EXPANSION PROPERTIES OF DOUBLE STANDARD MAPS

MICHAEL BENEDICKS, MICHAL MISIUREWICZ, AND ANA RODRIGUES

ABSTRACT. For the family of Double Standard Maps $f_{a,b} = 2x + a + \frac{b}{\pi} \sin 2\pi x \pmod{1}$ we investigate the structure of the space of parameters $a$ when $b = 1$ and when $b \in [0, 1)$. In the first case the maps have a critical point, but for a set of parameters $E_1$ of positive Lebesgue measure there is an invariant absolutely continuous measure for $f_{a,1}$. In the second case there is an open nonempty set $E_b$ of parameters for which the map $f_{a,b}$ is expanding. We show that as $b \uparrow 1$, the set $E_b$ accumulates on many points of $E_1$ in a regular way from the measure point of view.

1. INTRODUCTION

In one-dimensional dynamics, a lot is known about the families of smooth maps with a critical point, like quadratic maps, and about the maps that have no critical points (local diffeomorphisms of the circle). Here we start to investigate what happens at the interface of those two cases.

Consider the family of double standard maps of the circle onto itself, given by

$$f_{a,b} = 2x + a + \frac{b}{\pi} \sin 2\pi x \pmod{1},$$

where the parameters $a, b$ are real, $a \in [0, 1)$ and $b \in [0, 1]$. In fact, we consider $a$ from the circle $\mathbb{R}/\mathbb{Z}$, but since we are mostly working locally (and far from $a = 0$), considering $a$ real is simpler. These family of maps were introduced in [19].

For $b = 1$, maps of the family (1.1) have a unique cubic critical point (at $c = 1/2$) and negative Schwarzian derivative. Thus, they behave similarly to the quadratic maps. In particular, there is a set of parameters $a$ for which there is an invariant probability measure, absolutely continuous with respect to the Lebesgue measure. For $b < 1$, there is no critical point, so the the maps are local diffeomorphisms. Complexification of the maps, obtained by conjugacy via $e^{2\pi i x}$, gives the family

$$g_{a,b}(z) = e^{2\pi i a} e^{z} e^{b(z - \frac{1}{2})}.$$

Those maps are symmetric with respect to the unit circle, and factored by this symmetry, they have only one attracting point and no asymptotic values in $\mathbb{C}\setminus\{0\}$. Therefore a map $f_{a,b}$ has at most one attracting or neutral periodic orbit (see [19, 8, 10]).

One can also look at the family of double standard maps as a hybrid between the family of standard maps, studied by V. Arnold (see [1]) and important in the creation of the KAM theory, and expanding maps of the circle (see [22]). Of course instead of maps of degree 2 one can take maps of higher degrees and the results will be...
practically the same (but we would introduce one more parameter and loose a nice name of the family).

Some recent work has been done for classes of families that include double standard maps. Misiurewicz and Rodrigues studied them in [19, 20]. Benedicks and Rodrigues [4] investigated symbolic dynamics for this family. Universality for critical circle covers was studied by Levin and Świątek, [15]. Levin and van Strien [14] proved complex bounds, quasisymmetric rigidity and density of hyperbolicity for a class of real analytic maps which includes the double standard maps. Fagella and Garijo [10] studied a class of complex maps containing the maps \( g_{a,b} \). Dezotti [8] also considered maps \( g_{a,b} \), and using complex methods obtained important results on the real case.

As for the Arnold’s family, for the double standard family we call the sets for which there is an attracting periodic orbit of a given type (period plus combinatorics) tongues. Dezotti [8] proved that tongues are connected. The lowest tongue tip is at \( b = 1/2 \), for the period 1 tongue. If \( 0 < b < 1/2 \), the map \( f_{a,b} \) is expanding. At higher \( b \)-levels there may be finitely or infinitely many tongues (see [19]). In particular, at the critical level \( b = 1 \) all tongues are present, and it is easy to prove that they are dense at this level (see [14]). We show (in Theorem A) that at the lower levels \( f_{a,b} \) can have an attracting or neutral periodic orbit, and otherwise it is expanding. Moreover, the set of expanding maps is dense in the complement of the tongues.

For simplicity, we will be using notation \( f_a \) for \( f_{a,1} \). A parameter \( a_0 \) will be called an \textit{MT parameter} if the trajectory of the critical point \( c = 1/2 \) is preperiodic (but not periodic).

In this case \( f_{a_0} \) has an absolutely continuous invariant measure, [17], and it is also true that the critical value \( f_{a_0} (\frac{1}{2}) \) satisfies the Collet-Eckmann condition, i.e. that there is \( C > 0 \) and \( \kappa_1 > 0 \) such that for \( a = a_0 \)

\[
(f_a^n)'(f_a(c)) \geq Ce^{\kappa_1 n}, \quad \forall n \geq 0,
\]

which implies the existence of an absolutely continuous invariant measure, [23, 24].

Using the methods of [2] it is possible to prove

**Proposition 1.1.** There is a set of positive Lebesgue measure \( \tilde{E}_1 \) so that for all \( a \in \tilde{E}_1 \) there is \( n_0(a) \) so that

\[
(f_a^n)'(f_a(c)) \geq e^{n^{2/3}}, \quad \forall n \geq n_0(a),
\]

Here \( \frac{2}{3} \) can be replaced by any constant \( \sigma < 1 \).

A parameter exclusion requiring

\[
\text{dist}(f_a^j(c), c) \geq e^{-\sqrt{j}}, \quad j \geq 1,
\]

will be sufficient to prove (1.3) and then also Jacobson’s theorem follows.

Using the methods of Large deviations of [3] it is possible to prove

**Proposition 1.2.** There is a set of positive Lebesgue measure \( E_1 \) and some \( \kappa > 0 \) so that for all \( a \in E_1 \)

\[
(f_a^n)'(f_a(c)) \geq Ce^{\kappa n}, \quad \forall n \geq 0,
\]
For a similar result see [25].

In the present paper we will consider the non-critical case $0 < b < 1$ and use more elementary methods based on [2], which give stretched exponential growth of the type

\[(1.6) \quad (f_{a,b}^n)'(f_{a,b}(c)) \geq e^{n^{2/3}}, \quad n_0 \leq n \leq \hat{N}(a,b),\]

for all $a \in \tilde{E}_b$ for a set $\tilde{E}_b$, which is a finite union of intervals. To obtain this it is sufficient to do parameter exclusions of the type

\[(1.7) \quad \text{dist}(f_{a,b}^j(c), c) \geq C_1 e^{-\sqrt{j}}, \quad j \geq 1,\]

and then prove exponential expansion in Section 8. The discussion of the proof is more elaborated at the end of this section.

We will outline the proof of Proposition 1.1 after the proof of Theorem A.

By the results of [5] and [6], if $f_a(c)$ satisfies the Collet-Eckmann condition, then $f_a$ has an absolutely continuous invariant measure. This is the analogue of Jakobson’s theorem [11] in this case.

It is also possible to prove (1.2) for $a$ in a set $E_1$ of positive Lebesgue measure, but with the present setup this would require the method of Large deviations of [3], and this is not required when $0 < b < 1$.

Let us introduce some notations. For a fixed $b$, let us denote the sets of those parameters $a$ for which $f_{a,b}$ has an attracting (resp. neutral) orbit $T_b$ (resp. $TN_b$). Moreover, let $E_b$ be the set of those parameters $a$ for which $f_{a,b}$ is expanding, that is, there exist $C > 0$ and $\kappa > 0$ such that

\[(1.8) \quad (f_{a,b}^n)'(x) \geq Ce^{\kappa n}, \quad \forall n \geq 0 \quad \forall x \in T.\]

By the result of Mañé [16], if $a$ does not belong to $T_b$ or $TN_b$, then it belongs to $E_b$. Observe that by the definition, a small perturbation of an expanding map is also expanding, so $E_b$ is open. In fact, the set $E = \{(a,b) : a \in E_b, 0 \leq b < 1\}$ is open in $[0,1) \times [0,1)$.

Note that our definition of $E_1$ or $\tilde{E}_1$ is quite different from the noncritical case, i.e. the case of of $E_b$ for $b < 1$. Nevertheless, there are some common features of the noncritical case, because if $f_{a,b}$ is expanding, then by the results of Krzyżewski and Szlenk [12], or by the Lasota-Yorke Theorem [13], there exists an absolutely continuous invariant measure.

Extending the methods of the proof of (1.3), we prove the following theorem.

**Theorem A.** Let $a_0$ be a MT parameter for the family $\{f_a\}$. Denote $\omega(\varepsilon) = (a_0 - \varepsilon, a_0 + \varepsilon)$. Then for some $\varepsilon_0 > 0$ there is a function $b_0 : (0, \varepsilon_0) \to (0,1)$ such that

\[\lim_{\varepsilon \to 0} \inf \left\{ \frac{|E_b \cap \omega(\varepsilon)|}{|\omega(\varepsilon)|} : b \in (b_0(\varepsilon), 1] \right\} = 1.\]

Here $|A|$ denotes the Lebesgue measure of the set $A$.

This can be considered as the main result of the paper. It gives a quantitative relation between the behavior of the system for $b < 1$, where the maps are local diffeomorphisms, and for $b = 1$, the critical case.

Finally, we prove a topological result, using very different methods.
**Theorem B.** For each $b < 1$, the set $E_b$ is dense in the complement of $T_b$. In particular, every interval of the parameters $a$ either is contained in a closure of one tongue or intersects $E_b$.

The above theorem in some sense complements Theorem A. Locally it says less about the set $E_b$, but it applies to all $b < 1$, not only to $b$ sufficiently close to 1 (moreover, this closeness in Theorem A depends on $a_0$).

The paper is organized as follows. In Section 2 we introduce notation and some definitions that we will use throughout the paper. In Section 3 we prove the transversality condition for maps of the family (1.1). In Section 4 we prove that we have exponential growth of the derivative for an orbit of a map $f_{a,b}$ that moves outside an open interval containing the the critical point when $(a, b)$ is a small perturbation of an MT parameter $(a_0, 1)$. In Section 5 we describe the induction including its startup and prove that the conditions on the Induction Statement are satisfied for the first free return time. Furthermore, we define the bound period and prove some results concerning the derivative growth during the bound period. In Section 6 we prove the Global Distortion Lemma and in Section 7 we start the proof of Theorem A. In Section 8 we finish this proof. Finally, in Section 9 we prove Theorem B.

Let us indicate what is technically new in this paper compared to previous work. The proof of Theorem A is based on techniques in [2] and [3].

The main strategy the proof of Theorem A is the inductive proof of

\[(f_{a,b}^n)'(f_{a,b}(c)) \geq C_2 e^{n^{2/3}},\]

up to a certain time $\hat{N}$.

The set $\tilde{E}_1$ in the critical case $b = 1$ is a Cantor set of positive measure represented as

\[E_1 = \bigcap_{n=0}^{\infty} A_n,\]

where each $A_n$ is a disjoint union of intervals $\{I_n^j\}$. Here $A_{n+1} \subset A_n$ and the set $A_{n+1}$ are defined by removing subintervals of each $I_n^j$ according to two rules: First those subintervals that do not satisfy an approach rate condition (1.4) for the critical point (or inflexion point) $c = \frac{1}{2}$ are deleted. This replaces the basic assumption (BA) in [2] and [3]. In the non critical case $0 < b < 1$ this condition corresponds to (1.7).

The proof of Theorem A is different from that of Proposition 1.1 in the critical case due to the fact that the behaviour at the inflexion point $c = \frac{1}{2}$ is given by the Taylor series

\[f_{a,b}(x) = (2 - 2b)(x - \frac{1}{2}) + \sum_{k=1}^{\infty} a_{2k+1}(x - \frac{1}{2})^{2k+1}\]

\[= a + (2 - 2b)(x - \frac{1}{2}) + g(x).\]

Here $g(x)/(x - \frac{1}{2})^3$ is bounded above and below by strictly numerical positive constants, i.e they depending only on the MT map $f_{a_0}$. Similarly $g'(x)/(x - \frac{1}{2})^2$ has similar bounds from above and below.
This means that when a point \( y = f_{a,b}^n(x) \) is close to \( c = \frac{1}{2} \), the derivative \( f_{a,b}'(y) \) will either be dominated by \( 2 - 2b \) or by the quadratic term \( g'(y) \) and these two cases will be treated differently.

The induction in the non-critical case \( b < 1 \) can actually be terminated at the time \( \tilde{N} \) defined by the condition that constant term in the derivative \( 2 - 2b \) can be of size comparable to \( g'(x) \), which is quadratic, i.e.

\[
2 - 2b \sim (e^{-\sqrt{\tilde{N}}})^2,
\]

where \( \sim \) means that the two sides are comparable within fixed constants, which are only depending on \( f_{a_0} \). Defining \( \tilde{N} \) this way we can stop the induction at time \( \tilde{N} \) and the remaining set \( E_b = \bigcap_{n=0}^{\tilde{N}} A_n(b) \) is a finite union of intervals.

Another new aspect of the present paper is that for \( b < 1 \) that the condition of Collet-Eckmann type (again with \( c = \frac{1}{2} \))

\[
(f_{a,b}^n)'(f_{a,b}(c)) \geq C e^{\kappa n} \quad \forall n \geq 0,
\]

is no longer sufficient for the existence of absolutely continuous invariant measures for \( f_{a,b} \). There is however another argument given in Section 8 which uses (1.11) together with bound period estimates, see Section 4, which prove uniform hyperbolicity, (1.8).

2. Notation

Throughout this paper, \( C \) is a general numeric constant. For a set \( A \subset \mathbb{R} \) we will denote by \( |A| \) its Lebesgue measure.

Consider the family of double standard maps given by (1.1) with \( b = 1 \). Throughout this paper we denote \( f_a(x) = f_{a,1}(x) \) and \( \xi_j(a) \) the orbit of the critical point: \( \xi_j(a) = f_{a,j}^1(c) \).

For a general \( b \leq 1 \), we also use the notation \( \xi_j(a,b) = f_{a,b}^j(c) \). It is clear that when \( b < 1 \), the point \( c = \frac{1}{2} \) is an inflexion point. Sometimes we also use the notations \( f(x,a,b) \) for \( f_{a,b}(x) \) and \( f(x,a) \) for \( f_a(x) \).

By \( \partial_a f_{a,b}^j(x) \) we denote the partial derivative of \( f_{a,b}^j(x) \) with respect to \( a \) and by \( \partial_b f_{a,b}^j(x) \) the partial derivative of \( f_{a,b}^j(x) \) with respect to \( a \).

Definition 2.1. A parameter \( a = a_0 \) will be called an MT parameter, if there is an integer \( m \) and a period length \( \ell \) such that \( \xi_m(a_0) \) is a periodic point of \( f_{a_0} \) of period \( \ell \) and the multiplier \( \Lambda := (f_{a_0})'(\xi_m(a_0)) \) is larger than 1.

As in [2] and [3], we define a partition \( Q = \{I_{r,\ell}\} \) of the return interval \( I^* = (c - \delta, c + \delta) \), where \( \delta = e^{-r\delta} \). We first divide \( I^* \)

\[
I^* = \bigcup_{r=r_\delta}^{\infty} I_r \cup \bigcup_{r=r_\delta}^{\infty} I_{-r},
\]

where \( I_r \) is the interval \( (c + e^{-r-1}, c + e^{-r}) \) for \( 0 < r \leq r_\delta \) and \( I_{-r} \) is the interval \( (c - e^{-r}, c - e^{-r-1}) \).

We then subdivide \( I_r \) into \( r^2 \) intervals of equal length with disjoint interiors as follows

\[
I_r = \bigcup_{\ell=0}^{r^2-1} I_{r,\ell}.
\]
For convenience we also use the convention that $I_{r,r^2} = I_{r-1,0}$, $r > 0$, and the analogous convention for $r < 0$.

