with the kernel $h$ is given by

$$U_n(\mathbf{b}_K) = \binom{n}{2}^{-1} \sum_{1 \leq p < q \leq n} h(\mathbf{U}_p, \mathbf{U}_q)$$
Let \( b_k = \varphi_k \circ G \), \( \varphi_k(t) = \sqrt{2} \cos k\pi t \), \( k = 1, \ldots, K \).

\[
b_K = (b_1, \ldots, b_K)^\top.
\]

\[
G(u) = \binom{n}{2}^{-1} \sum_{1 \leq p < q \leq n} 1[\|U_p + U_q\| \leq u].
\]

Let the kernel function

\[
h(U_p, U_q) = b_K(\|U_p + U_q\|) \otimes ((U_p + U_q)/\|U_p + U_q\|),
\]

which is argument-symmetric.

The U-statistics with the kernel \( h \) is given by

\[
U_n(b_K) = \binom{n}{2}^{-1} \sum_{1 \leq p < q \leq n} h(U_p, U_q)
\]

The Jackknife pseudo values of the U-statistics is given by

\[
V_{ni} = nU_n(b_K) - (n - 1)U_{n-1}(b_K), \quad i = 1, \ldots, n.
\]
Let \( \mathbf{a}_J = (a_1, \ldots, a_J)^\top \).
Let $\mathbf{a}_J = (a_1, \ldots, a_J)^\top$.

Combine two parts together, we get the Jackknife empirical likelihood with side information as follows,

$$\mathcal{R}_n (\mathbf{h}, \mathbf{g}) = \sup \left\{ \prod_{i=1}^{n} n\pi_i : \pi \in \mathcal{P}_n, \sum_{i=1}^{n} \pi_i \mathbf{a}_J (V_i) \otimes f(U_i) = 0, \sum_{i=1}^{n} \pi_i V_{ni} = 0 \right\}$$
Let $\mathbf{a}_J = (a_1, \ldots, a_J)^\top$.

Combine two parts together, we get the Jackknife empirical likelihood with side information as follows,

$$
\mathcal{R}_n(h, g) = \sup \left\{ \prod_{i=1}^n n^{\pi_i} : \pi \in \mathcal{P}_n, \sum_{i=1}^n \pi_i a_J(V_i) \otimes f(U_i) = 0, \right. \\
\left. \sum_{i=1}^n \pi_i V_{ni} = 0 \right\}
$$

By Theorem 1, under certain conditions we have

$$
-2 \log \mathcal{R}_n(h, g) \to \chi^2_{(Je+Kd)}
$$
Simulation results of samples from normal distribution

- We still calculate powers of this test with different settings.
- For convenience, we set $d = e = \text{dim}$, and $J = K = r$.
- $r$ is basically the number of basis functions.
- The null hypothesis $H_0 : \theta = (0, 0, 0)^\top$. 
  
  $\text{rep}(x, \text{dim})$ denotes the alternative hypothesis $H_1 : \theta = (x, x, x)^\top$. 
## Multivariate Normal Distribution

**Comparison of Power for Different H1 with H0: theta = 0**

\[ n = 100 \quad m = 2000 \]

\[ d = e = \text{dim} \quad J = K = r \]

