**Directions**

1. Place your name and student ID number on this cover sheet, and put a check beside your section number.

2. You will have 2 hours to complete this examination.

3. **NO CALCULATORS.**

4. Cell phones and PDAs may not be turned on or used during the final.

5. No scrap paper, notes, or books are permitted.

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**Total** 116
Final Exam Spring 2012 (100 points)

Show your work please

Problem 1 (8 points) Compute the derivatives of
(a) \( \ln(\arcsin x) \).
(b) \( 2^{\arcsin x} \) (this is not \((2^x)^2\)).

\[
\frac{dy}{dx} = \frac{1}{\arcsin x \sqrt{1-x^2}}
\]

\[
y' = (2^{\arcsin x} \ln 2) 2x
\]

Answer:
(a) \( \frac{1}{\sqrt{1-x^2}\sin^{-1}x} \)
(b) \( 2x 2^{\arcsin x} \ln 2 \)
(c) \( x 2^{\arcsin x} + 1 \ln 2 \)

Problem 2 (8 points) In 1950 the US population was 151 (in millions of persons). In 1980 it was 226. If the population growth was perfectly exponential, what would be the population in 2020? You do not need to simplify the final answer.

\[
P(t) = P_0 e^{kt}
\]

\[
226 = 151 e^{30k}
\]

\[
\ln \frac{226}{151} = 30k
\]

\[
k = \frac{1}{30} \ln \frac{226}{151}
\]

\[
P(2020) = 151 e^{\frac{7}{3} \ln \frac{226}{151}}
\]

Answer:
\[
151 e^{\frac{7}{3} \ln \frac{226}{151}}
\]
Problem 3 (28 points) Evaluate

\[ \int \frac{\ln x}{x^5} \, dx. \]

Let \( u = \ln x \), \( du = \frac{1}{x} \, dx \)

\[ \int \frac{u}{x^5} \, dx = -\frac{\ln x}{4} + \frac{1}{4} \int x^{-5} \, dx = -\frac{\ln x}{4x^4} + \frac{1}{16x^4} + C. \]

\[ \text{Answer: } -\frac{\ln x}{4x^4} - \frac{1}{16x^4} + C. \]

\[ \int \tan 2\theta \, d\theta. \] Let \( u = 2\theta \), \( du = 2d\theta \)

\[ \frac{1}{2} \int \tan u \, du \]

\[ \text{Answer: } \frac{1}{2} \ln |\sec 2\theta| + C. \]

\[ \int \frac{\sin \theta}{\cos^2 \theta + 2 \cos \theta + 5} \, d\theta. \] Let \( u = \cos \theta \), \( du = -\sin \theta \, d\theta \)

\[ \int \frac{-u}{u^2 + 2u + 5} \, du = \int \frac{-1}{u + 1} \, du \]

\[ \text{Answer: } \frac{1}{2} \tan^{-1} \left( \frac{u + 1}{2} \right) + C. \]

\[ \int \frac{x^2 + x}{(x - 1)(x^2 + 1)} \, dx. \]

\[ \frac{x(x + 1)}{(x - 1)(x^2 + 1)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 1} \]

\[ x(x + 1) = A(x^2 + 1) + (Bx + C)(x - 1) \]

Let \( x = 1 \): \( 2 = 2A \Rightarrow A = 1 \)

Let \( x = 0 \): \( 0 = 1 + C \Rightarrow C = 1 \)

Let \( x = 2 \): \( 6 = 5 + 2B + 1 \Rightarrow B = 0 \)

\[ \text{Answer: } \ln |x - 1| + \tan^{-1} x + C. \]
Problem 4 (8 points)
Evaluate the following improper integral:
\[ \int_2^5 \frac{dx}{\sqrt{x-2}} \]
\[ \text{let } u = x-2 \quad du = dx. \]
\[ = \lim_{t \to 2^+} \int_t^5 \frac{1}{\sqrt{u}} \, du = \lim_{t \to 2^+} \int_{t-2}^3 u^{-\frac{1}{2}} \, du \]
\[ = \lim_{t \to 2^+} 2 u^{\frac{1}{2}} \bigg|_{t-2}^3 = \lim_{t \to 2^+} \left( 2\sqrt{3} - 2\sqrt{t-2} \right) \]

ANSWER:
\[ 2\sqrt{3} \]

Problem 5 (8 points) Find the limit
\[ \lim_{x \to \infty} \frac{\ln x^2}{x^{2/5}}. \]
\[ = \lim_{x \to \infty} \frac{2x}{x^{2/5}} = \lim_{x \to \infty} \frac{2}{x^{2/5}} = 0 \]
\[ = \lim_{x \to \infty} \frac{-2x^{-2}}{\frac{2}{5} (-\frac{3}{5}) x^{-3/5}} = \lim_{x \to \infty} \frac{1}{x^{-3/5}} = 0 \]

ANSWER:
\[ 0 \]
Problem 6 (8 points) Set up (but do NOT evaluate) the integral that represents the length of the curve 
\[ y = \sin^2 x, \quad 0 \leq x \leq 5. \]

\[ L = \int_0^5 \sqrt{1 + (2\sin x \cos x)^2} \, dx \]

**ANSWER:**
\[ L = \int_0^5 \sqrt{1 + 4 \sin^2 x \cos^2 x} \, dx \]

Problem 7 (8 points) Find the area of the surface that results from rotating the graph of 
\[ y = x^2, \quad 1 \leq x \leq 5, \]
about the y-axis.

\[ A = \int_1^5 2\pi x \sqrt{1 + (2x)^2} \, dx = 2\pi \int_1^5 x \sqrt{4x^2 + 1} \, dx \]