Note that we have $|I_{r,l}| = \frac{e^{-r}}{r}(1 - e^{-1})$ and $|I_r| = e^{-r}(1 - e^{-1})$. We will also need the extended interval

$$I_{r,l}^+ = I_{r,l-1} \cup I_{r,l} \cup I_{r,l+1}.$$  

For technical reasons we will also need a partition $Q' = \{I_{r,l}\}$, $|r| \geq r_\delta^1$, of an interval $I^{**} = (c - \delta_1, c + \delta_1)$, where $|r| \geq r_\delta^3$, for some $r_\delta^3 < r_\delta$, i.e. $\delta_1 = e^{-r_\delta^3} > \delta$.

A main tool in this paper is a sequence of partitions $P_n$, $n = 0, 1, 2, \ldots$ of the parameter space which is induced by the phase space partition. We define

$$E_n = \bigcup_{\omega \in P_n} \omega.$$  

We call a time $n$ a free return if there is a parameter interval $\omega$ belonging to a partition $P_n$ such that

$$\xi_n(\omega) = I_{r,l}.$$  

Similarly if we fix $b < 1$, we will have

$$\xi_n(\omega, b) = I_{r,l}.$$  

(In some cases these two conditions for technical reasons will have to be replaced by $I_{r,l} \subset \xi_n(\omega) \subset I_{r,l}^+$ or $I_{r,l} \subset \xi_n(\omega, b) \subset I_{r,l}^*$.)

3. Transversality

In this section we prove the transversality condition for maps belonging to the family (1.1).

Lemma 3.1. The following formula holds:

$$\partial_a f_{a,b}^n(x) = \sum_{k=0}^{n-1} (f_{a,b}^k)'(f_{a,b}^{n-k}(x)) = (f_{a,b}^{n-1})'(f_{a,b}(x)) \sum_{k=0}^{n-1} \frac{1}{(f_{a,b}^k)'(f_{a,b}(x))}.$$  

Proof. We have

$$\partial_a f_{a,b}^{n+1}(x) = 1 + f_{a,b}'(f_{a,b}(x)) \cdot \partial_a f_{a,b}^n(x)$$  

(note that $\partial_a f_{a,b}^0(x) = 0$ and $\partial_a f_{a,b}^1(x) = 1$). Using this formula, we prove by induction

$$\partial_a f_{a,b}^n(x) = \sum_{k=0}^{n-1} (f_{a,b}^k)'(f_{a,b}^{n-k}(x)).$$  

If $n = 0$, then both sides of (3.3) are 0. Assume now that (3.3) holds for some $n$ and prove it for $n + 1$ instead:

$$\partial_a f_{a,b}^{n+1}(x) = 1 + f_{a,b}'(f_{a,b}(x)) \cdot \sum_{k=0}^{n-1} (f_{a,b}^k)'(f_{a,b}^{n-k}(x)) = 1 + \sum_{k=0}^{n-1} (f_{a,b}^{k+1})'(f_{a,b}^{n-k}(x))$$

$$= 1 + \sum_{k=1}^{n} (f_{a,b}^k)'(f_{a,b}^{n-(k-1)}(x)) = \sum_{k=0}^{n} (f_{a,b}^k)'(f_{a,b}^{(n+1)-k}(x)).$$  

Thus, by induction, (3.3) holds for every $n$. 

Now, we have
\[
(f_{n-k-1}^n(a,b))' \cdot (f_{k}^n(a,b))' = (f_{n-1}^n(a,b)).
\]
From this and (3.3) we get
\[
\frac{\partial_a f_n(a,b)(x)}{(f_{n-1}^n(a,b))'} = \sum_{k=0}^{n-1} \frac{1}{(f_{n-k-1}^n(a,b))'}.
\]

We get the following immediate corollary.

**Corollary 3.2.** We have \(\partial_a f_{a,b}(x) \equiv 1\) and if \(n > 0\) then \(\partial_a f_{a,b}(x) \geq 1\). Moreover,
\[
\frac{\partial_a \xi_n(a,b)}{(f_{n-1}^n(a,b))'} = \sum_{k=0}^{n-1} \frac{1}{(f_{n-k-1}^n(a,b))'}.
\]
so, in particular,
\[
\partial_a \xi_n(a,b) \geq (f_{n-1}^n(a,b))'.
\]
In a special case, when there is a constant \(C_2\) so that
\[
(f_{n-1}^n(a,b))' \geq C_2 e^{2/3}, \quad \nu = 0, 1, 2, \ldots, n - 1
\]
then for all \(\nu \leq n\) we obtain by combining the inequality with the lower bound (3.5)
\[
1 \leq \frac{\partial_a \xi_n(a,b)}{(f_{n-1}^n(a,b))'} \leq q_*
\]
where
\[
q_* = \sum_{i=0}^{\infty} C_2^{-1} e^{-i/3}.
\]

**Remark 3.3.** We would like to emphasize the central rôle that Corollary 3.2 plays in this paper. We prove the estimate (3.6) successively by induction on \(\nu\). We can then conclude that (3.7) holds with \(n\) replaced by \(\nu\) holds for a given \(\nu\). From this estimate we conclude that the \(x\)- and \(a\)-derivative are comparable at a given time \(\nu\). It is important that we prove the \(x\)-expansion first and then verify the comparison. The constant \(q_*\) will be fixed, i.e. it only depends on \(f_{a_0}\).

We will also need the following lemma which can be viewed as a lower bound for the Radon-Nikodym derivative of \(\xi_n(a,b) \mapsto \xi_{\mu}(a,b)\), \(\nu < \mu\), (with respect to \(a \in \omega\)).

**Lemma 3.4.** Suppose that \(\omega\) is a parameter interval and \(\nu < \mu\). Assume further that there is a constant \(q'\) such that for all \(t \in \omega\)
\[
(f_{t,b}^{\nu-1})'(f_{t,b}(c)) \geq \frac{1}{q'} \partial_a \xi_{\nu}(t,b).
\]
Then
\[
|\xi_{\mu}(\omega, b)| \geq \frac{1}{q'} \inf_{a \in \omega} (f_{a,b}^{\mu-\nu})'(f_{a,b}(c)) \cdot |\xi_{\nu}(\omega, b)|.
\]
Proof. By Corollary 3.2,
\begin{equation}
|\xi_\mu(\omega, b)| = \int_\omega \partial_a \xi_\mu(t, b) \, dt \geq \int_\omega (f_{t,b}^{\mu-1})'(f_{t,b}(c)) \, dt.
\end{equation}

However, by (3.8) we have
\[(f_{t,b}^{\mu-1})'(f_{t,b}(c)) = (f_{t,b}^{\mu-\nu})'(f_{t,b}(c)) \cdot (f_{t,b}^{\nu-1})'(f_{t,b}(c)) \geq \inf_{a \in \omega} (f_{a,b}^{\mu-\nu})'(f_{a,b}(c)) \cdot \frac{1}{q} \partial_a \xi_\nu(t, b).
\]

Together with (3.9), we get
\[|\xi_\mu(\omega, b)| \geq \frac{1}{q} \inf_{a \in \omega} (f_{a,b}^{\mu-\nu})'(f_{a,b}(c)) \cdot \int_\omega \partial_a \xi_\nu(t, b) \, dt = \frac{1}{q} \inf_{a \in \omega} (f_{a,b}^{\mu-\nu})'(f_{a,b}(c)) \cdot |\xi_\nu(\omega, b)|.
\]

\[\square\]

4. THE OUTSIDE EXPANSION

The aim of this section is to prove that we have exponential growth of the derivative for an orbit of a map \(f_{a,b}\) that moves outside an open interval \(I\) containing \(c\), when \((a, b)\) is a small perturbation of an MT parameter \((a_0, 1)\). We consider the parameter space \(\mathbb{R}/\mathbb{Z} \times (0, 1]\), and when we speak of a neighborhood of \((a_0, 1)\), we mean its neighborhood in this space.

By \(|x - y|\) we denote the distance between \(x\) and \(y\) on the circle. Since the points \(x\) and \(y\) will be usually close to each other, this makes perfect sense. Denote
\begin{equation}
\overline{d} = \min_{j \geq 1} |c - f_{a_0}^j(c)|.
\end{equation}

By the definition of an MT parameter, we have \(\overline{d} > 0\).

Since \(f_{a_0}\) has negative Schwarzian derivative, the following lemma follows immediately from Theorem 1.3 of [17] (in a general case one could use also the result of Mañé (see [16])).

Lemma 4.1. Let \(I\) be an open interval containing \(c\). Then there exists a neighborhood \(\mathcal{N}\) of \((a_0, 1)\), positive constants \(C_3, \kappa_2\), and an integer \(M_1\) such that if \((a, b) \in \mathcal{N}\) then

(i) if \(x, f_{a,b}(x), \ldots, f_{a,b}^{n-1}(x) \notin I\), then
\[ (f_{a,b}^{n})'(x) > C_3 e^{\kappa_2 n}; \]

(ii) if \(x, f_{a,b}(x), \ldots, f_{a,b}^{n-1}(x) \notin I\) and \(n \geq M_1\), then
\[ (f_{a,b}^{n})'(x) > e^{\kappa_2 n}. \]

Proof. By Theorem 1.3 of [17] (or a result of Mañé [16]), there exists \(L > 0\) and \(\kappa'_2 > 0\) such that if \(x, f_{a_0}(x), \ldots, f_{a_0}^{L-1}(x) \notin I\), then \((f_{a_0}^L)'(x) > e^{\kappa'_2 L}\). Therefore, if \(\mathcal{N}\) is a sufficiently small neighborhood of \((a_0, 1)\), then for all \((a, b) \in \mathcal{N}\) and \(x\) such that \(x, f_{a,b}(x), \ldots, f_{a,b}^{L-1}(x) \notin I\), we have \((f_{a,b}^L)'(x) > e^{\kappa'_2 L}\). Since the infimum of \((f_{a,b}^i)'(x)\) over \((a, b) \in \mathcal{N}\), \(x \notin I\), and \(i = 0, 1, \ldots, L - 1\) is positive, there exists a positive constant \(C_3\) such that (i) holds with \(\kappa_2 = \kappa'_2\). Thus it also holds with \(\kappa_2 = \kappa'_2/2\), and then (ii) holds with any
\[ M_1 > \frac{-\log C_3}{\kappa_2}. \]

\[\square\]
Now we fix a positive constant \( \beta > 0 \). It will depend only on the unperturbed map \( f_{a_0} \) and can be chosen as, say \( \frac{1}{100} \min(\tilde{\kappa}, \kappa_3) \). Here \( \tilde{\kappa} = (1/\ell) \log \Lambda \), where \( \Lambda \) is the multiplier of the repelling fixed point of the MT-point, and \( \kappa_3 \) is the Lyapunov exponent in Lemma 4.6.

**Definition 4.2.** Let \( x \in I^{**} = (c - \delta_1, c + \delta_1) \). We say that \( x \) is \( \beta \)-bound to the critical point \( c \) up to time \( p \) for \( f_{a,b} \), if \( p \) is the maximal integer such that

\[
|f_{a,b}^j(x) - f_{a,b}^j(c)| \leq e^{-\beta j}, \quad \forall j \leq p.
\]

Observe that for every \( a, b \) (where \( b \leq 1 \)) and every \( x \) we have

\[
f_{a,b}'(x) \leq 4 \quad \text{and} \quad |f_{a,b}''(x)| \leq 4\pi < 13.
\]

Let us state a version of the Bound Distortion Lemma (see [2] and [3]).

**Lemma 4.3.** If \( \delta_1 \) is sufficiently small, then there is a constant \( C_4(\delta_1) > 1 \), which converges to 1 as \( \delta_1 \to 0 \), such that for every \( x \in I^{**} = (c - \delta_1, c + \delta_1) \) if \( x \) is \( \beta \)-bound to \( c \) up to time \( p \) for \( f_{a_0} \), then

\[
\frac{1}{C_4} < \frac{(f_{a_0}^k)'(f_{a_0}(x))}{(f_{a_0}^k)'(f_{a_0}(c))} < C_4
\]

for all \( k \leq p \). Moreover, there is a constant \( C_5 = C_5(\delta_1) > 0 \), which converges to 0 as \( \delta_1 \to 0 \), such that

\[
|f_{a_0}^k(x) - f_{a_0}^k(c)| < C_5
\]

for all \( k \leq p \).

**Proof.** Assume that \( x \) is bound to \( c \) up to time \( p \). Now choose \( p_1 = \frac{1}{10} \log(1/\delta_1) \).

Then by (4.3) we can estimate

\[
|f_{a_0}^j(x) - f_{a_0}^j(c)| \leq \delta_1 4^j \leq \delta_1 4^{p_1}
\]

if \( j \leq p_1 \) and

\[
|f_{a_0}^j(x) - f_{a_0}^j(c)| \leq e^{-\beta j} \leq e^{-\beta p_1}
\]

if \( p_1 < j \leq p \). Thus, if \( \delta_1 \) is sufficiently small then \( |f_{a_0}^j(x) - f_{a_0}^j(c)| \leq \frac{d}{2} \) and therefore

\[
|f_{a_0}^j(x) - c| \geq \frac{d}{2}
\]

for all \( j \leq p \).

This also proves the last statement of the lemma.

We have

\[
\frac{(f_{a_0}^k)'(f_{a_0}(x))}{(f_{a_0}^k)'(f_{a_0}(c))} = \prod_{j=1}^{k} \left( 1 + \frac{f_{a_0}'(f_{a_0}^j(x)) - f_{a_0}'(f_{a_0}^j(c))}{f_{a_0}'(f_{a_0}^j(c))} \right)
\]

\[
\leq \exp \left( \sum_{j=1}^{k} \frac{13|f_{a_0}^j(x) - f_{a_0}^j(c)|}{f_{a_0}'(f_{a_0}^j(c))} \right)
\]

\[
\leq \exp \left\{ K_1 \left( \delta_1 p_1 4^{p_1} + \sum_{j=p_1+1}^{k} e^{-\beta k} \right) \right\}
\]

The last sum in the exponential may be empty.
Similarly, using (4.6), we get
\begin{equation}
\frac{(f_{a_0}^k)'(f_{a_0}(c))}{(f_{a_0}^k)'(f_{a_0}(x))} \leq \exp \left\{ K_2 \left( \delta_1 p_1 4^{p_1} + \sum_{j=p_1+1}^{k} e^{-\beta j} \right) \right\}
\end{equation}

The sums in (4.7) and (4.8) are bounded by a constant independent of \( \delta_1 \), so we get (4.4). Moreover, by (4.7) and (4.8), \( C_4 > 1 \) converges to 1 as \( \delta_1 \to 0 \). \( \Box \)

Set \( \bar{\kappa} = (1/\ell) \log \Lambda \). Then there is a constant \( C_6 = C_6(a_0) \) so that
\begin{equation}
(f_{a_0}^j)'(f_{a_0}(c)) \geq C_6 \bar{\kappa}^j
\end{equation}
for all \( j \geq 1 \).