<table>
<thead>
<tr>
<th>dim</th>
<th>( \text{rep}(0, \text{dim}) )</th>
<th>( r = 1 )</th>
<th>( r = 2 )</th>
<th>( r = 3 )</th>
<th>( r = 4 )</th>
<th>( r = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( \text{rep}(0, \text{dim}) )</td>
<td>0.044</td>
<td>0.031</td>
<td>0.04</td>
<td>0.056</td>
<td>0.0785</td>
</tr>
<tr>
<td></td>
<td>( \text{rep}(0.1, \text{dim}) )</td>
<td>0.1215</td>
<td>0.084</td>
<td>0.081</td>
<td>0.1015</td>
<td>0.135</td>
</tr>
<tr>
<td></td>
<td>( \text{rep}(0.2, \text{dim}) )</td>
<td>0.4185</td>
<td>0.329</td>
<td>0.2825</td>
<td>0.2935</td>
<td>0.3595</td>
</tr>
<tr>
<td></td>
<td>( \text{rep}(0.3, \text{dim}) )</td>
<td>0.793</td>
<td>0.7365</td>
<td>0.663</td>
<td>0.6715</td>
<td>0.707</td>
</tr>
<tr>
<td></td>
<td>( \text{rep}(0.4, \text{dim}) )</td>
<td>0.9225</td>
<td>0.971</td>
<td>0.928</td>
<td>0.9215</td>
<td>0.9265</td>
</tr>
<tr>
<td></td>
<td>( \text{rep}(0.5, \text{dim}) )</td>
<td>0.901</td>
<td>0.9985</td>
<td>0.9955</td>
<td>0.9945</td>
<td>0.9935</td>
</tr>
<tr>
<td>3</td>
<td>( \text{rep}(0, \text{dim}) )</td>
<td>0.0325</td>
<td>0.0345</td>
<td>0.0715</td>
<td>0.1355</td>
<td>0.2825</td>
</tr>
<tr>
<td></td>
<td>( \text{rep}(0.1, \text{dim}) )</td>
<td>0.0775</td>
<td>0.097</td>
<td>0.131</td>
<td>0.201</td>
<td>0.363</td>
</tr>
<tr>
<td></td>
<td>( \text{rep}(0.2, \text{dim}) )</td>
<td>0.293</td>
<td>0.335</td>
<td>0.381</td>
<td>0.471</td>
<td>0.6185</td>
</tr>
<tr>
<td></td>
<td>( \text{rep}(0.3, \text{dim}) )</td>
<td>0.531</td>
<td>0.7425</td>
<td>0.7355</td>
<td>0.7995</td>
<td>0.8665</td>
</tr>
<tr>
<td></td>
<td>( \text{rep}(0.4, \text{dim}) )</td>
<td>0.6055</td>
<td>0.954</td>
<td>0.9365</td>
<td>0.956</td>
<td>0.9665</td>
</tr>
<tr>
<td></td>
<td>( \text{rep}(0.5, \text{dim}) )</td>
<td>0.566</td>
<td>0.9465</td>
<td>0.9395</td>
<td>0.9695</td>
<td>0.9795</td>
</tr>
<tr>
<td>4</td>
<td>( \text{rep}(0, \text{dim}) )</td>
<td>0.0235</td>
<td>0.054</td>
<td>0.145</td>
<td>0.349</td>
<td>0.639</td>
</tr>
<tr>
<td></td>
<td>( \text{rep}(0.1, \text{dim}) )</td>
<td>0.068</td>
<td>0.109</td>
<td>0.214</td>
<td>0.443</td>
<td>0.717</td>
</tr>
<tr>
<td></td>
<td>( \text{rep}(0.2, \text{dim}) )</td>
<td>0.209</td>
<td>0.351</td>
<td>0.477</td>
<td>0.6965</td>
<td>0.8825</td>
</tr>
<tr>
<td></td>
<td>( \text{rep}(0.3, \text{dim}) )</td>
<td>0.3425</td>
<td>0.7225</td>
<td>0.772</td>
<td>0.9135</td>
<td>0.9595</td>
</tr>
<tr>
<td></td>
<td>( \text{rep}(0.4, \text{dim}) )</td>
<td>0.374</td>
<td>0.836</td>
<td>0.8735</td>
<td>0.9515</td>
<td>0.981</td>
</tr>
<tr>
<td></td>
<td>( \text{rep}(0.5, \text{dim}) )</td>
<td>0.3155</td>
<td>0.757</td>
<td>0.842</td>
<td>0.933</td>
<td>0.9745</td>
</tr>
</tbody>
</table>
Simulation results of samples from $t$ distribution

- We still calculate powers of this test with different settings.
- For convenience, we set $d = e = dim$, and $J = K = r$, $df$ denotes the degrees of freedom of $t$ distribution.
- $r$ is basically the number of basis functions.
- The null hypothesis $H_0 : \theta = (0, 0, 0)^\top$.
- $rep(x, dim)$ denotes the alternative hypothesis $H_1 : \theta = (x, x, x)^\top$. 
## Multivariate t Distribution

**Comparison of Power for Different H1 with H0: \( \theta = 0 \)**

**n=100 m=2000 df=1**

<table>
<thead>
<tr>
<th>d=e=dim</th>
<th>J=K=r</th>
</tr>
</thead>
<tbody>
<tr>
<td>rep(0, dim)</td>
<td>0.044</td>
</tr>
<tr>
<td>rep(0.1, dim)</td>
<td>0.0745</td>
</tr>
<tr>
<td>rep(0.2, dim)</td>
<td>0.2145</td>
</tr>
<tr>
<td>rep(0.3, dim)</td>
<td>0.484</td>
</tr>
<tr>
<td>rep(0.4, dim)</td>
<td>0.733</td>
</tr>
<tr>
<td>rep(0.5, dim)</td>
<td>0.8725</td>
</tr>
</tbody>
</table>