Let \( u = 2x \tan \theta \)
\[ du = 2 \sec^2 \theta d\theta \]
\[ \frac{dx}{\sec^2 \theta} = \frac{du}{2} \]

\[ = \frac{\pi}{2} \int \frac{u^2}{2} \, du = \frac{\pi}{2} \left( \frac{1}{3} u^3 \right) \bigg|_1^5 = \frac{\pi}{6} \left( 5^3 - 1^3 \right) = \frac{\pi}{6} (125 - 1) = \frac{\pi}{6} (124) \]

**ANSWER:**
\[ A = \frac{\pi}{6} (124) \]

\[ \frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} \]
\[ \tan^2 \theta + 1 = \sec^2 \theta \]
Problem 8 (8 points) Split the series \( \sum_{n=0}^{\infty} \frac{4^n + 5^n}{6^n} \) into geometric series, find their respective \( a, r \), and use them to compute the sum of the above given series.

\[
\sum_{n=0}^{\infty} \frac{4^n}{6^n} + \sum_{n=0}^{\infty} \frac{5^n}{6^n} = \sum_{n=1}^{\infty} \left( \frac{4}{6} \right)^{n-1} + \sum_{n=1}^{\infty} \left( \frac{5}{6} \right)^{n-1}
\]

\[
r = \frac{4}{6} < 1 \quad \text{and} \quad r = \frac{5}{6} < 1
\]

\[
S = \frac{\frac{1}{1 - \frac{4}{6}}}{1 - \frac{4}{6}} + \frac{\frac{1}{1 - \frac{5}{6}}}{1 - \frac{5}{6}} = \frac{6}{2} + \frac{6}{1} = 3 + 6 = 9
\]

**ANSWER:** 9

Problem 9 (8 points)
Test the following series for convergence or divergence (don't forget to say which test are you using):

\[
\sum_{n=1}^{\infty} (-1)^n n e^{-n}.
\]

\[
b_n = \frac{n}{e^n} \quad \text{pos, decr.}
\]

**Ratio Test**

\[
\lim_{n \to \infty} \frac{b_n}{b_{n+1}} = \lim_{n \to \infty} \frac{n e^n}{(n+1) e^n} = \lim_{n \to \infty} \frac{1}{e} = 0 \quad \text{convq. A.S.T.}
\]

**ANSWER:** convq by A.S.T. or Absolute convq.
Problem 10 (8 points)
Find the radius of convergence and the interval of convergence of the series below. If the interval is finite, do not forget to test the end points.
\[
\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2 - 2}
\]
Ratio Test: 
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(x-5)^{n+1}}{(n+1)^2 - 2} \cdot \frac{n^2 - 2}{(x-5)^n} \right|
\]
\[
= \lim_{n \to \infty} \left| \frac{x-5}{n+1} \right| \frac{n^2 - 2}{n^2 - 2} = \lim_{n \to \infty} \left| \frac{x-5}{n+1} \right| \frac{n^2 - 2}{n^2 - 2} = 1
\]
\[
|x-5| < 1 = R
\]
\[-1 < x < 6\]
when \(x = 4\), \(\sum (-1)^n \frac{n}{n^2 - 2}\) convergs by AST.
when \(x = 6\), \(\sum \frac{n}{n^2 - 2}\) converg to \(\frac{1}{1/n^2} = \frac{1}{1/6^2} = \frac{1}{1/36} = 36\).

\[
\text{ANSWER:}
\]
\[
\text{Convg}
\]
\[
R = 1
\]
\[
[4, 6] 2 + 2
\]

Problem 11 (8 points)
Find the Maclaurin series and its radius of convergence for \(f(x) = \cosh x\).
\[
cosh x = \frac{e^x + e^{-x}}{2}
\]
\[
\sinh x = \frac{e^x - e^{-x}}{2}
\]
\[
f(n)(0) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}
\]
\[
f(x) = \cosh x = \sum_{n=0}^{\infty} \frac{f(n)(0) x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}
\]
\[
\text{ANSWER:}
\]
\[
\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \quad ; \quad R = \infty
\]
Bonus Problem (8 points)

Find the area of the inner loop of the curve given by the polar coordinates equation $r = 1 - 2\sin \theta$.

\[
A = \int_{\pi/6}^{5\pi/6} \frac{1}{2} [1 - 2\sin \theta]^2 d\theta
\]

\[
= \int_{\pi/6}^{5\pi/6} \frac{1}{2} [1 - 4\sin \theta + 4\sin^2 \theta] d\theta
\]

\[
= \int_{\pi/6}^{5\pi/6} \left[ \frac{1}{2} - 2\sin \theta + 1 - \cos 2\theta \right] d\theta
\]

\[
= \left[ \frac{1}{2}\theta + 2\cos \theta + \theta - \frac{1}{2} \sin 2\theta \right]_{\pi/6}^{5\pi/6}
\]

\[
= \left( \frac{5\pi}{12} + 2\cos \frac{5\pi}{6} + \frac{5\pi}{6} - \frac{1}{2} \sin \frac{5\pi}{3} 
- \frac{\pi}{12} + 2\cos \frac{\pi}{6} - \frac{\pi}{6} + \frac{1}{2} \sin \frac{\pi}{6} \right)
\]

\[
= \frac{4\pi}{12} + \frac{4\pi}{6} - 2\sqrt{3} + \frac{1}{4} + \frac{1}{4}
\]

\[
= \pi - 2\sqrt{3} + \frac{1}{2}
\]

\[
0 = 1 - 2\sin \theta
\]

\[
2\sin \theta = 1
\]

\[
\sin \theta = \frac{1}{2}
\]

\[
\theta = \frac{\pi}{6}, \frac{5\pi}{6}
\]

\[
\text{ANSWER:} \quad \pi - 2\sqrt{3} + \frac{1}{2}
\]