At \( c \), the first two derivatives of \( f_a \) vanish, but the third one does not. Therefore, there are positive constants \( C_7, C_8 \) such that for all \( a \) sufficiently close to \( a_0 \)
\begin{equation}
C_7 |x - c|^3 < |f_a(x) - f_a(c)| < C_8 |x - c|^3
\end{equation}
whenever \( x \in I^{**} \). If \( \delta_1 \) is small, the constants \( C_7 \) and \( C_8 \) can be made close to each other. Similarly, for some positive constants \( C_9', C_{10}' \),
\begin{equation}
C_9'(x - c)^2 < f_a'(x) < C_{10}'(x - c)^2
\end{equation}
whenever \( x \in I^{**} \). If instead of \( f_a \) we consider \( f_{a,b} \) with \( b \) sufficiently close to 1, we similarly obtain
\begin{equation}
|x - c| \left( 2 - b + C_7(x - c)^2 \right) < |f_{a,b}(x) - f_{a,b}(c)| < |x - c| \left( 2 - b + C_8(x - c)^2 \right)
\end{equation}
and
\begin{equation}
2 - 2b + C_9(x - c)^2 < f_{a,b}'(x) < 2 - 2b + C_{10}(x - c)^2,
\end{equation}
and we chose \( C_9 \) and \( C_{10} \) so that these estimates are valid for all \( b \leq 1 \). Moreover, we have
\begin{equation}
|f_{a,b}''(x)| \leq 8\pi^2 |x - c| < 80|x - c|.
\end{equation}

In the following lemma we estimate the length of the bound period.

**Lemma 4.4.** If \( \delta_1 \) is sufficiently small, \( x \) is \( \beta \)-bound to \( c \) up to time \( p \) for \( f_{a_0} \), and \( x \in I^{**} \setminus \{ c \} \), then
\begin{equation}
p < -\frac{4}{\bar{\kappa}} \log |x - c|.
\end{equation}
In the particular case when \( x \in I_{+r} \) we obtain
\begin{equation}
p \leq \frac{4r}{\bar{\kappa}}.
\end{equation}

**Proof.** By Lemma 4.3, we have
\[
|f_{a_0}^p(x) - f_{a_0}^p(c)| > \frac{(f_{a_0}^p)'(f_{a_0}(c))}{C_4}|f_{a_0}(x) - f_{a_0}(c)|.
\]
Taking into account (4.9) and (4.10), we get
\[
1 > |f_{a_0}^p(x) - f_{a_0}^p(c)| > \frac{C_6 C_7}{C_4} e^{\bar{\kappa} p} |x - c|^3.
\]
If $\delta_1$ is small, then $C_4 < 2$, so taking logarithms gives us
\[ \log \frac{C_6C_7}{2} + \delta p + 3 \log |x - c| < 0. \]
If $\delta_1$ is small, then $- \log |x - c|$ is large, so we get $- \log(C_6C_7/2) < - \log |x - c|$, and therefore $\delta p < -4 \log |x - c|$. \hfill \square

We need a derivative estimate for an orbit of $f_a$ that moves completely outside $I^* = (c - \delta, c + \delta)$ or $I^{**} = (c - \delta_1, c + \delta_1)$ but returns to one of these intervals at time $n$.

In the proof of the next lemma we will use the fact that $f_a$ has negative Schwarzian derivative. This result can be generalized to the $C^2$ case (see van Strien [24]).

**Lemma 4.5.** Let $\overline{d}$ be as in (4.1). For every $\delta_1 \in (0, \overline{d}/2)$ and for every $n \geq 1$, if $x$ is such that $f_a^n(x) \notin I^{**}$ for $j = 0, \ldots, n - 1$, and $f_a^n(x) \in I^{**}$, then $(f_a^n)'(x) > \overline{d}/2$.

**Proof.** On each side of $x$ there are the two closest preimages of $c$ of order less than $n$: $\eta_1 < x$ and $\eta_2 > x$. Then $f_a^n$ has positive derivative on $(\eta_1, \eta_2)$ and has negative Schwarzian derivative on that interval. Therefore on one of the intervals $[\eta_1, x]$ and $[x, \eta_2]$ the maximum of the derivative $(f_a^n)'$ is attained at $x$. We may assume that this is the interval $[\eta_1, x]$. Then $f_a^n(\eta_1) = f_a^k(c)$ for some $k > 0$, so
\[ |f_a^n(\eta_1) - f_a^n(x)| \geq \overline{d} - \delta_1 > \overline{d}/2. \]

By the Mean Value Theorem,
\[ |f_a^n(\eta_1) - f_a^n(x)| = (f_a^n)'(t)|\eta_1 - x| \leq (f_a^n)'(t) \]
for some $t \in (\eta_1, x)$, and thus,
\[ (f_a^n)'(x) \geq (f_a^n)'(t) > \overline{d}/2. \] \hfill \square

In the following lemma we consider what we call a free period.

**Lemma 4.6.** There is a neighborhood $\mathcal{N}$ of $(a_0, 1)$ in the parameter space and positive constants $C^*$ and $\kappa_3$, such that for sufficiently small $\delta_1$, if $(a, b) \in \mathcal{N}$ then if $x, f_a(x), \ldots, f_a^{q-1}(x) \notin I^{**}$ and $f_a^q(x) \in I^{**}$ then
\[ (f_a^q)'(x) \geq C^* e^{\kappa_3 q}. \]

**Proof.** By Lemma 4.1, if $q \geq M_1$, then $(f_a^q)'(x) \geq e^{\kappa_2 q}$. If $q < M_1$, then by Lemma 4.5, $(f_a^q)'(x) > \overline{d}/2$ (here we can extend the estimate to a neighborhood of $(a_0, 1)$ because we consider only finitely many iterates of the map). Thus, if we set
\[ C^* = \frac{\overline{d}}{2} e^{-\kappa_2 M_1}, \]
then since $q < M_1$,
\[ (f_a^q)'(x) > \frac{\overline{d}}{2} = \frac{\overline{d}}{2} e^{-\kappa_2 M_1} e^{\kappa_2 M_1} e^{-\kappa_3 q} e^{\kappa_2 q} = C^* e^{\kappa_3 (M_1 - q)} e^{\kappa_2 q} \geq C^* e^{\kappa_2 q} \]
and we have proven (4.17) with $\kappa_3 = \kappa_2$. \hfill \square
We will also need an estimate of the derivative during the bound period.

Lemma 4.7. Assume that $9\beta \leq \tilde{\kappa}$. Let $C^*$ be the constant from Lemma 4.6. Then there is an arbitrarily small $\delta_1$ such that if $x$ is $\beta$-bound to $c$ up to time $p$ for $f_{a_0}$ and $x \in I^{**} = (c - \delta_1, c + \delta_1)$ then

\begin{equation}
(f_{a_0}^p)'(x) > \frac{1}{C^*} e^{\frac{p}{2}}.
\end{equation}

Proof. By the Mean Value Theorem, there is an $y$ between $f_{a_0}(x)$ and $f_{a_0}(c)$, such that

\[ |f_{a_0}^p(x) - f_{a_0}^p(c)| = (f_{a_0}^{p-1})'(y)|f_{a_0}(x) - f_{a_0}(c)|. \]

By this, Lemma 4.3 and (4.10), there exists a constant $K_3$ such that if $\delta_1$ is sufficiently small then

\[ |f_{a_0}^p(x) - f_{a_0}^p(c)| < K_3 e^{\beta p} |x - c|^3. \]

Similarly, since $(f_{a_0}^p)'(x) = f_{a_0}^p(x) \cdot (f_{a_0}^{p-1})'(f_{a_0}(x))$, we get by Lemma 4.3 and (4.11) that there exists a constant $K_4$ such that if $\delta_1$ is sufficiently small then

\[ (f_{a_0}^p)'(x) > K_4 e^{\beta p} |x - c|^2. \]

By the definition of $p$ we have

\[ |f_{a_0}^{p+1}(x) - f_{a_0}^{p+1}(c)| > e^{-\beta (p+1)}, \]

and therefore for some constant $K_5$

\[ |f_{a_0}^p(x) - f_{a_0}^p(c)| > K_5 e^{-\beta p}. \]

Thus,

\[ K_3 e^{\beta p} |x - c|^3 > K_5 e^{-\beta p}, \]

so

\[ |x - c|^2 > K_5^{2/3} K_3^{-2/3} e^{(2/3)(-\beta - \tilde{\kappa})}. \]

Together with an earlier estimate, this gives us

\[ (f_{a_0}^p)'(x) > K_4 e^{\beta p} K_5^{2/3} K_3^{-2/3} e^{(2/3)(-\beta - \tilde{\kappa})} = K_4 K_5^{2/3} K_3^{-2/3} e^{(1/3) (\tilde{\kappa} - 2\beta)}. \]

Since $9\beta \leq \tilde{\kappa}$, we have

\[ \frac{1}{3} (\tilde{\kappa} - 2\beta) > \frac{7}{27} \tilde{\kappa}, \]

and therefore (4.18) holds if

\begin{equation}
C^* > K_4^{-1} K_5^{-2/3} K_3^{-2/3} e^{-\frac{7}{108} \tilde{\kappa} p(\delta_1)},
\end{equation}

where $p(\delta_1)$ is the bound period associated with $\delta_1$. Since $p(\delta_1) \to \infty$ as $\delta_1 \to 0$, the above inequality holds if $\delta_1$ is sufficiently small.

Remark 4.8. If in (4.18) we replace (on both sides of the inequality) $p$ by $p - 1$ or $p + 1$, then at the right-hand side of in (4.19) there will be one more multiplicative constant. Since it was irrelevant in the proof what constant is there, Lemma 4.7 still holds with a suitably modified inequality (4.18).

The following lemma is very similar to Lemma 4.1, but the exponent in the estimate does not depend on the size of the neighborhood of $c$ that we consider. In this lemma we assume that $\delta_1$ is sufficiently small (so that the lemmas that we use in the proof hold) but fixed.
Lemma 4.9. Let $I$ be an open symmetric interval around $c$, whose closure is contained in $I^{**}$. Fix a sufficiently small neighborhood $\mathcal{N}$ of $(a_0, 1)$ (depending on $I$). Then there are constants $C_{11} > 0$ and $\kappa_4 > 0$ (independent of $I$) and an integer $M$ (depending on $I$) such that for $(a, b) \in \mathcal{N}$

(i) if $x, f_{a,b}(x), \ldots, f_{a,b}^{n-1}(x) \notin I$ and $f_{a,b}^n(x) \in I$, then

$$(f_{a,b}^n)'(x) \geq C_{11} e^{\kappa_4 n};$$

(ii) if $x, f_{a,b}(x), \ldots, f_{a,b}^{n-1}(x) \notin I$ and $n \geq M$, then

$$(f_{a,b}^n)'(x) \geq e^{\kappa_4 n}.$$

Proof. Let $0 = t_0 < t_1 < t_2 < \cdots < t_S < t_{S+1} = n$, where $t_i$ for $i \in \{1, 2, \ldots, S\}$ are the times when $f_{a,b}^{t_i}(x) \in I^{**} \setminus I$. We want to estimate

$$(f_{a,b}^n)'(x) = \prod_{j=0}^{S} (f_{a,b}^{t_{j+1}-t_{j}})'(f_{a,b}^{t_j}(x)).$$

The times from $[t_0, t_1)$ form a free period; let $t_1 - t_0 = q_0$. Hence, by Lemma 4.6,

$$(f_{a,b}^{q_0})'(x) \geq C^* e^{\kappa_3 q_0}.$$

Consider now times from $[t_j, t_{j+1})$, where $j > 0$. We can write it as a union of a bound period $[t_j, t_j + p_j)$ and a free period $[t_j + p_j, t_{j+1})$, and we write its length as $t_{j+1} - t_j = p_j + q_j$. For the bound periods $[t_j, t_j + p_j)$ we can use the estimate from Lemma 4.7 if $\mathcal{N}$ is sufficiently small, because by Lemma 4.4 we work only with the finite number of iterates (for $I$ fixed; this is why $\mathcal{N}$ depends on $I$). Although $p$ may depend on the map that we are using, if $\mathcal{N}$ is sufficiently small, it only may change to $p \pm 1$, and then by Remark 4.8 we can still use Lemma 4.7.

Thus, for the bound periods $[t_j, t_j + p_j)$ we get

$$(f_{a,b}^{p_j})'(f_{a,b}^{t_j}(x)) \geq \frac{1}{C^*} e^{\tilde{\kappa} p_j}$$

and for the free period, as before, the estimate from Lemma 4.6 gives us

$$(f_{a,b}^{q_j})'(f_{a,b}^{t_j+p_j}(x)) \geq C^* e^{\kappa_3 q_j}.$$

Combining these estimates we get

$$(f_{a,b}^n)'(x) \geq C^* e^{\kappa_3 q_0} \prod_{j=1}^{S} \frac{1}{C^*} e^{\tilde{\kappa} p_j} \cdot C^* e^{\kappa_3 q_j} \geq C^* e^{\kappa_4 n},$$

with $\kappa_4 = \min(\kappa_3, \tilde{\kappa}/4)$. This completes the proof of (i).

Under the assumptions of (ii) instead of (i) we make the same construction and estimates. The only difference is that we do not know whether $f_{a,b}^n(x) \in I^{**}$, so we lose information about the last period.

If the last period is free, then we usually cannot use the estimate from Lemma 4.6. Instead, we use the estimate from Lemma 4.1 (with $I = I^{**}$), so $C^* e^{\kappa_3 q_S}$ gets replaced by $C_{12} e^{\kappa_2 q_S}$, where $C_{12} = C_3(I^{**})$. In this way, we get the estimate

$$
(f_{a,b}^n)'(x) \geq C_6 e^{\kappa_5 n},
$$

where $\kappa_5 = \min(\kappa_3, \tilde{\kappa}/4, \kappa_2)$. Note that $C_6$ and $\kappa_2$ (and therefore $\kappa_4$) do not depend on $I$ and $\kappa_3 = \kappa_2$. We now have (i) and (ii) with $\kappa_4 = \min(\kappa_4', \kappa_5)$.
Remark 4.10. Note that we in this setting will have an analogy of Lemma 4.4 and the estimate

\[(4.21)\quad p \leq \frac{4r}{\kappa_4}\]

holds.

Claim. At the end of the last period \(f_{n,a,b}(x)\) cannot be bound.