**dim=2**

<table>
<thead>
<tr>
<th>d=e=dim</th>
<th>J=K=r</th>
</tr>
</thead>
<tbody>
<tr>
<td>rep(0, dim)</td>
<td>0.036</td>
</tr>
<tr>
<td>rep(0.1, dim)</td>
<td>0.06</td>
</tr>
<tr>
<td>rep(0.2, dim)</td>
<td>0.198</td>
</tr>
<tr>
<td>rep(0.3, dim)</td>
<td>0.4035</td>
</tr>
<tr>
<td>rep(0.4, dim)</td>
<td>0.619</td>
</tr>
<tr>
<td>rep(0.5, dim)</td>
<td>0.734</td>
</tr>
</tbody>
</table>

**dim=3**

<table>
<thead>
<tr>
<th>d=e=dim</th>
<th>J=K=r</th>
</tr>
</thead>
<tbody>
<tr>
<td>rep(0, dim)</td>
<td>0.0295</td>
</tr>
<tr>
<td>rep(0.1, dim)</td>
<td>0.062</td>
</tr>
<tr>
<td>rep(0.2, dim)</td>
<td>0.205</td>
</tr>
<tr>
<td>rep(0.3, dim)</td>
<td>0.4385</td>
</tr>
<tr>
<td>rep(0.4, dim)</td>
<td>0.6315</td>
</tr>
<tr>
<td>rep(0.5, dim)</td>
<td>0.7275</td>
</tr>
</tbody>
</table>
# Multivariate t Distribution

Comparison of Power for Different H1 with H0: theta=0

\[ n=100 \quad m=2000 \quad df=2 \]

\[ d=e=dim \quad J=K=r \]

<table>
<thead>
<tr>
<th></th>
<th>rep(0, dim)</th>
<th>rep(0.1, dim)</th>
<th>rep(0.2, dim)</th>
<th>rep(0.3, dim)</th>
<th>rep(0.4, dim)</th>
<th>rep(0.5, dim)</th>
</tr>
</thead>
<tbody>
<tr>
<td>dim=2</td>
<td>r=1</td>
<td>0.042</td>
<td>0.0805</td>
<td>0.2415</td>
<td>0.514</td>
<td>0.7585</td>
</tr>
<tr>
<td></td>
<td>r=2</td>
<td>0.033</td>
<td>0.055</td>
<td>0.1925</td>
<td>0.462</td>
<td>0.7725</td>
</tr>
<tr>
<td></td>
<td>r=3</td>
<td>0.0375</td>
<td>0.0615</td>
<td>0.15</td>
<td>0.38</td>
<td>0.664</td>
</tr>
<tr>
<td></td>
<td>r=4</td>
<td>0.056</td>
<td>0.0795</td>
<td>0.176</td>
<td>0.3995</td>
<td>0.6695</td>
</tr>
<tr>
<td></td>
<td>r=5</td>
<td>0.092</td>
<td>0.1265</td>
<td>0.241</td>
<td>0.452</td>
<td>0.708</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>rep(0, dim)</th>
<th>rep(0.1, dim)</th>
<th>rep(0.2, dim)</th>
<th>rep(0.3, dim)</th>
<th>rep(0.4, dim)</th>
<th>rep(0.5, dim)</th>
</tr>
</thead>
<tbody>
<tr>
<td>dim=3</td>
<td>r=1</td>
<td>0.034</td>
<td>0.0605</td>
<td>0.14</td>
<td>0.2845</td>
<td>0.3895</td>
</tr>
<tr>
<td></td>
<td>r=2</td>
<td>0.041</td>
<td>0.068</td>
<td>0.222</td>
<td>0.4875</td>
<td>0.883</td>
</tr>
<tr>
<td></td>
<td>r=3</td>
<td>0.065</td>
<td>0.0945</td>
<td>0.236</td>
<td>0.5135</td>
<td>0.8785</td>
</tr>
<tr>
<td></td>
<td>r=4</td>
<td>0.129</td>
<td>0.1775</td>
<td>0.3485</td>
<td>0.6155</td>
<td>0.934</td>
</tr>
<tr>
<td></td>
<td>r=5</td>
<td>0.2615</td>
<td>0.332</td>
<td>0.5085</td>
<td>0.7195</td>
<td>0.964</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>rep(0, dim)</th>
<th>rep(0.1, dim)</th>
<th>rep(0.2, dim)</th>
<th>rep(0.3, dim)</th>
<th>rep(0.4, dim)</th>
<th>rep(0.5, dim)</th>
</tr>
</thead>
<tbody>
<tr>
<td>dim=4</td>
<td>r=1</td>
<td>0.027</td>
<td>0.049</td>
<td>0.107</td>
<td>0.191</td>
<td>0.2765</td>
</tr>
<tr>
<td></td>
<td>r=2</td>
<td>0.05</td>
<td>0.101</td>
<td>0.2535</td>
<td>0.5205</td>
<td>0.735</td>
</tr>
<tr>
<td></td>
<td>r=3</td>
<td>0.1375</td>
<td>0.2045</td>
<td>0.3665</td>
<td>0.624</td>
<td>0.8075</td>
</tr>
<tr>
<td></td>
<td>r=4</td>
<td>0.3335</td>
<td>0.4035</td>
<td>0.5895</td>
<td>0.8155</td>
<td>0.9265</td>
</tr>
<tr>
<td></td>
<td>r=5</td>
<td>0.636</td>
<td>0.691</td>
<td>0.824</td>
<td>0.925</td>
<td>0.9725</td>
</tr>
</tbody>
</table>
### Multivariate t Distribution

