Open problem session

Boris Solomyak:
Let $\| \cdot \|$ denotes the distance to the closest integer.
For which $\theta > 1$ can one find $C > 0$ and $\rho = \rho(\theta) > 1$ such that for all $N$

$$\max_{\tau \in [1, \theta]} \exp \left( - \sum_{n=1}^{N} \| \tau \theta^n \|^2 \right) < C \rho^{-N}?$$

False for
- $\theta$ Pisot (all conjugates inside the unit circle),
- $\theta$ Salem (all conjugates on unit circle except $1/\theta$),
- $G_\delta$ set.

True for almost all $\theta$ (Erdős 1940).

Theorem (Pisot). There exists such $\tau > 0$ that $\sum_{n=1}^{\infty} \| \tau \theta^n \|^2 < \infty$ if and only if $\theta$ is Pisot number.

Motivation: $\lambda = 1/\theta$. Let $\nu_\lambda$ be the distribution of the random series $\sum_{n=1}^{\infty} \pm \lambda^n$ where the signs are chosen independently with probabilities $(1/2, 1/2)$ (this is the Cantor-Lebesgue measure when $\lambda < 1/2$ and is usually called ‘infinite Bernoulli convolution’ for an arbitrary $\lambda < 1$). Then $\hat{\nu}_\lambda(t) = \prod_{n=0}^{\infty} \cos(2\pi t \lambda^n)$. Answering the Question would have implications for absolute continuity and smoothness properties of $\nu_\lambda$.

For more details and references see

Rodrigo Treviño:
Is there a translation surface of infinite genus, which is finite with respect to the planar area form, and has a lattice Veech group?

Martin Möller:
Consider the rel foliation of a stratum $\Omega \mathcal{M}_g(\xi)$ with leaves of complex dimension $|\xi| - 1$. These leaves carry a natural flat structure, and in fact, when $|\xi| = 2$, have the structure of a quadratic differential.

- Understand leaves with compact completion.

Theorem of M. Schmoll: Whenever a surface is square-tiled with $|\xi| = 2$, the leaf completion is compact and square-tiled. In $\Omega \mathcal{M}_2(1, 1)$ if the leaf is over a square-tiled surface
then the Veech group of the completion is $\text{Sl}(2, \mathbb{Z})$. Are there other compact square-tiled leaf-completions?

- What are closures of leaves (in strata/in moduli space)?

McMullen has an example of a straight trajectory in a leaf with wild closure.

Over primitive Teichmüller curves i.e. when a leaf is contained in $\Omega \Sigma D$ then $\mathbf{P} \Omega \Sigma D = H \times H / \text{Sl}(2, \mathbb{O}_D)$) and by Mautner all leaves are dense.

- Is the REL-foliation ergodic in every stratum with $|\kappa| \geq 2$?

**Barak Weiss:**

Is there a classification of square-tiled surfaces with Veech group equal to $\text{Sl}(2, \mathbb{Z})$?

**Mike Hochman:**

Let $\Pi_n = \{ \sum_{k=0}^n \sigma_k x^k | \sigma_k \in \{0, \pm 1\} \}$. Does there exist $c$ such that if $\alpha$ and $\beta$ are real roots of some polynomials from $\Pi_n$ then either $\alpha = \beta$ or $|\alpha - \beta| > c^n$?

Motivation: Let $\nu_\lambda$ denote the distribution measure of the random series $\sum_{n=1}^{\infty} \pm 1 \lambda^n$, where the signs are chosen i.i.d. with equal probabilities (this is the Bernoulli convolution with parameter $\lambda$). The measure $\nu_\lambda$ has dimension 1 if $\nu(E) = 0$ for every Borel set $E$ of Hausdorff dimension $< 1$. A positive answer to the question above would imply that $\dim(\nu_\lambda) = 1$ unless the parameter $\lambda$ is algebraic. The best current result allows a dimension 0 (but possibly uncountable) set of such parameters.
Uri Shapira:

Let

\[ \text{QI} = \{ \alpha \in \mathbb{R} : \alpha \text{ is a quadratic irrational} \} \] .

Given \( \alpha \in \text{QI} \) we denote by \( a_i(\alpha) \) the \( i \)'th digit of the continued fraction expansion of \( \alpha \) and by \( \nu_\alpha \) the normalized counting measure supported on the (eventual) period of \( \alpha \) under the Gauss map \( T \) in the unit interval (\( T : x \mapsto 1/x - \lfloor 1/x \rfloor \)). Let \( \nu \) denote the Gauss-Kuzmin measure on the unit interval.

**Definitions.** Let \( \alpha_n \in \text{QI} \) be a sequence.

(i) We say that \( \alpha_n \) is *asymptotically Gauss-normal* if \( \nu_{\alpha_n} \) converges in the weak* topology to \( \nu \).

(ii) We say that \( \alpha_n \) is *uniformly bounded* if there exists \( M \) such that \( \sup_n \limsup_k a_k(\alpha_n) \leq M \).

(iii) We say that \( \alpha_n \) is *uniformly divergent* if \( \liminf_n \liminf_k a_k(\alpha_n) = \infty \).

**Problem.** Let \( \alpha \in \text{QI} \).

- Can one always find a sequence of primes \( p_n \) such that \( p_n \alpha \) is asymptotically Gauss-normal?
- Can one always find a sequence of primes \( p_n \) such that \( p_n \alpha \) is uniformly bounded?
- Can one always find a sequence of primes \( p_n \) such that \( p_n \alpha \) is uniformly divergent?