Proof. Suppose by contradiction that \(f_{n,a,b}(x)\) is still bound to \(f_{n-tS,a,b}(c)\) then

\[\text{dist}(f_{n-tS,a,b}(x), c) \geq C_2 e^{-\alpha(n-tS)} - e^{-\beta(n-tS)}\]

Since \(f_{n,a,b}^{tS}(x) \in I_{\pm r}, e^{-r} \geq \delta\) and by Remark 4.10, \(p \leq 4r/\kappa_4\). We conclude that

\[\text{dist}(f_{n-tS,a,b}(x), c) \geq \frac{C_2}{2} e^{-4r\alpha/\kappa_4} \geq \frac{C_2}{2} \delta^{4\alpha/\kappa_4} \gg \delta.\]

It follows that \(f_{n,a,b}(x) \not\in I\). The last bound period satisfies (4.18) and for the final free period we use Lemma 4.5. This completes the proof of the lemma.

\[\square\]

Remark 4.11. We need in the future in several occasions a distortion estimate in the situation of Lemma 4.9, i.e. for orbits located outside of \(I\). We need the estimate for parameter dynamics, i.e. we have a parameter interval \(\omega\) in the space of \(a\)-parameters, and we consider \(\xi_j(\omega, b)\) for \(j\) satisfying \(\nu \leq j < \mu = n\), where \(\xi_j(\omega, b) \cap I = \emptyset\) for \(j = \nu, \ldots, n-1\) and \(\xi_n(\omega, I) \neq \emptyset\). Let \(\omega' \subset \omega\) be the interval that is mapped onto \(I\). Then Lemma 4.9 (i) implies that

\[(4.22)\quad \inf_{a \in \omega}(f_{n-a,b}^{\nu-\nu}(f_{n-a,b}^\nu(c)) \geq C_{11} e^{\kappa_4(n-\nu)}\]

We also assume that (3.6) hold at time \(\nu\) i.e. for \(a \in \omega\)

\[(4.23)\quad (f_{j,a,b}^\nu(c) \geq C_2 e^{j/3}, \quad 1 \leq j \leq \nu - 1\]

Then by Corollary 3.2

\[(4.24)\quad 1 \leq \frac{\partial_a \xi(\omega, b)}{(f_{a,b}^{\nu-1})'(f_{a,b}(c))} \leq q.*\]

Then we conclude from Lemma 3.4 that

\[(4.25)\quad |\xi(\omega', b)| \geq \frac{1}{q.*} \inf_{a \in \omega}(f_{a,b}^{\nu-\nu})(f_{a,b}^\nu(c)) \cdot |\xi(\omega', b)|.\]

Lemma 4.12. There exists a constant \(C_{13}\), such that in the situation of Remark 4.11, if \(a', a'' \in \omega\) then

\[(4.26)\quad \frac{(f_{a',b}^{\nu-\nu})(f_{a',b}^\nu(c))}{(f_{a'',b}^{\nu-\nu})(f_{a'',b}^\nu(c))} \leq \exp \left(C_{13} \frac{|f_{a',b}^\nu(c) - f_{a'',b}^n(c)|}{\delta}\right).\]
Proof. Set \( x_k = f_{a',b}^{\nu+k}(c) \) and \( y_k = f_{a''\nu}^{\nu+k}(c) \). Note that that \( f_{a',b}^{\nu+k}(x) \) is independent of \( a \). Therefore
\[
\log \left( \frac{(f_{a',b}^{\nu})'(x_0)}{(f_{a''\nu}^{\nu})'(y_0)} \right) = \sum_{k=0}^{n-\nu-1} (\log f_{a',b}^{\nu}(x_k) - \log f_{a',b}^{\nu}(y_k)) \\
\leq \sum_{k=0}^{n-\nu-1} \left( \frac{|f_{a',b}^{\nu}(\eta_k)|}{f_{a',b}^{\nu}(\eta_k)} \cdot |x_k - y_k| \right)
\]
for some \( \eta_k \) between \( x_k \) and \( y_k \). Since \( \eta_k \notin I^* \) for \( k = 0, \ldots, n-\nu-1 \), we get, by (4.13) and (4.14),
\[
|f_{a',b}^{\nu}(\eta_k)| < \frac{80}{C_9 \delta}.
\]
Therefore,
\[
\log \left( \frac{(f_{a',b}^{\nu})'(x_0)}{(f_{a''\nu}^{\nu})'(y_0)} \right) \leq \frac{80}{C_9 \delta} \sum_{k=0}^{n-\nu-1} |x_k - y_k|.
\]
We have \( x_k = \xi_{\nu+k}(a',b) \) and \( y_k = \xi_{\nu+k}(a'',b) \). Therefore, by Remark 4.11,
\[
|x_k - y_k| \leq \frac{q_*}{C_{11}} e^{\kappa_4(n-\nu-k)} |x_{\nu} - y_{\nu}|.
\]
Thus,
\[
\sum_{k=0}^{n-\nu-1} |x_k - y_k| \leq \frac{q_*}{C_{11}} \sum_{m=0}^{\infty} e^{-\kappa_4 m} |x_{\nu} - y_{\nu}| = \frac{q_*}{C_{11}(1 - e^{-\kappa_4})} |x_{\nu} - y_{\nu}|.
\]
Together with (4.28), we get
\[
\log \left( \frac{(f_{a',b}^{\nu})'(x_0)}{(f_{a''\nu}^{\nu})'(y_0)} \right) \leq \frac{80q_*}{C_9 C_{11}(1 - e^{-\kappa_4}) \delta} |x_{\nu} - y_{\nu}|.
\]
and we have proved (4.26) with
\[
C_{13} = \frac{160q_*}{C_9 C_{11}(1 - e^{-\kappa_4})}.
\]

\[ \square \]

Remark 4.13. We note that the distortion in Lemma 4.12 is uniformly bounded since \( |f_{a,b}^n(c) - f_{a',b}^n(c)| \leq 2\delta \).

We will need a distorsion estimate of the same type as Lemma 4.12 in the situation when we only assume estimates as (4.22) for all \( \nu < n \) and with another Lyapunov exponent \( \kappa_5 > 0 \), together with (4.23). This is the case of hyperbolic times in the sense of Alves.

Lemma 4.14. Assume that \( \xi_j(\omega, b), j = \nu, \ldots, n, \) is located in \( U = S^1 \setminus I_{**} \) and
\[
\inf_{a \in \omega} (f_{a,b}^{\nu-j})' (f_{a,b}^j(c)) \geq C_{11} e^{\kappa_5(n-j)} \quad \text{for all } j, \nu \leq j < n.
\]
Furthermore assume that (4.23) is satisfied.

Then
(4.30) \[
\frac{(f^{n}_{a,b})'(f^{n}_{a,b}(c))}{(f^{n}_{a,b})'(f^{n}_{a,b}(c))} \leq \exp \left( C_{14} (f^{n}_{a,b}(c) - f^{n}_{a,b}(c)) \right). 
\]

Here the constant \( C_{14} \) can be chosen as \( C_{14} = C_{14}' \frac{N(f_{\mathcal{N}}, U)}{(1 - e^{-\kappa_4})} \), where \( N(f_{\mathcal{N}}, U) \) is the maximal nonlinearity

\[
N(f_{\mathcal{N}}, U) = \sup_{(a,b) \in \mathcal{N}} \max_{x \in U} \frac{|f''_{a,b}(x)|}{f_{a,b}(x)}.
\]

\( N(f_{\mathcal{N}}, U) \) depends only on \( f_{a_0} \) and hence not on \( \delta \) and \( C_{14}' \) is a constant that only depends on \( f_{a_0} \).

**Proof.** We will not give the proof since it is virtually word by word the same as that of Lemma 4.12. The only difference that the upper bound \( 80/(C_0 \delta) \) in (4.27) is replaced by \( N(N, U) \)

\[ \square \]

## 5. Induction

Recall that the partition of the return interval \( I^* = (c - \delta, c + \delta) \) was introduced on Section 2.

Recall also that we defined \( \xi_a(a, b) = f^n_{a,b}(c) \).

The next lemma will be used for the startup of the induction.

**Lemma 5.1.** Assume that \( \delta_1 \) is sufficiently small and the neighborhood \( \mathcal{N} \) of \( (a_0, 1) \) is sufficiently small. Then there are constants \( C_1, C_2, \kappa_6 > 0 \) so that for every \( \varepsilon = 2^{-N_0} \) sufficiently small, there is a function \( b_0(\varepsilon) \) so that for every \( b_0(\varepsilon) \leq b < 1 \) one can partition \( (a_0 - \varepsilon, a_0 - \varepsilon^2) \) into a partition \( \mathcal{Q} \) of countable number of parameter intervals \( \omega \) and an exceptional set \( \mathcal{E} \) of measure \( o(\varepsilon) \), so that for all \( \omega \in \mathcal{Q} \) there is an \( n_0 = n_0(\omega) \) so that for some \( (r, \ell) \), with \( r \leq n_0^2 \), (or equivalently \( e^{-r} \geq e^{-\sqrt{n_0}} \))

\[ I_{r,\ell} \subset \xi_{n_0}(\omega, b) \subset I_{r,\ell}^+. \]

and such that for every \( a \in \omega_b \)

(a) \( (f_{a,b}^j)'(f_{a,b}(c)) \geq C_2 e^{\kappa_6 j} \) for \( 0 \leq j \leq n_0 - 1 \);
(b) \( \partial_a f_{a,b}^j(c) \geq C_2 e^{\kappa_6(j-1)} \) for \( 1 \leq j \leq n_0 \);
(c) \( |\xi_a(a, b) - c| > C_1 e^{-\sqrt{j}} \) for \( 1 \leq j < n_0 \);
(d) \( (f_{a,b}^{n_0-1})'(f_{a,b}(c)) \geq e^{(n_0 - 1)^{2/3}} \);
(e) \( |\xi_{n_0}(a, b) - c| \geq e^{-\sqrt{n_0}} \).

The corresponding statement holds also for the interval \( (a_0 + \varepsilon^2, a_0 + \varepsilon) \).

**Proof.** We partition \( (a_0 - \varepsilon, a_0 - \varepsilon^2) \) into subintervals \( \eta_j = (a_0 - 2^{-j}, a_0 - 2^{-j-1}) = (a_j', a_j''), j = J_0, \ldots, 2J_0 - 1 \). The critical point \( c \) of unperturbed map \( f_{a_0} \) is mapped to a repelling periodic point \( P \) in \( m \) iterates. Let \( U_0 \) be a symmetric interval contained in the linearization domain of \( P \) so that

\[ (f_{a,b}^j)'(x) \geq \lambda_1 = \Lambda_1 > 1 \quad \text{for} \ x \in U_0. \]
Let $\tilde{n}_j = (a_0 - 2^{-j}, a_0)$. Then there is a constant $C_{15}$ so that
\[ (f_{a,b}^j)(\xi_m(a, b)) \geq C_{15} \lambda_1^j, \quad \text{for all } a \in \tilde{n}_j \]
as long as $\xi_{m+i}(\tilde{n}_j, b) \subset U_0$. By Lemma 3.4
\[ |\xi_{m+i}(\tilde{n}_j, b)| \geq \frac{C_{15}}{q_*} \lambda_1^j \cdot |\xi_m(\tilde{n}_j, b)| \]
and $q_*$ has a uniform control by Corollary 3.2. It follows that there is a first time $L$ so that $\xi_{n_0+L+1}(\tilde{n}_j, b) \not\in U_0$. Let $\text{orb}(P)$ be the orbit of the repelling periodic point $P$.
By Lemma 4.14 $|\xi_{m+L}(\eta_j, b)|$ and dist$(\xi_{m+L}(\eta_j, b), \text{orb}(P))$ are comparable within a fixed constant $C_{16}$, which can expressed in terms of the distorsion constant of (4.30), $C_{14}$ and the upper bound 4 of $(f_{a,b}')(x)$. Thus $C_{16}$ only depends on $f_{a_0}$.
We continue to iterate $\xi_{m+L+i}(\eta_j, b)$ for $i = 1, 2, \ldots$. By Lemma 4.9, Lemma 3.4 and the control of the constant $q_*$ it follows that at the first time $J$ such that $\xi_{m+L+J}(\eta_j, b) \cap I^* \neq \emptyset$
\[ |\xi_{m+L+J}(\omega, b)| \geq \frac{C_{11}}{q_*} e^{\kappa_4 J} |\xi_{m+L}(\tilde{\omega}, b)|. \]
$\kappa_6 = \min(\log \Lambda, \kappa_4)$ is then the required Lyapunov exponent in (a). It follows by Lemma 4.9 (ii), Lemma 3.4 and the control of $q_*$ that the time $J$ will be finite. At time $N_0 = m + L + J$, we partition
\[ (c - \delta, c + \delta) \cap \xi_{n_0}(\eta_j, b) \]
into preimages $\{\omega\}$ under the map $a \mapsto \xi_{n_0}(a, b)$ of the partition $Q = \{I_r\}$ and define $n_0 = N_0$ for these $\omega$'s. In the special case when $\xi_{n_0}(\eta_j, b)$ only intersects partially an end interval of $Q = \{I_r\}$, we just keep iterating until we cover complete intervals of $Q$. In other special case when $\xi_{n_0}(\eta_j, b)$ only partially covers a $I_r$ interval we adjoin the corresponding preimage to the adjacent interval. Simultaneously we delete the part of $\eta_j$ that is mapped to $(c - e^{-\sqrt{m}}, c + e^{-\sqrt{m}})$. By the uniform distorsion of both the $x$-derivative and $a$-derivative which follows from Lemma 4.9, Lemma 4.12 and Corollary 3.2 a proportion of at most $C_{17} e^{-\sqrt{m}/\delta}$ of the piece of $\eta_j$ mapped into $(c - \delta, c + \delta)$. Here $C_{17}$ is a constant only depending on $f_{a_0}$. We continue to iterate $\xi_{n_0}(\eta_j, b) \setminus (c - \delta, c + \delta)$, still using Lemma 4.9, Lemma 4.12 and Corollary 3.2. For the new returning interval $\omega$ formed in this way $n_0(\omega) > N_0$ and still only a quantity proportional to $e^{-\sqrt{m}/\delta}$ is deleted. The conclusions (a)–(e) of Lemma 5.1 are immediately verified. □

**Remark 5.2.** The startup argument is essentially the same as the free period argument in the main induction and the argument in Lemma 4.9 in Section 4. See the main induction below in this section for a more thorough discussion. The only difference is the initial period that is spent close to the repelling periodic point which in some sense replaces the bound period. The expansive behaviour close to the repelling periodic point allows us to avoid inessential free returns and gives the initial exclusion ratio of at most $C_{17} e^{-\sqrt{m}/\delta}$.

Let us now fix $b$, $0 < b < b_0(\varepsilon)$. Note that for every positive integer $n$ we have a family $\mathcal{P}_n$ of subintervals of $(a_0 - \varepsilon, a_0 + \varepsilon)$ (as in Lemma 5.1) with pairwise disjoint interiors, such that each element of $\mathcal{P}_{n+1}$ is contained in some element of $\mathcal{P}_n$. In the
set of pairs \((n, \omega)\) such that \(\omega \in \mathcal{P}_n\) there is a natural structure of a combinatorial tree, that goes down with its branches. Pairs \((n, \omega)\) are vertices of this tree; \(n\) is the level on which the vertex lies; there is an edge from \((n, \omega)\) to \((n + 1, \omega')\) if and only if \(\omega' \subset \omega\).

Certain pairs with the property \(\xi_n(\omega, b) \subset I^*\) will be called free return pairs. The induction will be separate on every branch of the tree. Fixing the branch results in considering a descending sequence of intervals \(\omega_n \in \mathcal{P}_n\). If \((n, \omega_n)\) is a free return pair, then we will call \(n\) a free return time. An important feature of the construction is that if \(n\) is not a free return time then \(\omega_n = \omega_{n-1}\). The main induction step will be from a free return time to the next free return time. The constants \(C_2\) and \(C_1\) are as in Lemma 5.1. In the whole induction they will stay the same.

Our **Induction Statement** is the following. If \(n\) is a free return time and and \(a \in \omega\), then:

(i) we have

\[
(f_{a,b}^{n-1})' (f_{a,b}(c)) \geq e^{2(n-1)^{2/3}},
\]

(ii) for every \(\nu \in [n_0, n)\)

\[
(f_{a,b}^\nu)' (f_{a,b}(c)) \geq e^{\nu^{2/3}},
\]

(iii) for every \(\nu \in [1, n)\)

\[
(f_{a,b}^\nu)' (f_{a,b}(c)) \geq C_2 e^{\nu^{2/3}},
\]

(iv) if \(\nu < n\) is also a free return time, then

\[
(f_{a,b}^{n-\nu})' (f_{a,b}^\nu(c)) \geq C(\delta) >> 1,
\]

(v) for every \(\nu \in [n_0, n]\)

\[
|\xi_\nu(a, b) - c| \geq e^{-\sqrt{\nu}},
\]

(vi) for every \(\nu \in [0, n]\)

\[
|\xi_\nu(a, b) - c| \geq C_1 e^{-\sqrt{\nu}},
\]

In [2] and [3] statements (v) and (vi) is called the basic assumption (BA).

Remember that \(b\) sufficiently close to 1 is fixed. We set \(\mathcal{P}_n = \{\omega_b\}\) for \(n = 1, 2, \ldots, N_0\). Thus, this is the beginning of every branch. Then we declare \(n_0 = n_0(\omega)\) to be the first free return time according to the startup construction. Thus, for every branch we have to start induction by checking that that the above conditions are satisfied for \(n = n_0(\omega)\).