**Comparison of Power for Different H1 with H0: \theta=0**

**n=100, m=2000, df=3**

<table>
<thead>
<tr>
<th>d=e=dim, J=K=r</th>
<th>r=1</th>
<th>r=2</th>
<th>r=3</th>
<th>r=4</th>
<th>r=5</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>dim=2</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>rep(0, dim)</td>
<td>0.0425</td>
<td>0.0365</td>
<td>0.0385</td>
<td>0.054</td>
<td>0.092</td>
</tr>
<tr>
<td>rep(0.1, dim)</td>
<td>0.102</td>
<td>0.0565</td>
<td>0.0655</td>
<td>0.083</td>
<td>0.118</td>
</tr>
<tr>
<td>rep(0.2, dim)</td>
<td>0.2615</td>
<td>0.199</td>
<td>0.1665</td>
<td>0.197</td>
<td>0.24</td>
</tr>
<tr>
<td>rep(0.3, dim)</td>
<td>0.5625</td>
<td>0.5165</td>
<td>0.435</td>
<td>0.4375</td>
<td>0.476</td>
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<tr>
<td>rep(0.4, dim)</td>
<td>0.7745</td>
<td>0.821</td>
<td>0.7565</td>
<td>0.7355</td>
<td>0.738</td>
</tr>
<tr>
<td>rep(0.5, dim)</td>
<td>0.8225</td>
<td>0.957</td>
<td>0.9245</td>
<td>0.912</td>
<td>0.8975</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>d=e=dim, J=K=r</th>
<th>r=1</th>
<th>r=2</th>
<th>r=3</th>
<th>r=4</th>
<th>r=5</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>dim=3</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>rep(0, dim)</td>
<td>0.029</td>
<td>0.039</td>
<td>0.075</td>
<td>0.135</td>
<td>0.2665</td>
</tr>
<tr>
<td>rep(0.1, dim)</td>
<td>0.0575</td>
<td>0.0705</td>
<td>0.108</td>
<td>0.186</td>
<td>0.3405</td>
</tr>
<tr>
<td>rep(0.2, dim)</td>
<td>0.1405</td>
<td>0.198</td>
<td>0.251</td>
<td>0.3485</td>
<td>0.5215</td>
</tr>
<tr>
<td>rep(0.3, dim)</td>
<td>0.2545</td>
<td>0.4995</td>
<td>0.5325</td>
<td>0.6355</td>
<td>0.7465</td>
</tr>
<tr>
<td>rep(0.4, dim)</td>
<td>0.3165</td>
<td>0.804</td>
<td>0.7775</td>
<td>0.84</td>
<td>0.902</td>
</tr>
<tr>
<td>rep(0.5, dim)</td>
<td>0.305</td>
<td>0.874</td>
<td>0.8835</td>
<td>0.932</td>
<td>0.9585</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>d=e=dim, J=K=r</th>
<th>r=1</th>
<th>r=2</th>
<th>r=3</th>
<th>r=4</th>
<th>r=5</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>dim=4</strong></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>rep(0, dim)</td>
<td>0.031</td>
<td>0.0505</td>
<td>0.1425</td>
<td>0.332</td>
<td>0.642</td>
</tr>
<tr>
<td>rep(0.1, dim)</td>
<td>0.047</td>
<td>0.093</td>
<td>0.191</td>
<td>0.4155</td>
<td>0.705</td>
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<td>rep(0.2, dim)</td>
<td>0.0915</td>
<td>0.231</td>
<td>0.3715</td>
<td>0.6105</td>
<td>0.817</td>
</tr>
<tr>
<td>rep(0.3, dim)</td>
<td>0.147</td>
<td>0.5055</td>
<td>0.619</td>
<td>0.7995</td>
<td>0.935</td>
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<tr>
<td>rep(0.4, dim)</td>
<td>0.1705</td>
<td>0.6985</td>
<td>0.7985</td>
<td>0.9105</td>
<td>0.9775</td>
</tr>
<tr>
<td>rep(0.5, dim)</td>
<td>0.1745</td>
<td>0.7145</td>
<td>0.7995</td>
<td>0.9435</td>
<td>0.981</td>
</tr>
</tbody>
</table>
Testing Rotational Symmetry