**Lemma 5.3.** The Induction Statement conditions (i)-(vi) are satisfied for \(n = n_0\).

This is a consequence of the startup construction, Lemma 5.1.

Now we make a small modification of Definition 4.2.

**Definition 5.4.** Let \(a'\) be the midpoint of the interval \(\omega\) such that

\[
\xi_n(\omega, b) \subset I_{r,l}^+ = I_{r,l-1} \cup I_{r,l} \cup I_{r,l+1}
\]
for some \( n, r, l \). We define the bound period as the maximal integer \( p \) so that for all \( j \leq p, a \in \omega, \) and \( x \in \xi_n(\omega, b) \)
\[
|f_{a,b}^j(x) - f_{a',b}^j(c)| \leq e^{-4\sqrt{j}}. 
\]

By (4.3) and Lemma 3.1, we get for every \( n, a, b, x \)
\[
\partial_a f_a^n(x) \leq \sum_{k=0}^{n-1} 4^k = \frac{4^n - 1}{3} < 4^n. 
\]

In the several next lemmas we will be using the same set of assumptions. We formalize these in the following definition

**Definition 5.5.** We say that Condition (*) is satisfied if

- \( \omega, n, r, l, p \) and \( a' \) are as in Definition 5.4,

- conditions (iii), (v) and (vi) of the Induction Statement hold.

Next we formulate another version of the Bound Distorsion Lemma

**Lemma 5.6.** There is a constant \( C_{18} \) such that if Condition (*) holds, then
\[
\frac{1}{C_{18}} \leq \frac{(f_{a,b}^k)'(f_{a,b}(y))}{(f_{a,b}^k)'(f_{a,b}(c))} \leq C_{18} 
\]
and
\[
\frac{1}{C_{18}} \leq \frac{(f_{a,b}^k)'(f_{a,b}(y))}{(f_{a',b}^k)'(f_{a',b}(c))} \leq C_{18} 
\]
for every \( x \in I_{r,l}, y \) between \( x \) and \( c, a \in \omega, \) and \( k \leq p \). By making \( \delta \) sufficiently small, the constant \( C_{18} = C_{18}(\delta) > 1 \) can be chosen arbitrarily close to 1.

**Proof.** The proof will proceed by induction on \( k \). Using (4.3), we get in the same way as in the proof of Lemma 4.3
\[
\frac{(f_{a,b}^k)'(f_{a,b}(y))}{(f_{a,b}^k)'(f_{a,b}(c))} \leq \exp \left( \sum_{j=1}^{k} \frac{13|f_{a,b}^j(y) - f_{a,b}^j(c)|}{f_{a,b}'(f_{a,b}(c))} \right). 
\]

Furthermore,
\[
|f_{a,b}^j(x) - f_{a,b}^j(c)| \leq |f_{a,b}^j(x) - f_{a,b}^j(c)| \leq |f_{a,b}^j(x) - f_{a,b}^j(c)| + |f_{a',b}^j(c) - f_{a,b}^j(c)|, 
\]
and by (5.7) we have
\[
|f_{a,b}^j(x) - f_{a',b}^j(c)| \leq e^{-4\sqrt{j}}. 
\]
Thus, we need to estimate \( |f_{a',b}^j(c) - f_{a,b}^j(c)| \). Note that by the mean value theorem there is \( a'' \) between \( a \) and \( a' \) so that
\[
|f_{a',b}^j(c) - f_{a,b}^j(c)| = \partial_a f_{a'',b}^j(c) \cdot |a - a'|. 
\]
Note that \( |a - a'| \) can be interpreted as \( |\xi_1(a, b) - \xi_1(a', b)| \). By Lemma 3.4
\[
|\xi_n(a, b) - \xi_n(a', b)| \geq \frac{1}{q_*} \inf_{a \in [a,a']}(f_{a,b}^{n-1}')(f_{a,b}(c)) \cdot |\xi_1(a, b) - \xi_1(a', b)|. 
\]
By the induction statement (i), \((f_{a,b}^n)'(f_{a,b}(c)) \geq e^{(n-1)^2/3}\). We may therefore conclude that \(|a - a'| \leq q_n e^{-(n-1)^2/3} \cdot |\xi_n(a, b) - \xi_n(a', b)|\). But by the mean value theorem

\[|f_{a,b}^j(x) - f_{a,b}^j(c)| = |f_{a,b}^{j-1}(f_{a,b}(x)) - f_{a,b}^{j-1}(f_{a,b}(c))| = (f_{a,b}^{j-1})'(y) \cdot |f_{a,b}(c) - f_{a,b}(x)|.\]

However since \(|\xi_n(a, b) - \xi_n(a', b)| \leq e^{-r}\), we have

\[|f_{a,b}(c) - f_{a,b}(x)| \geq C_7 \cdot |c - x|^{3} \geq C_7 e^{2(-r-1)} \cdot e^{-1} \cdot |\xi_n(a, b) - \xi_n(a', b)|.\]

By the basic assumption \(e^{-r} \geq C_1 e^{-\sqrt{n}}\). Note also that by Corollary 3.2, \(\partial_a f_{a,b}^j(c)\) is comparable within the multiplicative constant \(q_n\) to \((f_{a,b}^{j-1})'(f_{a,b}(c))\). But this quantity is in itself by induction comparable within a multiplicative constant \(C_{18}\) to \(\inf_{a \in (a, a')} (f_{a,b}^{n-1})'(f_{a,b}(c))\). We use the statement of our result for \(k = j - 1\). We finally obtain

\[|f_{a,b}^j(c) - f_{a,b}^j(y)| \leq \frac{1}{2} |f_{a,b}(x) - f_{a,b}(c)|;\]

since \(n \geq n_0(a)\), and \(n_0(a)\) can be chosen arbitrarily large.

When inserting this estimate in (5.11) we conclude that

\[(5.14) \quad |f_{a,b}^j(y) - f_{a,b}^j(c)| \leq 2 e^{-4\sqrt{j}}.\]

To estimate \(f_{a,b}(f_{a,b}^j(c))\) from below, we use (iv) of the Induction Statement and (4.13). We get

\[(5.15) \quad f_{a,b}'(f_{a,b}^j(c)) \geq C_9 C_1^2 e^{-2\sqrt{j}}.\]

Putting together (5.11), (5.14) and (5.15), we obtain

\[(5.16) \quad \frac{(f_{a,b}^k)'(f_{a,b}(y))}{(f_{a,b}^k)'(f_{a,b}(c))} < \exp \left( \frac{13 \cdot 2}{C_9 \cdot C_1^2} \sum_{j=1}^{k} e^{-2\sqrt{j}} \right).\]

For the lower bound, we obtain in a similar way as (5.11)

\[\frac{(f_{a,b}^k)'(f_{a,b}(c))}{(f_{a,b}^k)'(f_{a,b}(y))} \leq \exp \left( \sum_{j=1}^{k} \frac{13|f_{a,b}^j(y) - f_{a,b}^j(c)|}{(f_{a,b}^j)'(f_{a,b}(y))} \right).\]

Note however that

\[(f_{a,b})'(f_{a,b}(y)) \geq C_9 (f_{a,b}^j(y) - c)^2 \geq C_9 \left(|c - f_{a,b}^j(c)| - |f_{a,b}^j(c) - f_{a,b}^j(y)|\right)^2,\]

and using (5.5) and (5.14) we get

\[f_{a,b}'(f_{a,b}^j(y)) > C_9 (C_1 e^{-\sqrt{j}} - (1 + K_6) e^{-4\sqrt{j}})^2\]

whenever \(C_1 e^{-\sqrt{j}} > (1 + K_6) e^{-4\sqrt{j}}\). Now, \(C_1\) is fixed and thus, there is a positive integer \(\tilde{N}\) such that if \(j \geq \tilde{N}\) then \(C_1 e^{-\sqrt{j}} > 2(1 + K_6) e^{-4\sqrt{j}}\), and then

\[f_{a,b}'(f_{a,b}^j(y)) > \frac{C_9 C_1^2}{4} e^{-2\sqrt{j}}.\]
By making $N$ and $I^*$ sufficiently small, we can make $|f_{a,b}^j(c) - f_{a,b}^j(y)|$ smaller than $C_1 e^{-4\sqrt{j}}$ instead of $(1 + K_6)e^{-4\sqrt{j}}$, and then we get

$$f'_{a,b}(f_{a,b}(x)) \geq C_9 C_1^2 K_7 e^{-2\sqrt{j}}$$

for some constant $K_7$ depending only of $a_0$. In such a way, in the same way as we obtained (5.16), we get a similar estimate for the reciprocal ratio, but with a different constant. We choose then as $C_{18}$ the larger of those constants and we get (5.9). The proof of (5.10) is completely analogous and will be omitted. As for the statement that $C_{18}$ can be chosen arbitrarily close to $1$ (but larger than $1$), we refer to the argument in Lemma 4.3.

\[\Box\]

**Lemma 5.7.** Assume that Condition (*) holds. Then the bound period $p$ in the sense Definition 5.4 of satisfies

$$p \leq 8r^{3/2}.$$

**Proof.** Let $k \leq p$. Take a point $x \in I_{r,l}$. By the Mean Value Theorem, there is a point $y$ between $x$ and $c$ such that

$$|f_{a,b}^k(x) - f_{a,b}^k(c)| = |f_{a,b}(x) - f_{a,b}(c)| \cdot (f_{a,b}^{k-1})'(f_{a,b}(y)).$$

By Lemma 5.6 and (iii) of the Induction Statement,

$$(f_{a,b}^{k-1})'(f_{a,b}(y)) \geq \frac{1}{C_{18}} (f_{a,b}^{k-1})'(f_{a,b}(c)) \geq \frac{C_2}{C_{18}} e^{(k-1)^2/3}.$$ 

Since $|x - c| \geq e^{-r-1}$, by (4.13) we get

$$|f_{a,b}(x) - f_{a,b}(c)| \geq \frac{C_9}{3} e^{-3r-3}.$$ 

Putting the last three inequalities together, we get

$$|f_{a,b}^k(x) - f_{a,b}^k(c)| \geq \frac{C_2}{C_{18}} e^{(k-1)^2/3} \cdot \frac{C_9}{3} e^{-3r-3}.$$ 

Taking into account (5.14) (which is valid also for $y = x$), we get

$$(1 + K_6) > (1 + K_6)e^{-\sqrt{k}} > \frac{C_2 C_9}{3C_{18}} e^{(k-1)^2/3} e^{-3r-3}.$$ 

Therefore,

$$(k - 1)^{2/3} < 3r - \log K_8$$

where $K_8$ is a constant only depending on $a_0$.

If $\delta$ is sufficiently small, then $\frac{1}{2}r > -\log K_8$, and we get $k^{2/3} < (k - 1)^{2/3} + \frac{2}{3}k^{-1/3} < 4r$.

By Definition 4.2 we conclude that $p \leq 8r^{3/2}$.

\[\Box\]

Let us prove an elementary lemma about our family.

**Lemma 5.8.** For the family of double standard maps, if $0 < |x - c| < 1/2$ then

$$f'_{a,b}(x) > \frac{|f_{a,b}(x) - f_{a,b}(c)|}{|x - c|}.$$

(5.17)
Proof. We have $c = 1/2$ and $f''_{a,b}(t) = -4\pi b \sin(2\pi t)$. Therefore $f_{a,b}$ is strictly convex in $(c, c + 1/2)$, and thus for $x \in (c, c + 1/2)$ we get (5.17). Similarly, in $(c - 1/2, c)$ the function $f_{a,b}$ is strictly concave, and (5.17) follows similarly. □

Lemma 5.9. There exists a positive constant $C_{19}$ such that if Condition $(\ast)$ holds, then

$$ (f_{a,b}^{p+1})'(x) > C_{19} e^r \cdot e^{-\sqrt{p}}. \leqno{(5.18)} $$

Proof. By Definition 5.4, there exists $a \in \omega$ such that

$$ |f_{a,b}^{p+1}(x) - f_{a,b}^{p+1}(c)| \geq e^{-\sqrt{p+1}}. \leqno{(5.19)} $$

We have

$$ |f_{a,b}^{p+1}(x) - f_{a,b}^{p+1}(c)| \leq |f_{a,b}^{p+1}(x) - f_{a,b}^{p+1}(c)| + |f_{a,b}^{p+1}(c) - f_{a,b}^{p+1}(c)|. \leqno{(5.20)} $$

By the Mean Value Theorem, there is a point $y$ between $x$ and $c$ such that

$$ |f_{a,b}^{p+1}(x) - f_{a,b}^{p+1}(c)| = |f_{a,b}(x) - f_{a,b}(c)| \cdot (f_{a,b}^{p+1})(f_{a,b}(y)). \leqno{(5.21)} $$

Now we estimate the second summand in (5.20). As in the proof of Lemma 5.6, we can prove that

$$ |f_{a,b}^{p+1}(c) - f_{a,b}^{p+1}(c)| \leq \frac{1}{2} |f_{a,b}^{p+1}(c) - f_{a,b}^{p+1}(x)|. $$

Therefore

$$ |f_{a,b}^{p+1}(x) - f_{a,b}^{p+1}(c)| \geq \frac{1}{2} |f_{a,b}^{p+1}(x) - f_{a,b}^{p+1}(c)| \geq \frac{1}{2} e^{-\sqrt{p+1}}. $$

From Lemma 5.6 we get

$$ (f_{a,b}^{p})(f_{a,b}(x)) \geq \frac{1}{C_{18}^2} (f_{a,b}^{p})(f_{a,b}(y)), $$

so

$$ (f_{a,b}^{p})(f_{a,b}(x)) > \frac{1}{2C_{18}^2} \cdot e^{-\sqrt{p+1}} \cdot \frac{1}{|f_{a,b}(x) - f_{a,b}(c)|}. $$

By the Chain Rule and Lemma 5.8 we get from this inequality

$$ (f_{a,b}^{p+1})(x) > \frac{1}{2C_{18}^2} \cdot e^{-\sqrt{p+1}} \cdot \frac{1}{|x - c|}. $$

Since $x \in I_r$, we have $|x - c| \leq e^{-r}$, and we get (5.18) with a suitable choice of $C_{20}$. □

Let $(n, \omega)$ be a free return pair. Consider the intervals $\xi_{n+p+1+s}(\omega, b)$, $s = 0, \ldots, s_0 - 1$, where $s_0$ is the smallest nonnegative integer such that

$$ \xi_{n+p+1+s}(\omega, b) \cap I^* \neq \emptyset. $$

For $0 \leq s < s_0$, we say that $\xi_{n+p+1+s}(\omega, b)$ is in free orbit and the length of this orbit is $s_0$. We also use the notation $n' = n + p + 1 + s_0$ and it is our new free return time.

At the first free return there are different cases that can occur.

Case 1. $\Omega_{n'} = \xi_{n'}(\omega, b)$ is completely contained in $I^*$ but does not contain a complete interval $I_{r, t}$. Then either $\xi_{n+p+1+s_0}(\omega, b)$ is contained in an interval $I_{r, t}$ or it is contained in the union of two adjacent intervals $I_{r, t} \cup I_{r, t+1}$. 


This is called an *inessential free return*. In this case \( \omega \in \mathcal{P}_{n'} \), and we just continue to iterate. This also applies if \( \Omega_{n'} \) intersects the boundary of \( I^* \) but does not contain any of the end intervals.

**Case 2.** \( \Omega_{n'} \) contains at least one of the partition intervals \( I_{r,\ell} \). This is the case of an *essential free return*. We then proceed to define a new partition on a subset of \( \omega \) according to the following algorithm.

- We do not include the preimage of \( (c - C_2e^{-\sqrt{m}}, c + C_2e^{-\sqrt{m}}) \) under \( a \mapsto \xi_{n'}(a, b) \) in \( \bigcup_{\omega' \in \mathcal{P}_{n'}, \omega'} \), in order that (BA) should be satisfied.
- The intervals \( \omega_{r,\ell} \) and \( \omega'_{r,\ell} \) are defined as the preimages of \( I_{r,\ell} \) under \( \omega \ni a \mapsto \xi_{n'}(a, b) \). Because of the double covering property of \( f_{a, b} \), there could be 0, 1 or 2 such intervals. These will be new partition intervals of \( \mathcal{P}_{n'} \). At the two ends of \( \omega \) we could have the property that some intervals only partially cover \( I_{r,\ell} \). In that case we use the special rule that we adjoin the corresponding subintervals to the adjacent intervals of \( \mathcal{P}_{n'} \).
- There may be at most three subintervals of \( \omega \), call them \( \omega_1, \omega_2 \) and \( \omega_3 \) which are mapped outside \( I^* \) by \( \omega \ni a \mapsto \xi_{n'}(a, b) \). In the beginning of the procedure there are at most two intervals mapped outside, but in later stages because of the double covering property of \( f_{a, b} \), there can be three. In this case these intervals are *long*, i.e. they are not contained in intervals adjacent to the end intervals in the partition of \( (c - e^{-r_3+1}, c + e^{-r_3+1}) \), they are considered to be still free, and the free period continues for these intervals. If one or more of the intervals \( \omega_1, \omega_2 \) or \( \omega_3 \) are *short*, i.e. not long, they are adjoined to their adjacent neighbors.

Let \( X_{BA} \) be the set that is mapped to \( (c - e^{-\sqrt{m}}, c + e^{-\sqrt{m}}) \). Then we define the partition \( \mathcal{P}_{n'}|(\omega \setminus X_{BA}) \) as the intervals \( \{\omega_{r,\ell}\}, \{\omega'_{r,\ell}\} \) and \( \omega_i, i = 1, 2, 3 \). Some of these intervals may be empty.

Later we will see that deletions because of (BA) do not happen in Case 1, because the interval \( \Omega_{n'} \) is too long.

In order to proceed, we need to verify, at least partially, the induction step from time \( n \) to time \( n' \). Here \( n' \) is interpreted as the first free return to \( I^* \) after \( n \). There may be previous returns \( \nu \), where another partition element of \( \mathcal{P}_{n} \) has a free return, while the present parameter interval does not return.

**Lemma 5.10.** Assume the Induction Statement (i)-(vi). Then the Induction Statement conditions (ii), (iii) and (iv) hold for any free return pair \( (n', \omega') \), where \( n' \) is as above.

**Proof.** Let \( \eta \) be the distance from \( \xi_{n}(\omega, b) \) to \( c \). Therefore, by Induction Statement (i) and (4.13),

\[
(f_{a, b}^n)'(f_{a, b}(c)) > C_9 \eta^2 e^{2(n-1)^2/3}.
\]

However, by (v), \( \eta \geq C_1 e^{-\sqrt{n}} \), so we get

\[
(f_{a, b}^n)'(f_{a, b}(c)) > C_9 C_1^2 e^{2(n-1)^2/3} e^{-2\sqrt{n}}.
\]

After time \( n \) there follows the bound period \( p \), and by Lemma 5.6 and (iii) we get

\[
(f_{a, b}^n)'(f_{a, b}^{n+1}(c)) \geq C_9^{-1} (f_{a, b}^k)'(f_{a, b}(c)) \geq C_9^{-1} C_2 e^{k^2/3}
\]
for all \(k \leq p\). Combining (5.22) and (5.23), we conclude that
\[
(f_{a,b}^{n+k})'(f_{a,b}(c)) = (f_{a,b}^n)'(f_{a,b}(c)) \cdot (f_{a,b}^k)'(f_{a,b}(c)) > C_9 C_1^2 C_{18}^{-1} C_2 e^{2(n-1)^{2/3} - 4\sqrt{p} + k^{2/3}}.
\]

For \(k \leq p \leq 8n^{3/2} \leq 8n^{3/4}\), we conclude that
\[
(f_{a,b}^{n+k})'(f_{a,b}(c)) \geq e^{(n+k)^{2/3}}
\]

At time \(n + p + 1\) the bound period has expired and we have for all \(a \in \omega\)
\[
(f_{a,b}^p)'(f_{a,b}^n(c)) e^{-3r} \geq C_{21} e^{-4\sqrt{p+1}},
\]
where \(C_{21}\) is a constant only depending on \(f_{a_0}\).

For the total derivative we obtain
\[
(f_{a,b}^{n+p})'(f_{a,b}^n(c)) \geq C_{21} e^{2n^{2/3} - 3r} (f_{a,b}^p)'(f_{a,b}^{n-1}(c))
\]

After raising (5.25) to the power \(\frac{2}{3}\) we obtain
\[
(f_{a,b}^{n+p})'(f_{a,b}^n(c)) \geq C_{21}^2 e^{2n^{2/3} - \frac{3r}{2}} (f_{a,b}^p)'(f_{a,b}^{n-1}(c))^{1/3} e^{-\frac{r}{3} \sqrt{p+1}}
\]
Looking at the exponents, we get using (5.3) the lower bound
\[
2n^{2/3} + \frac{1}{3} p^{2/3} - \frac{8}{3} \sqrt{p+1} \geq 2(n+p)^{2/3} + \frac{1}{10} p^{2/3}
\]
Here we have used the information from Lemma 5.7, \(p \leq 8n^{3/4}\) and that if \(p \leq \frac{1}{100} n\)
\[
2n^{2/3} + \frac{1}{3} p^{2/3} \geq 2(n+p)^{2/3} + \frac{1}{5} p^{2/3}.
\]

Now, if \(k = p + s\) and \(0 < s \leq s_0\), then we can use Lemma 4.9 (ii). If \(s \geq M\) then
\[
(f_{a,b})'(f_{a,b}^{n+p+1}(c)) \geq e^{\kappa s}.
\]
If \(s < M\) we use Lemma 4.5, which allows a perturbation to \((a, b) \in \mathcal{N}\) with a worse constant \(\overline{d}/4\) instead of \(\overline{d}/2\) and we get
\[
(f_{a,b})'(f_{a,b}^{n+p+1}(c)) \geq \overline{d}/4.
\]
Thus, independently whether \(s \geq M\) or \(s < M\), we have
\[
(f_{a,b}^s)'(f_{a,b}^{n+p+1}(c)) \geq \overline{d}/4 e^{(s-M)\kappa 4}.
\]
Together with (5.24) (where we substitute \(k = p + s\)), we get
\[
(f_{a,b}^{n+k})'(f_{a,b}(c)) = (f_{a,b}^{n+p})'(f_{a,b}(c)) \cdot (f_{a,b}^s)'(f_{a,b}^{n+p+1}(c))
\]
\[
> C_9 C_1^2 C_{18}^{-1} C_2 \overline{d}/4 e^{2(n-1)^{2/3} - 4\sqrt{p} + k^{2/3} + (s-M)\kappa 4}.
\]

Note that since the constants \(C_3, C_1, C_{18}, C_2\) are absolute constants and \(p\) can be made arbitrarily large by making \(\delta\) sufficiently small. Doing this, we conclude that (ii) and (iii) of the induction statement holds for \(\nu\) satisfying \(n + p < \nu < n'\) where \(n'\) is the next free return time.
We now turn to verifying (i) at the next free return time \( n' \). Using the previous derivative estimates and (i) of Lemma 4.9 we get after writing \( n' = n + p + 1 + q \) that

\[
(f_{a,b}^{n'-1})'(f_{a,b}(c)) \geq e^{2(n-1)^{2/3} + \frac{1}{30}p^{2/3} - \frac{2}{3} \sqrt{p+1} C_{11} e^{\kappa_4(q-1)}},
\]

where \( C_{11} \) is an absolute constant only depending on \( a_0 \). Arguing in different cases depending on the relative sizes of \( n \) and \( q \), one can verify that (i) of the induction with \( n \) replaced by \( n' \) holds.

Since \( C_{11} \) and \( C_{19} \) do not depend on \( \delta \), while by making \( \delta \) sufficiently small we can make \( p \) as large as we want, we conclude using Lemma 5.7 that

\[
C_{11} C_{19} e^p \cdot e^{-\sqrt{p+1}} \geq C_{11} C_{19} \exp\{\frac{1}{2}p^{2/3} - 4\sqrt{p+1}\} \geq 1.
\]

This proves (iv) for \( n' \).

Since we have deleted the set of parameters that are mapped to

\[
(c - C_2 e^{-\sqrt{n'}}, c + C_2 e^{-\sqrt{n'}}),
\]

where \( C_2 = \max(C_2, 1) \), we conclude that (v) and (vi) of the induction also hold. This completes the proof of the induction step.

We note that the proof also gives the information

\[
(5.28) \quad (f_{a,b}^{n'-n})'(f_{a,b}(c)) \geq e^\frac{1}{6}(n'-n)^{2/3}.
\]

\[\square\]

**Remark 5.11.** Deon (5.28) it immediately follows that

\[
(5.29) \quad (f_{a,b}^{n'-1})'(f_{a,b}(c)) \geq (f_{a,b}^{n-1})'(f_{a,b}(c)) \cdot e^{\frac{1}{3}p^3},
\]

which will be used later. We will also later use (5.28).

**Remark 5.12.** Clearly, in Lemma 3.4, \( \omega \) can be replaced by any subinterval \( \omega' \subset \omega \).

If we choose \( \mu = n' \) and if \( \xi_{n'}(\omega', b) \subset I^* \), and \( n + p + 1 \leq \nu < \mu = n' \), we can use Lemma 4.9 (i) to estimate \( \inf_{a \in \omega}(f_{a,b}^{n'-\nu})'(f_{a,b}(c)) \) from below by \( C_{11} e^{\kappa_4(\nu'-\nu)} \). Moreover by Lemma 5.10 we know that (iii) of the induction holds for \( \nu < n' \). Therefore we conclude from (3.7) that (3.8) holds with \( q' = q_* \). In such a way we get

\[
|\xi_{n'}(\omega', b)| \geq \frac{C_{11}}{q_*} e^{\kappa_4(n'-\nu)}|\xi_{\nu}(\omega', b)|.
\]

6. The global distortion Lemma

**Lemma 6.1.** There exists a constant \( C_{14} \), such that if \( a \) and \( a' \) are two parameter points, so that \( a, a' \in \omega \subset P_n \), where \( n \) is a free return time and Induction Statement for \( n \) (and all smaller free return times) holds, then

\[
(6.1) \quad \frac{(f_{a,b}^k)'(f_{a,b}(c))}{(f_{a',b}^k)'(f_{a',b}(c))} \leq C_{14}
\]

for all \( k \leq n - 1 \).
Proof. Let us fix \( k \leq n - 1 \). Set \( t_0 = 1 \) and let \( \{t_j\}_{j=1}^m \) be the free return times arranged in an increasing order. Here \( m \) is defined by the condition \( t_{m-1} < k \leq t_{m-1} - 1 \), and we can assume that \( t_m = n \). Observe that for all free returns \( t_j \) there is \( r_j \) so that \( \eta_{t_j}(a, b), \xi_{t_j}(a', b) \in I_{r_j} \).

Note that \( f_{a,b}'(x) = f_{a',b}'(x) \). Thus, using the Mean Value Theorem, we can write the logarithm of the left hand side of (6.1) as

\[
\log \frac{(f_{a,b})'(c)}{(f_{a',b})'(c)} = \log \frac{\prod_{j=1}^k f_{a,b}'(\xi_j(a, b))}{\prod_{j=1}^k f_{a,b}'(\xi_j(a', b))}
\]

for some \( \eta_j \) between \( \xi_j(a, b) \) and \( \xi_j(a', b) \), where \( p_i \) is the corresponding bound time and \( p_0 = -1 \). We will denote the first sum in the parenthesis above by \( S'_1 \) and the second term in parenthesis by \( S''_1 \). Note that the sum \( S'_0 \) is empty.

By (4.13) and (4.14) we get

\[
f''_{a,b}(\eta_j) \leq \frac{80}{C_9|\eta_j - c|}.
\]

Set \( \sigma_i = |\xi_{t_i}(a, b) - \xi_{t_i}(a', b)| \). We claim that the sum \( S'_i \) can be estimated from above by a constant times \( \sigma_i e^{r_i} \).

First we note that by (6.2), the first term of \( S'_i \) can be estimated by \( 80\sigma_i/(C_9e^{r_i}) \).

For the remaining terms we introduce the reference interval \( \Omega_{t_i} = I_{r_i+1} \) and intervals \( \Omega_{t_i+\nu} = f_{a,b}'(\Omega_{t_i}), \nu = 0, 1, 2, \ldots, p_i \). We have

\[
\xi_{t_i+1}(a, b) - \xi_{t_i+1}(a', b) = (f_{a,b}(\xi_{t_i}(a, b)) - f_{a,b}(\xi_{t_i}(a', b)))
+ (f_{a,b}(\xi_{t_i}(a', b)) - f_{a,b}(\xi_{t_i}(a', b)))
= f_{a,b}'(y)(\xi_{t_i}(a, b) - \xi_{t_i}(a', b)) + (a - a')
\]

for some \( y \) between \( \xi_{t_i}(a, b) \) and \( \xi_{t_i}(a', b) \). Furthermore, \( |\Omega_{t_i+1}| = f_{a,b}'(y')|\Omega_{t_i}| \) for some \( y' \in \Omega_{t_i} \).