- Suppose a random vector $X \in S^{d-1}$ is rotationally symmetric about direction $\theta$, that is,

$$X - \theta \overset{d}{=} O(X - \theta),$$

for every $d \times d$ rotation matrix $O$ about a fixed direction $\theta$ in $\mathcal{R}^d$. 

Let $T = \theta^\top X$ be the projection of $X$ onto the direction $\theta$.

Let $\xi$ be the unit tangent at $\theta$ to $S^{d-1}$.

$\xi \sim U(S^{d-2}(\theta))$, where $S^{d-2}(\theta) = \{x \in \mathbb{R}^d : |x| = 1, x^\top \theta = 0\}$. 

Rotational Symmetry $\Rightarrow$ $T = \theta^\top X \perp \perp \xi = X - T\theta \parallel X - T\theta$. 

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Lingnan Li
Independence implies
\[ E(a(T)b(\xi)) = 0, \quad a \in L_{2,0}(F_T), \ b \in L_{2,0}(G_\xi). \]

Similar to the spherical symmetry case, take \( a_j = \varphi_j \circ F_T, \ j = 1, \ldots, J \), and a vector function \( \xi \rightarrow f(\xi) \) for some known function \( f : \mathcal{R}^d \rightarrow \mathcal{R}^e \).
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We have \( E(a_j(T)\xi) = 0. \) Let \( a_J = (a_1, \ldots, a_J)^\top. \) The empirical likelihood ratio takes the form

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R_{n}^{rsh} = \sup \left\{ \prod_{i=1}^{n} n\pi_i : \pi \in \mathcal{P}_n, \sum_{i=1}^{n} \pi_i a_J(T) \otimes \xi = 0 \right\}
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\]

Under certain conditions we have

\[ -2 \log \mathcal{R}^{rsh}_n \Rightarrow \chi^2_{Je} \]
Simulations with Jackknife pseudo values

- We construct the same U-statistics and Jackknife pseudo values as the spherical symmetry case.
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- In our simulation, we generated the data distributed from Von Mises-Fisher distribution.
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$$-2 \log \mathcal{R}_n(h, g) \rightarrow \chi^2(J_0 + Kd)$$

- In our simulation, we generated the data distributed from Von Mises-Fisher distribution.
- We are testing

\[ H_0 : \theta = (0, 0, 1)^\top \quad \text{V.S.} \quad H_1 : \theta = (0.14, 0.14, 0.98)^\top. \]
Calculate the powers of this test with different settings.

For convenience, take \( d = e = 3, J = K = r \).

\( r \) is basically the number of basis functions.

The results of simulations are showed below:
The level of significance of the testing of rotational symmetry

<table>
<thead>
<tr>
<th></th>
<th>r=1</th>
<th>r=3</th>
<th>r=5</th>
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<tbody>
<tr>
<td>n=50</td>
<td>0.0505</td>
<td>0.099</td>
<td>0.2275</td>
</tr>
<tr>
<td>n=100</td>
<td>0.053</td>
<td>0.0685</td>
<td>0.0865</td>
</tr>
</tbody>
</table>

The power of the testing of rotational symmetry

<table>
<thead>
<tr>
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<th>r=1</th>
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<tr>
<td>n=50</td>
<td>0.949</td>
<td>0.9995</td>
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</tr>
<tr>
<td>n=100</td>
<td>0.9995</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Reference

Thank you very much!