We get

\[
|\xi_{t_i+1}(a, b) - \xi_{t_i+1}(a', b)| \leq \frac{f_{a,b}'(y)}{|\Omega_{t_i+1}|} \cdot \frac{\sigma_i}{f_{a,b}'(y') \cdot |\Omega_{t_i}|} \cdot \frac{|a - a'|}{|\Omega_{t_i+1}|}.
\]

By the Mean Value Theorem

\[
\frac{|a - a'|}{f_{a,b}'(y)\sigma_i} = \frac{1}{f_{a,b}'(y)\partial_a \xi_{t_i}(a'', b)}
\]

for some \( a'' \) between \( a \) and \( a' \). By (3.5) and the induction statement (iii),

\[
\partial_a \xi_{t_i}(a'', b) \geq (f_{a',b}'(c)) \geq C_2 e^{r_i - 1}.
\]

By (4.13) and the induction statement (v),

\[
f_{a,b}'(y) \geq C_9 C_1^2 e^{-2\sqrt{r_i}}.
\]
Therefore we get
\[ \frac{|a - a'|}{f_{a,b}'(y)\sigma_i} \leq \frac{e}{C_9 C_i^2 C_2} e^{(2\sqrt{\pi} - (t_i)^{2/3})}. \]
Since \( t_i \) can be made as large as we want (because \( t_i \geq n_0 \)), we get \(|a - a'| < f_{a,b}'(y)\sigma_i/2.
Therefore from (6.3) we get
\[ \frac{1}{2} \frac{f_{a,b}'(y)}{f_{a,b}'(y')} \cdot \sigma_i < \frac{|\xi_{t_i+1}(a,b) - \xi_{t_i+1}(a',b)|}{|\Omega_{t_i+1}|} < 2 \frac{f_{a,b}'(y)}{f_{a,b}'(y')} \cdot \sigma_i. \]

Since \( y, y' \in I_{t_i} \cup I_{t_i+1} \), we have \(|y - c| \in [e^{-r_1}, e^{-r_1}]\), and the same holds for \( y' \) . Therefore, by (4.13), we get
\[ \frac{f_{a,b}'(y)}{f_{a,b}'(y')} < \frac{2 - 2b + C_{10} e^{-2r}}{2 - 2b + C_{9} e^{-2r-4}}. \]
The right-hand side above is a weighted average between \((2 - 2b)/(2 - 2b) = 1\) and \((C_{10} e^{-2r})/(C_6 e^{-2r-4}) = C_{10} e^4/C_9 > 1\), so it is smaller than \( C_{10} e^4/C_9\). Since we can switch \( y \) and \( y' \), we get
\[ \frac{C_9}{C_{10} e^4} < \frac{f_{a,b}'(y)}{f_{a,b}'(y')} < \frac{C_{10} e^4}{C_9}. \]
In such a way we get the following inequality with \( C_{22} = 2C_{10} e^4/C_9\).
\[ C_{22}^{-1} \frac{\sigma_i}{|\Omega_{t_i}|} \leq \frac{|\xi_{t_i+1}(a,b) - \xi_{t_i+1}(a',b)|}{|\Omega_{t_i+1}|} \leq C_{22} \frac{\sigma_i}{|\Omega_{t_i}|}. \]

Now we want to estimate from above \[ \frac{|\xi_{t_i+\nu}(a,b) - \xi_{t_i+\nu}(a',b)|}{|\Omega_{t_i+\nu}|}. \]
The numerator can be estimated as follows:
\[ |\xi_{t_i+\nu}(a,b) - \xi_{t_i+\nu}(a',b)| \leq |f_{a,b}'^{-1}(f_{a,b}'^{t_i+1}(c)) - f_{a,b}'^{-1}(f_{a,b}'^{t_i+1}(c))| + |f_{a,b}'(f_{a,b}'^{t_i+1}(c)) - f_{a,b}'(f_{a,b}'^{t_i+1}(c))|. \]
By a similar argument as in the proof of Lemma 5.7, in particular estimating \(|a - a'|\) as in that lemma we obtain that
\[ |f_{a,b}'(f_{a,b}'^{t_i+1}(c)) - f_{a,b}'(f_{a,b}'^{t_i+1}(c))| \leq \frac{1}{2} |\xi_{t_i+\nu}(a,b) - \xi_{t_i+\nu}(a',b)|. \]
Therefore we get the estimate
\[ |\xi_{t_i+\nu}(a,b) - \xi_{t_i+\nu}(a',b)| \leq 2 |f_{a,b}'^{-1}(f_{a,b}'^{t_i+1}(c)) - f_{a,b}'^{-1}(f_{a,b}'^{t_i+1}(c))| = 2 |f_{a,b}'^{-1}(y)| |\xi_{t_i+1}(a,b) - \xi_{t_i+1}(a',b)|, \]
where \( y \) is between \( \xi_{t_i+1}(a,b) \) and \( \xi_{t_i+1}(a',b) \) (using the Mean Value Theorem). Again by the same theorem there is \( y'' \in \Omega_{t_i+1} \) such that
\[ |\Omega_{t_i+\nu}| = (f_{a,b}'^{-1})'(y'')|\Omega_{t_i+1}|. \]
By Lemma 5.6, \((f_{a,b}'^{-1})'(y)/(f_{a,b}'^{-1})'(y'') \leq C_{22}, \) so we get
\[ \frac{|\xi_{t_i+\nu}(a,b) - \xi_{t_i+\nu}(a',b)|}{|\Omega_{t_i+\nu}|} \leq 2C_{18} \frac{|\xi_{t_i+1}(a,b) - \xi_{t_i+1}(a',b)|}{|\Omega_{t_i+1}|}. \]
Let us consider the interval $\omega_i \in P_{t_i}$ containing $\omega$ and denote its midpoint by $\hat{a}_i$. Then by (5.7),
\begin{equation}
|\xi_{t_i+\nu}(\hat{a}, b) - \xi_{\nu}(\hat{a}_i, b)| \leq e^{-4\sqrt{\nu}}
\end{equation}
for all $\hat{a} \in \omega_i$ and $\nu \leq p_i$.

Moreover, by Induction Statement (v), we may assume that
\begin{equation}
|\xi_{t_i+\nu}(\hat{a}, b) - c| \geq \frac{1}{2}|\xi_{\nu}(\hat{a}_i, b) - c|.
\end{equation}

There is an integer $\nu_0$ such that
\begin{equation}
\frac{1}{2} \cdot C_1 e^{-\sqrt{\nu}} \geq e^{-4\sqrt{\nu}}
\end{equation}
for all $\nu \geq \nu_0$. Note that $\nu_0$ depends only on $C_1$, which is independent of $\delta$. Therefore, we may assume that $\delta$ is so small that
\begin{equation}
2\delta \cdot 4^{\nu_0} \leq \frac{1}{2} \cdot C_1 e^{-\sqrt{\nu_0}}.
\end{equation}

Moreover, by Induction Statement (v),
\begin{equation}
|\xi_{\nu}(\hat{a}_i, b) - c| \geq C_1 e^{-\sqrt{\nu}}.
\end{equation}

Consider $\nu \leq p_i$. If $\nu \geq \nu_0$, then by (6.8), (6.10) and (6.12), we get
\begin{equation}
|\xi_{t_i+\nu}(\hat{a}, b) - \xi_{\nu}(\hat{a}_i, b)| \leq \frac{1}{2}|\xi_{\nu}(\hat{a}_i, b) - c|.
\end{equation}

Therefore
\[|\xi_{t_i+\nu}(\hat{a}, b) - c| \geq |\xi_{\nu}(\hat{a}_i, b) - c| - |\xi_{t_i+\nu}(\hat{a}, b) - \xi_{\nu}(\hat{a}_i, b)| \geq \frac{1}{2}|\xi_{\nu}(\hat{a}_i, b) - c| \]
and (6.9) follows.

If $\nu < \nu_0$ then by (4.3) and (5.8)
\[|\xi_{t_i+\nu}(\hat{a}, b) - \xi_{\nu}(\hat{a}_i, b)| \leq |\xi_{t_i+\nu}(\hat{a}, b) - \xi_{\nu}(\hat{a}, b)| + |\xi_{\nu}(\hat{a}, b) - \xi_{\nu}(\hat{a}_i, b)| \leq 4^\nu|\xi_t(\hat{a}, b) - c| + 4^\nu|\hat{a} - \hat{a}_i| \leq 4^\nu(\delta + |\omega_i|).
\]

By Lemma 5.1 (b) for $j = n_0$ and by making $n_0$ sufficiently large, we get $\partial_a \xi_j(a, b) \geq 1$. Therefore
\[|\omega| \leq |\xi_{n_0}(\omega, b)| \leq \delta.
\]
Thus, $|\omega_i| \leq \delta$, and we get
\[|\xi_{t_i+\nu}(\hat{a}, b) - \xi_{\nu}(\hat{a}_i, b)| \leq 2\delta \cdot 4^\nu.
\]

Together with (6.11) and (6.12) we get also in this case (6.13), and (6.9) follows.

Now for each $\nu$ we choose $\hat{a}_{\nu}$ between $a$ and $a'$, so that $\xi_{t_i+\nu}(\hat{a}_{\nu}, b) = \eta_{t_i+\nu}$. Thus, by (6.9) and (6.12),
\begin{equation}
|\eta_{t_i+\nu} - c| = |\xi_{t_i+\nu}(\hat{a}_{\nu}) - c| \geq \frac{1}{2}|\xi_{\nu}(\hat{a}_i, b) - c| \geq \frac{1}{2} \cdot C_1 e^{-\sqrt{\nu}}.
\end{equation}
By (6.2) we have

\[
\sum_{j=t_i+1}^{t_i+p_i} \left| f''_{a,b}(\eta_j) \right| |\xi_j(a, b) - \xi_j(a', b)| \\
\leq \sum_{\nu=1}^{p_i} \frac{80}{C_9} \cdot \frac{|\Omega_{t_i+\nu}|}{|\eta_{t_i+\nu} - c|} \cdot \frac{|\xi_{t_i+\nu}(a, b) - \xi_{t_i+\nu}(a', b)|}{|\Omega_{t_i+\nu}|}
\]

By the definition of bound periods and the definition of \( \Omega_{t_i+\nu} \), we have

\[
|\Omega_{t_i}| \leq e^{-4\sqrt{\nu}}.
\]

Moreover,

\[
|\Omega_{t_i}| = e^{-r_{t_i} - 1} e^{-r_{t_i} - 2}.
\]

Substituting those two inequalities, (6.14), (6.7) and (6.4) into the right-hand side of (6.15), we get

\[
\sum_{j=t_i+1}^{t_i+p_i} \left| f''_{a,b}(\eta_j) \right| |\xi_j(a, b) - \xi_j(a', b)| \leq \sum_{\nu=1}^{p_i} C_{23} \cdot \frac{e^{-4\sqrt{\nu}}}{e^{-\sqrt{\nu}}} \cdot \frac{\sigma_i}{e^{-r_{t_i}}}
\]

for some constant \( C_{23} \). This implies that there is a constant \( C_{24} \), such that

\[
\sum_{j=t_i+1}^{t_i+p_i} \left| f''_{a,b}(\eta_j) \right| |\xi_j(a, b) - \xi_j(a', b)| \leq C_{24} \frac{\sigma_i}{e^{-r_{t_i}}}.
\]

Together with the estimate on the first term of \( S'_i \), that we obtained long ago, we get a constant \( C_{25} \) such that

\[
S'_i \leq C_{25} \frac{\sigma_i}{e^{-r_{t_i}}}. \tag{6.16}
\]

Note that \( C_{25} \) depends on \( C^{st} \), but not on \( \delta \).

To estimate \( S''_i \), we use Lemma 4.12, and get immediately

\[
S''_i \leq C_{13} \frac{\sigma_{i+1}}{\delta}.
\]

However, \( \delta > e^{-r_{t_i+1}} \), so

\[
S''_i \leq C_{13} \frac{\sigma_{i+1}}{e^{-r_{t_i+1}}}. \tag{6.17}
\]

This estimate also applies to \( S''_0 \).

By Lemma 3.4 applied to a subinterval \( \omega' = [a, a'] \) (or \( [a', a] \)) of \( \omega \) (we can do it by Remark 5.12) and (5.28), see Remark 5.11, we get

\[
\sigma_{i+1} \geq \frac{1}{q_s} \inf_{a \in \omega'} (f_{a,b}^{t_{i+1} - t_i})(f_{a,b}^{t_i}(c)) \cdot \sigma_i \geq \frac{C_{11} C_{19}}{q_s} e^{(b_i^{2/3} - 4\sqrt{p_i + 1})} \cdot \sigma_i.
\]

As we already noticed in the proof of Lemma 5.10, by taking \( \delta \) sufficiently small we can make \( p_i \) as large as we need and we may assume that

\[
\frac{C_{11} C_{19}}{q_s} \exp\{p_i^{2/3} - 4\sqrt{p_i + 1}\} \geq 2,
\]

and therefore we get

\[
\sigma_{i+1} \geq 2\sigma_i. \tag{6.18}
\]
Now we are ready to estimate the logarithm of the left-hand side of (6.1), which is less then or equal to \( \sum_{i=0}^{m-1} (S'_i + S''_i) \). By (6.16) and (6.17), we get

\[
\sum_{i=0}^{m-1} (S'_i + S''_i) \leq (C_{25} + C_{13}) \sum_{i=0}^{m} \frac{\sigma_i}{e^{-r_i}}.
\]

Rearrange the sum \( \sum_{i=0}^{m} \sigma_i/e^{-r_i} \) and group it according to the values of \( r_i \). Set \( W_k = \{ i \in [1, m] : r_i = k \} \). Consider \( k \) such that \( W_k \) is nonempty. Then we can write

\[
\sum_{i \in W_k} \sigma_i e^{-r_i} \leq 2 \frac{\sigma_{\mu_k}}{e^{-k}},
\]

where \( \mu_k \) is the largest element of \( W_k \). However, \( \sigma_{\mu_k} \) is the length of an interval which is contained in the union of 3 subintervals of \( I_k \), and the length of each of those subintervals is \( |I_k|/k^2 \). Moreover, \( |I_k| < e^{-k} \). Thus,

\[
\sum_{i \in W_k} \sigma_i e^{-r_i} \leq \frac{6}{k^2}.
\]

If \( W_k \) is empty, then of course (6.19) also holds. In such a way we get

\[
\sum_{i=0}^{m-1} (S'_i + S''_i) \leq 6(C_{25} + C_{13}) \sum_{k=1}^{\infty} \frac{1}{k^2}.
\]

The right-hand side of the above inequality is finite, so we can denote its exponential by \( C_{14} \) and then (6.1) holds.

Lemma 6.2. There exists a constant \( C_{26} \), such that if \( a \) and \( a' \) are two parameter points, so that \( a, a' \in \omega \in P_n \), where \( n \) is a free return time and Induction Statement for \( n \) (and all smaller free return times) holds, then

\[
\frac{\partial_a f^k}{\partial_a f^k_{a',b}}(c) \leq C_{26}
\]

for all \( k \leq n \).

Proof. The lemma follows immediately from Lemma 6.1, Corollary 3.2 and Induction Statement (iii).

7. Part I of the proof of Theorem A

In this section we prove a proposition, which is an essential part of the proof of Theorem A, and is stated as follows.

Proposition 7.1. Let \( a = a_0 \) be an MT-parameter for \( f_a \) and let \( \varepsilon > 0 \) be given. There is a function \( \eta(\varepsilon) \to 0 \) and a function \( b_0(\varepsilon) \to 1 \) as \( \varepsilon \to 0 \) such that if \( b_0(\varepsilon) < b < 1 \), if \( \omega_0 \) is a parameter interval such that

\[
\omega_0 \subset (a_0 - \varepsilon, a_0 - \varepsilon^2) \cup (a_0 + \varepsilon^2, a_0 + \varepsilon),
\]

(7.1)
such that $I_{r,\ell} \subset \xi_{n_0}(\omega_0, b) \subset I_{r,\ell}^+$ and such that induction assumptions (i)-(vi) are satisfied for $n = n_0$. Then there is a set $\tilde{E}_b \subset \omega_0$ so that $|\tilde{E}_b| \geq (1 - \eta(\varepsilon))|\omega_0|$, $C = C(a_0)$ and $\tilde{\eta} = \tilde{\eta}(a_0) > 0$ so that

\begin{equation}
(7.2) \quad (f_{a,b}^n)'(f_{a,b}(c)) \geq Ce^{\tilde{\eta}n}, \quad \forall n \geq 0 \quad \forall a \in \tilde{E}_b.
\end{equation}

Note that the assumptions of Proposition 7.1 is satisfied by Lemma 5.1.

This, together with Proposition 8.1 in Section 8 immediately lead to the following

**Corollary 7.2.** The set $E$ of parameters for which the double standard map is uniformly expanding accumulates on the MT points $(a_0, 1)$ in the parameter space.

However we will need more general formulation of the propositions given above in order to prove Theorem A.

The proofs will be based on the induction formulated in Section 5. In the critical case $b = 1$, which we are not treating in detail, the remaining parameter set is of positive measure, while in the non-critical case $b < 1$ the remaining parameter set is a finite union of intervals.

We first discuss the parameter deletion due to the (BA) assumption.

If $n$ is a free essential return time for a partition element $\omega = (a, a')$ of a partition $\mathcal{P}_{n''}$, where $n''$ is the essential free return immediately before $n$.

At each time we may have to omit a fraction of the parameter interval because of (BA). Assume that the previous free return occurred in the interval $I_{r''}$. Its length is $\frac{C}{r''} |I_{r''}|$, $1 \leq c \leq 3$. By the (BA) assumption applied to time $n''$, we have

\[e^{-r''} \geq e^{-\sqrt{n''}}\]

Since $n - n''$ has a minimal length with estimate $|n - n''| \geq C \log(1/\delta)$, where $C$ is a constant only depending on $a_0$. Not also that $r'' \leq \sqrt{n}$. During the bound period the interval $K_r = (c, c + e^{-r'' - 1})$ of size $e^{-r'' - 1}$ is increased to size $e^{-v/r'' + 1}$, where $p'' \leq (r'')^{3/2}$ by Lemma 5.7. Our present interval is of length $\frac{C}{r''} |I_{r''}|$, $1 \leq c' \leq 3$.

For $a = a'$ the size of $f_{a,b}'(K_{r'' + 1})$ can be estimated by formula (4.12) as follows

\[|x - c|(2 - 2b + C_9(x - c)^2) \leq |f_{a,b}'(x) - f_{a,b}'(c)| \leq |x - c|(2 - 2b + C_{10}(x - c)^2).\]

By inserting $x = c + e^{-r'' - 1}$ we obtain an estimate for $|f_{a,b}(K_{r'' + 1})|$ as follows:

\begin{equation}
(7.3) \quad e^{-r'' - 1}(2 - 2b + C_7e^{-2r'' - 2}) \leq |f_{a,b}(K_{r'' + 1})| \leq e^{-r'' - 1}(2 - 2b + C_8e^{-2r'' - 2}).
\end{equation}

For the image of $\omega$ at time $n'' + 1$ we obtain the estimate

\begin{equation}
(7.4) \quad |\xi_{n'' + 1}(a, b) - \xi_{n'' + 1}(a', b)| = f_{a,b}'(y) \cdot |\xi_{n''}(a, b) - \xi_{n''}(a', b)| \pm |a - a'|
\end{equation}

Here $y \in I_r$ so it follows from (4.13) that $|a - a'|$ can be estimated by the first term as in the estimate of (6.3) and we obtain

\[2 - 2b + C_9e^{-2r'' - 2} < f_{a,b}'(y) < 2 - 2b + C_{10}e^{-2r''}.\]

By the definition of a free return we also have the estimate

\[\frac{1}{p''^2} e^{-r''} \leq |\xi_{n''}(a, b) - \xi_{n''}(a', b)| \leq \frac{3}{p''^2} e^{-r''}.
\]
By Lemma 5.6 (the bound distortion lemma) and comparison with the orbit of $K_r$, the size of $|\xi_{n'+p'+1}(a, b) - \xi_{n'+p'+1}(a', b)|$ has the lower bound.

$$\frac{1}{C_{18}} \cdot \frac{1}{r'} e^{-4\sqrt{p' + 1}} \geq \frac{1}{C_{18}} \cdot \frac{1}{r'} e^{-\sqrt{8(pr' + 3/2)}} \geq \frac{1}{C_{18}} e^{-83/2\cdot n^{3/8}}.$$  

We have again used that $\delta$ may be chosen arbitrarily small. Using (6.18), Lemma 4.9 and Lemma 3.4 and it follows that the relative fraction to be deleted is at most

$$C_{11}^{-1} \cdot \frac{1}{q} e^{-\sqrt{n}} \leq e^{-\frac{1}{2} \sqrt{n}},$$

since $n \geq n_0$ which at each time $n$ we in principle may have to do such a deletion. The remaining fraction of the parameter interval can then be estimated from below as

$$\geq \prod_{n=n_0}^{\infty} \left( 1 - e^{-\frac{1}{2} \sqrt{n}} \right)$$

Note that this is arbitrarily close to 1 as $N_0(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

**Proof of Proposition 7.1.** The proof of Proposition 7.1 is based on the induction. Note that the Cantor Set construction can be stopped at a finite stage $\hat{N}$, which is defined by the relation

$$2 - 2b \geq C_8 e^{-2\sqrt{N}}.$$  

After this time the term $2 - 2b$ of equation 4.12 dominates in the derivative.

$$\square$$

**Outline of the proof of Proposition 1.1.** We proceed as in the proof of Theorem A. In this case the time $\hat{N}$, after which the linear term $2 - 2b$ dominates in the derivative does not exist and the induction proceeds to infinite time. However the estimate (7.6) is still valid.

8. **Part II of the proof of Theorem A — the uniform expansion**

In this section we consider $b < 1$, and we construct a non-empty union of open intervals $\hat{E}_b \supset \tilde{E}_b$ so that for $(a, b) \in \hat{E}_b$ there is an integer $N$ so that for $(a, b) \in \hat{E}_b$, $f_a^N$ is uniformly expanding. This is formulated in Proposition 8.1. The set $\hat{E}_b$ is obtained by stopping the construction of the parameter set $\tilde{E}_b$ of Proposition 7.1 at a finite stage.

Let us outline the main idea of the proof of the uniform expansion. We will heavily use that the fact that $d = 2 - 2b > 0$, i.e. that the inflexion point is non-critical. In the case the starting point $x$ is outside the return interval $I^*$ we can use (i) of Lemma 4.1 to conclude that if $x, f_{a,b}(x), \ldots, f_{a,b}^{n-1}(x) \notin I^*$, then

$$(f_{a,b}^n)'(x) \geq C_2 e^{q_2 n};$$

Here it is important that the constant $C_2$ does not depend on $\delta$.

At the return time $n$ we have a derivative loss but this derivative loss is compensated during the bound period by Lemma 4.7. Since $p \to \infty$ as $\delta \to 0$, we can make the factor $e^{p^{2/3}}$ compensate $C_2/C^*$ by making $\delta$ sufficiently small. We also use that the
derivative of $f_{a,b}$ is bounded below by $f_{a,b}'(\frac{1}{2}) = 2 - 2b$, and we will also denote this number by $d$.

We state this result as follows.

**Proposition 8.1.** Let $a = a_0$ be a MT parameter. Then if $b_0 = b_0(a_0) < 1$ is sufficiently close to 1 then for all $b \in (b_0, 1)$ there is a set $\hat{E}_b$, which is a finite union of intervals $\{\omega_j\}_{j=0}^{q_b}$ so that for $a \in \omega_j$, there is an integer $M_j$ so that for all $x \in T$,

\[(f_{a,b}^M)'(x) \geq \lambda_j > 1.\]

**Proof.** The proof is really the same as the proof of Proposition 7.1 initially.

As before, we carry out the construction only until time $\hat{N}$. Here $\hat{N}$ is the smallest integer $\hat{N}$ satisfying

\[e^{-\sqrt{\hat{N}}} \leq d.\]

At time $\hat{N}$ we have a partition $\mathcal{P}_{\hat{N}}$ consisting of finitely many intervals $\{\omega_j\}_{j=1}^{M_\hat{N}}$.

We now aim to prove that the hyperbolicity statement (8.1) is true.

We first recall the two outside expansion statements of Lemma 4.1. Suppose that $(a,b) \in \mathcal{N}$. Then the following holds.

1) If $x, f_{a,b}x, \ldots, f_{a,b}^{n-1}x \notin I^*$ and $f_{a,b}^nx \in I^*$,

\[(f_{a,b})'(x) \geq C_2 e^{\kappa_2 n}.\]

2) There is an integer $M$ so that if $x, f_{a,b}x, \ldots, f_{a,b}^{M-1}x \notin I^*$ then

\[(f_{a,b}^M)'(x) \geq e^{\kappa_2 M}.\]

Let us define $R_0$ as the smallest integer $R_0$ satisfying $e^{-2R_0} \leq e^{-\sqrt{N}}$, i.e. $R_0$ corresponds to the $r$ where the square term in the expression for the derivative is of the same size as the constant term $d = 2 - 2b$. The bound period $p(x), x \in (c - e^{-R_0}, c + e^{-R_0})$ is chosen to be the infimum of the bound period for $y \in I_{\pm R_0}$.

We also know by (5.2) and Lemma 5.6 that

\[(f_{a,b}^p)'(x) \geq \frac{1}{C_{18}} e^{r^2/3}, \quad x \in I_{\pm r},\]

and it holds as well that

\[(f_{a,b}^p)'(x) \geq \frac{1}{C_{18}} e^{r^2/3}, \quad x \in (c - e^{-R_0}, c + e^{-R_0}).\]

Introducing $\kappa_7$ as

\[\kappa_7 = \frac{1}{2} \min_{r_s \leq |r| \leq R_0} \min_{x \in I_r} \frac{1}{p(x)^{1/3}},\]

we can in all cases write these estimates as

\[(f^p)'(x) \geq e^{\kappa_7 p}.\]

The factor $\frac{1}{2}$ is here used to absorb the constant $C_{18}$.

Let us in the following let us use the notation $\hat{I}_R$ for the union of $(c - e^{-R_0}, c + e^{-R_0})$ and the previously defined $I_{-R_0}$ and $I_{R_0}$. The idea is that the derivative
recovery has the same estimate for these three (original) intervals since \((f_{a,b})'(x) \sim d = 2 - 2b\) in \(\hat{I}_{R_0}\) and the bound period is defined in terms of \(I_{R_0}\).

Divide the set \(T \setminus I^*\) into several pieces.

We first consider the set
\[
X_M = \{x : x, f_{a,b}x, \ldots, f^M_{a,b}x \notin I^*\}.
\]
For \(x \in X_M\), hyperbolicity is valid by Lemma 4.9, (ii):
\[
(f^M_{a,b})'(x) \geq e^{\kappa_4 M}.
\]

We also introduce the sets
\[
X_k = \{x : x, \ldots, f^{k-1}_{a,b}x \notin I^*\text{ but } f^k_{a,b}x \in I^*\}, \quad 1 \leq k \leq M - 1.
\]
Pick a \(k \geq 1\). Now write the set
\[
X_k = \bigcup_{r \leq |r| \leq R_0} X_{k,r},
\]
where \(X_{k,r} = \{x \in X_k : f^k_{a,b}x \in I_r, |r| \leq |r| < R_0\}\) and
\[
X_{k,\pm R_0} = \{x \in X_k : f^k_{a,b}x \in I_{\pm R_0}\}.
\]

We then know that for \(x \in X_{k,r}\)
\[
(f^{k+p}_{a,b})'(x) \geq C_{11} e^{\kappa_2 k} e^{\kappa_7 p} \geq e^{\kappa_8 (k+p)},
\]
where \(\kappa_8 = \min(\kappa_1, \kappa_7 / 2)\).

Here we have used the fact that also for the minimal possible \(p\) the factor \(e^{\kappa_7 p}\) always compensates the constant \(C_{11}\) of Lemma 4.9, and this constant is independent of \(\delta\).

Hence we know that the entire set \(T\) can be written as a disjoint union of sets \(\{Y_j\}_{j=1}^d\) so that for some \(\kappa_9\) and all \(x \in Y_j\)
\[
(f^{n_j}_{a,b})'(x) \geq e^{\kappa_9 n_j}.
\]

We start with an \(x \in Y_{j_0}\). After \(n_{j_0}\) steps we will end up in \(Y_{j_1}\) and after another \(n_{j_1}\) steps we will end up in \(Y_{j_2}\) etc. The total time will be \(n_{j_0} + n_{j_1} + n_{j_2} + \cdots + n_{j_s}\) and
\[
(f^{n_{j_0} + \cdots + n_{j_s}}_{a,b})'(x) \geq e^{\kappa_9 (n_{j_0} + \cdots + n_{j_0})},
\]
where
\[
n_{j_0} + \cdots + n_{j_0} = \sum_{i=1}^{m} k_i n_i.
\]
Let \(n_{\text{max}} = \max_{1 \leq j \leq d} n_j\) and pick an integer \(N\) very large so that
\[
e^{\kappa_9 N} \cdot d_1^{n_{\text{max}}} \geq e^{\kappa_{10} N}.
\]
Here \(d_1 = 1/B\), where \(B = 4 \geq \max_{x \in T} |f'_{a,b}(x)|\).

For each point \(x\) there is an \(n = n(x) = n_{j_0} + n_{j_1} + n_{j_2} + \cdots + n_{j_s}\) so that
\[
N \leq n \leq N + n_{\text{max}}.
\]

We claim that
\[
(f^N_{a,b})'(x) \geq e^{\kappa_{11} N}.
\]
This follows since
\[(f_{a,b}^N)'(x) = (f_{a,b}^n)'(x)/(f_{a,b}^{n-N})'(f_{a,b}^N(x)) \geq e^{\kappa_9 N} d_1^{n_{\max}} \geq e^{\kappa_{11} N},\]
for a suitably \(\kappa_{11}\). We conclude that the statement of Proposition 8.1 holds. \(\square\).

We now have all ingredients for the proof of Theorem A.

Let \(\omega_0\) be an interval as defined in Proposition 7.1 satisfying (7.1) and let \(\tilde{E}_b\) be the set defined in this proposition. Let \(\tilde{E}_b = \tilde{E}_b^N \supset \tilde{E}_b\) be the set corresponding to the \(\tilde{N}\):th order construction of Proposition 8.1. \(\tilde{N}\) is here determined as the smallest integer satisfying \(e^{-\tilde{N}} \leq d\) as in the proof of Proposition 8.1. By (8.1) it then follows that the conclusion of Theorem A holds.

9. Proof of Theorem B

In this section we are going to prove the last result of the paper. The methods of its proof will be completely different than the ones used in the rest of the paper. We will use the term “countable” in the sense “at most countable.” For the definitions, see Introduction.

Proof of Theorem B. Fix \(b < 1\). Each tongue is open, so the set \(T_b\) is open. Therefore it is the union of countably many components, each of them an open interval. Since the points on the boundary of a tongue belong to \(TN\), and the sets \(T\) and \(TN\) are disjoint, each component is contained in one tongue.

We claim that the intersection of the closures of two distinct components \(A_1\) and \(A_2\) is empty. Suppose it is not and that \(a\) belongs to this intersection. Then \((a, b) \in TN\), so it has its type. This type must be the same as the type of each of the tongues containing \(A_1\) and \(A_2\), so those types are the same, that is, \(A_1\) and \(A_2\) are contained in the same tongue. If \(n\) is the period of the neutral periodic orbit of \(f_{a,b}\), the map \(f_{a,b}^n\) has an interval on which it looks like one of the Cases 1, 2 or 4 of Lemma 4.1 of [19]. By Theorem 4.1 and Lemma 2.6 of [20], this cannot be Case 4 (a neutral periodic point repelling from both sides), and by Lemma 4.2 of [19] it cannot be Case 1 or 2 (a neutral periodic point repelling from one side). This proves our claim.

If a parameter \(a \in TN_b\) does not belong to a boundary of a component of \(T_b\), then by Lemma 4.2 of [19] the neutral periodic orbit of \(f_{a,b}\) is repelling from both sides (Case 4), so by Theorem 4.1 and Lemma 2.6 of [20] \(a\) is isolated in the set of elements of \(T_b \cup TN_b\) which have type of the same period. This proves that there are only countably many such values of \(a\).

By the claim, the complement of \(T_b\) is a closed set without isolated points. The set \(TN_b\) is countable. Therefore \(E_b\) (which is the complement of \(T_b\) minus \(TN_b\)) is dense in the complement of \(T_b\).

The second part of the statement follows from the first one and the fact that each component of \(T_b\) is contained in one tongue. \(\square\)

References


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Matematiska institutionen, KTH, Lindstedtsvägen 25, S-100 44 Stockholm, Sweden
E-mail address: michaelb@math.kth.se

Department of Mathematical Sciences, IUPUI, 402 N. Blackford Street, Indianapolis, IN 46202, USA
E-mail address: mmisiure@math.iupui.edu

Department of Mathematics, Exeter University, Exeter EX4 4QF, UK
E-mail address: A.Rodrigues@exeter.ac.uk